NIU Ph.D. Candidacy Examination Fall 2018 (8/21/2018)

Electricity and Magnetism

You may solve ALL FOUR problems if you choose. The points of the best three problems will be counted towards your final score of this part of the examination. (40 points each. Total of 120 points)

Do not just quote a result. Show your work clearly step by step.

1. [40 points] A metal sphere of radius a carries a charge Q (Fig. 1). It is surrounded, out to radius b, by linear dielectric material of permittivity ε, forming a nanoparticle with a core/shell structure.

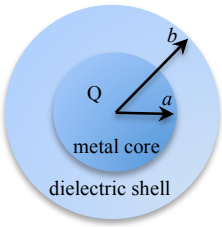

Fig. 1

(a) Calculate the electric displacement $\vec{D}$ for all points $r > a$, and inside the metal sphere. [6 points]
(b) Obtain $\vec{E}$ for $a < r < b$ and $r > b$, using $\vec{D}$ calculated in (a). [6 points]
(c) Compute the potential at the center of this sphere (core/shell structure). Put the reference point for the potential to be at infinity. [12 points]
(d) Calculate the polarization $\vec{P}$. [4 points]
(e) Calculate the bound charge in the dielectric at the outer surface at b and at the inner surface at a. Take the normal vector $\hat{n}$ points outward with respect to the dielectric, which is $+\hat{r}$ at b but $-\hat{r}$ at a. [12 points]

2. [40 points] A cylindrically-symmetric resonant radiofrequency cavity support an axial electric field (on its axis of symmetry) (Fig. 2) given by the time-dependent field $E_z = E_0(\mathrm{z})\cos(\omega_0 t + \phi)$ where $E_0(\mathrm{z})$ is the field spatial profile, t the time, and ϕ an arbitrary phase offset. The field is non vanishing within $-\frac{\lambda_0}{2} < z < \frac{L}{2}$. The cavity is made of a perfect metal.

   (a) Describe the meaning of the parameter ω_0. [5 points]
   (b) Give the other components of the electric field E_r and E_ϕ in cylindrical coordinate close to the axis of the cavity (that is under the paraxial approximation and ignoring the boundary conditions imposed by the cylindrical surface of the cavity wall). Confirm the electric-field components are consistent with the boundary conditions imposed by the cavity side walls [you do not need to consider the cylindrical surface since the fields are derived under the paraxial approximation (far from the cylindrical surface)]. [10 points]
   (c) Find the associated magnetic-field components close to the cavity axis. From your answer to (b) and (c) what is the name of the mode described by the derived electromagnetic fields? [10 points]

   For the remainder of the problem we consider the field spatial profile to be $E_0(z) = E_0 cos(kz)$.
   (d) How is k related to L? We further take $L = \frac{\lambda_0}{2}$ with $\lambda_0 = 2\pi c/\omega_0$. Consider an ultra-relativistic particle with charge $q>0$ and mass m entering the cavity at the time $t = 0$ on its axis at $z = -L/2$. What is the change in momentum? What happens if the particle has a radial offset r? For which value of ϕ the final axial momentum is maximized? What happens to the radial momentum for this value of ϕ? [10 points]
   (e) The cavity described in this problem is typically used to accelerate charged particles in accelerators. What can you conclude regarding the transverse effects of such a cavity, e.g. if one considers a realistic bunch of particle with a finite transverse dimension? How are the results modified if $E_0(\mathrm{z}) = \mathrm{E}_0$ is a constant? [5 points]

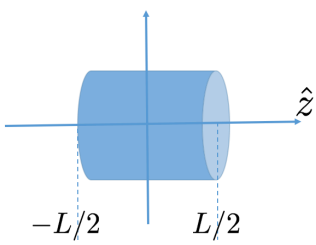

Fig. 2

3. [40 points] A rod of length L is lying on two perfectly conducting rails. The potential difference between the rails is held at V_0. Let R be the resistance of the rod. The rod is also connected to a mass m with a rope and pulley.

   (a) Find the terminal velocity of the rod if a magnetic field B is applied. [20 points]
   (b) Also calculate the fraction of the power supplied by the battery that is converted to mechanical power. [20 points]

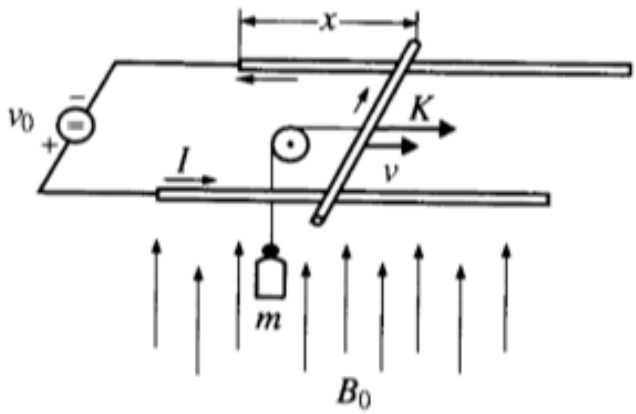

Fig. 3

4. [40 points] A very long cylindrical solenoid has radius a, length d, and N turns of wire. The axis of symmetry is the z axis, and the wire surrounding the solenoid carries a current $I(t)$ in the $\hat{\phi}$ direction, where

$$I(t) = \begin{cases} I_0 & \text{for } t < 0; \\ I_0 e^{-t/T} & \text{for } t \geq 0, \end{cases}$$

and t is the time and T is a constant. You may treat the current as slowly-varying in this problem.

   (a) What is $\vec{B}$ everywhere both inside and outside of the solenoid, as a function of time t? [10 points]
   (b) Find $\vec{E}$ everywhere both inside and outside of the solenoid, as a function of time t. [14 points]
   (c) Find the total energy stored in the electromagnetic fields, as a function of t. [8 points]
   (d) Consider an imaginary long cylinder of length d and radius r, coaxial with and inside the solenoid (so that $r < a$). Find the power being transmitted through the surface of this cylinder, as a function of time. (Give your answer as positive if energy is flowing out, or negative if energy is flowing in.) [8 points]

NIU Ph.D. Candidacy Examination Spring 2018 (1/9/2018)

Electricity and Magnetism

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Do not just quote a result. Show your work clearly step by step.

1. [40 points] **Particle refraction at the boundary between two regions:** We consider a charged particle with mass m, charge q, and total energy E, crossing the boundary between two regions (1) and (2) subjected to different electrostatic potentials $\Phi_1$and Φ_2 taken to be constants; see Figure 1. The particle's trajectory is taken to be in the (x, y) plane and crosses the boundary defined by the plane $x = 0$.

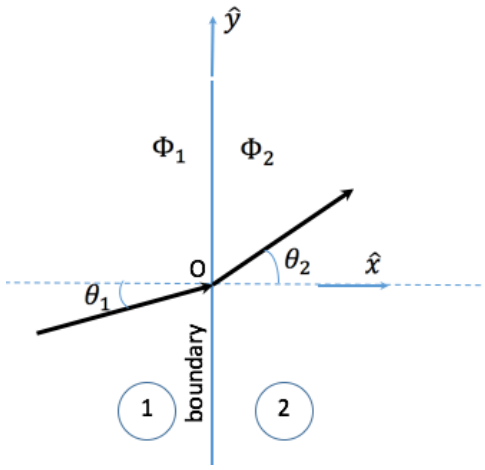

Figure 1

   (a) Given the geometry of the problem, give two conserved quantities. [5 points]

   (b) From (a), show that the particle angles θ_1 and θ_2 are related by a condition similar to Snell's law in Optics. Identify the quantities that play the role of the index of refraction and relate them to the particle properties and potentials $\Phi_1$and Φ_2. [15 points]

   (c) Discuss the evolution of θ_2 (for $\theta_1 \neq 0$) as a function of the potential difference $\Delta\Phi = \Phi_1 - \Phi_2$ identifying the different possible cases. Especially, give a condition between the angle θ_1, $\Delta\Phi$, and the incident momentum p_1 to have a "grazing" refraction ($\theta_2 = \frac{\pi}{2}$). [13 points]

   (d) Discuss how the boundary could be practically implemented, and what is the needed surface-charge density to establish the needed potential difference. [7 points]

2. [40 points] **Electric multipole expansion**
The potential of an arbitrary localized charge distribution can be expressed in the following electric multipole expansion in powers of $1/r$.

$$V(r)=\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int(r')^n P_n(\cos\theta')\rho(r')d\tau'.$$

Figure 2 defines the appropriate variables: θ' is the angle between **r** and **r'**. $d\tau'$ is an infinitesimal volume element. s is the distance between $d\tau'$ and P. $P_n(\cos\theta')$ is the Legendre polynomials in terms of $\cos\theta'$. $P_n(x)$ in general is defined by

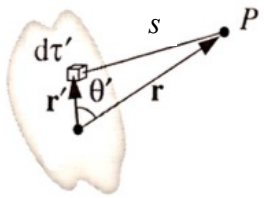

Figure 2

$$P_n(x)=\frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n\left(x^2-1\right)^n.$$

A sphere of radius R, centered at the origin, carries the charge density

$$\rho(r,\theta)=k\frac{R}{r^2}(R-2r)\sin\theta,$$

where k is a constant, and r, θ are the standard spherical coordinates. Using the above multipole expansion of V in powers of $1/r$, calculate
(a) the monopole term, [10 points]
(b) dipole term, [10 points]
(c) quadruploe term. [10 points]
Then,
(d) find the approximate potential (the lowest non-zero term) for points on the z axis, far from the sphere, and briefly comment on the meaning of the successive terms. [10 points]

3. [40 points] **Moving loop in a constant magnetic field:** A circular loop with initial radius r_0 is inside a solenoid with constant magnetic field $\boldsymbol{B}_0$; see Figure 3. The loop is connected to a bulb and the whole circuit has a resistance R. At time $t=0$ the loop is moved by pulling the two ends through a hole in the solenoid at constant speed u maintaining the circular shape of the loop (the whole loop remains inside the solenoid).

(a) What is the current in the loop $I(t)$ and the power $P(t)$ dissipated in the circuit? [12 points]

(b) What is the total energy dissipated in the circuit? [8 points]

(c) What is the force one should apply at point A to pull the wires at speed u? [10 points]

(d) Show that the work done mechanically equals the energy dissipated in the lamp circuit computed in question (b). [10 points]

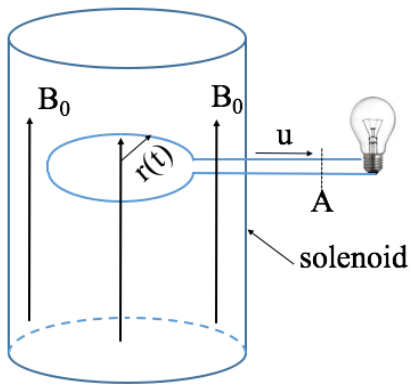

Figure 3

4. [40 points] An infinitely long, non-magnetic, solid cylinder has radius a. Consider a finite co-axial cylinder of smaller radius r and length ℓ. For each of the independent situations described in parts (a) and (b), find the time derivative of the total electromagnetic field energy inside, and the flux of energy through the surface, for the finite cylinder. If they do not agree, provide a reason and account for the difference quantitatively.

   (a) The infinite cylinder of radius a is wrapped with n turns of wire per unit length along the axis, carrying a slowly varying current $I(t) = I_0 + kt$, where k is a constant. Work to first order in k, dropping all quantities of order k^2. [20 points]

   (b) The infinite solid cylinder is not wrapped with wire, but is made from material with finite conductivity σ, and carries a current I along its axis of symmetry, with magnitude that is uniform (in space) and constant (in time). [20 points]

NIU Ph.D. Candidacy Examination Fall 2017 (8/22/2017)

Electricity and Magnetism

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Do not just quote a result. Show your work clearly step by step.

1. [40 points] A static electric field with spherical symmetry is described by

$$\vec{E}(r) = \left(cre^{-\frac{r}{R}} \right) \hat{\mathbf{r}}$$, where c and R are constants.

   (a) Determine the charge density, $\rho(r)$. [8 points]
   (b) Find the total charge of the system. [8 points]
   (c) Find the static electric potential $V(r)$. [8 points]
   (d) Find the total energy stored in the electric field. [8 points]
   (e) A small test charge $+q$ is released at rest at the radial location $r = R$. What is the kinetic energy when it reaches a point far away? [8 points]

2. [40 points] A toy consists of identical donut-shaped permanent magnets (magnetization parallel to the axis), which slide on a vertical rod without friction (Fig.1). Treat the magnets as dipoles with mass m_d md and dipole moment $\vec{\mathbf{m}}$. (a) and (b) are questions on a magnetic dipole in general, and (c) and (d) are specific to this magnet toy, using the results of the general dipole.

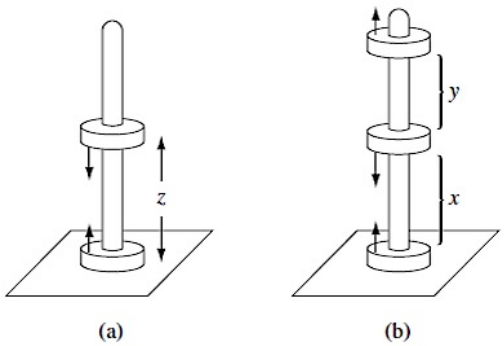

Fig.1 Donut-magnet toys for problem 2(c) and (d)

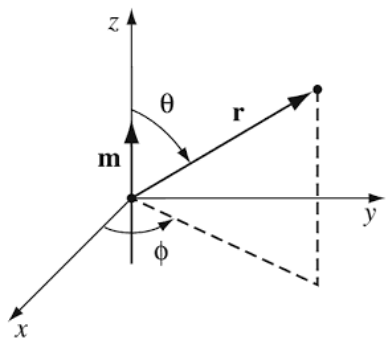

Fig.2. Geometry of magnetic dipole and coordinates for problem 2 (a) and (b).

   (a) In general, what is the vector potential of a (pure) magnetic dipole $\vec{\mathbf{A}}_{dip}(\mathbf{r})$ if the magnetic dipole $\vec{\mathbf{m}}$ is at the origin and points in the z-direction (Fig.2)? Express in the spherical coordinates. [6 points]

   (b) Then, still in general, what is the magnetic field of a (pure) dipole $\vec{\mathbf{B}}_{dip}(\mathbf{r})$? Express in the spherical coordinates. [4 points]

Now let's consider the toy mentioned above .

(c) If one puts two back-to-back magnets on the rod, the upper one will "float" – the magnetic force upward balancing the gravitational force downward (Fig. 1a). At what height (z) does it float? [15 points]

(d) Now one adds a third donut-shaped magnet (parallel to the bottom one) as in Fig. 1b. Show that the ratio of the two heights ($\alpha \equiv x/y$) is expressed as

$2 = \left(1/\alpha^4\right) + \left(1/(\alpha+1)^4\right)$. $\quad * \alpha = 0.850115$ (numerically calculated). [15 points]

3. [40 points] A rectangular loop (with sides a and b) is located the magnetic field produced by an infinitely long current-carrying straight wire; see Fig. 3.
   (a) Find the induced voltage in the loop if it is fixed in space while the current in the long wire is time dependent: $I(t) = I_0 \cos(\omega t)$. [15 points]
   (b) Find the magnitude and direction of the induced voltage as a function of r when the current is constant in time [$I(t) = I_0$], while the loop moves *towards* the wire with velocity u. [15 points]
   (c) Find the induced voltage in the loop when the current is $I(t) = I_0 \cos(\omega t)$ and loop moves towards the wire with velocity u. [10 points]

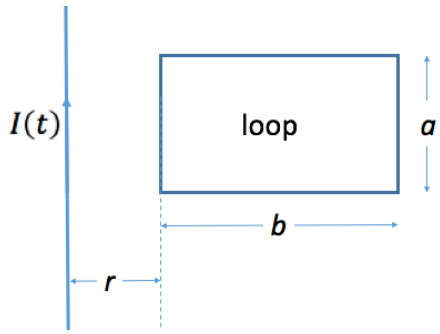

Fig. 3. Diagram for Problem 3.

4. [40 points] An electromagnetic wave propagates in a long coaxial cable consisting of an inner cylinder of radius a and an outer cylinder of radius b. Suppose that the magnetic field for $a < r < b$ is given in cylindrical coordinates by

$$\vec{B} = B_0 (r/a)^n \cos(kz - \omega t)\hat{\phi} \qquad (1)$$

where B_0, ω, and k are positive constants and n is a rational number. The region $a < r < b$ is vacuum.
   (a) Solve for the constant n. [8 points]
   (b) Find the electric field in the region $a < r < b$. [8 points]
   (c) Find the relation between k and ω. [6 points]
   (d) Find the charge density and the current density on the inner cylinder. [12 points]
   (e) Find the total time-averaged power transmitted by the cable. [6 points]

NIU Ph.D. Candidacy Examination Spring 2017 (1/10/2017)

Electricity and Magnetism

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Do not just quote a result. Show your work clearly step by step.

1. [40 points] A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω.
   (a) Find the vector potential it produces at point $\boldsymbol{r}$ (both $r \le R$ and $r \ge R$). [20 points]
   (b) Find the magnetic field inside this spherical shell. [10 points]
   (c) Find the magnetic field outside this spherical shell. [10 points]

2. [40 points] A sphere of radius a has a bound charge Q distributed uniformly over its surface. The sphere is surrounded by a uniform fluid dielectric medium with fixed dielectric constant ε, as shown in Fig. 1. The fluid also contains a free charge density given by
$$\rho(\bar{r}) = -kV(\bar{r}),$$
where k is a constant and V(r) is the electric potential at r relative to infinity.

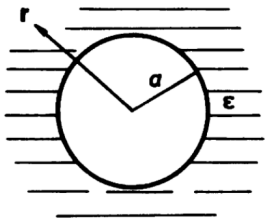


Fig. 1

   (a) Find the potential everywhere, letting $V = 0$ at $\mathbf{r} \rightarrow \infty$. [20 points]
   (b) Find the pressure as a function of $\mathbf{r}$ in the dielectric. [20 points]

3. [40 points] Consider a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical shells of length l. One is inside the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface. The other is outside the solenoid at radius b, carries charge $-Q$. $l >> b$. When the current in the solenoid is gradually reduced, the cylinders begin to rotate.
   (a) Sketch the set up of the above solenoid-two cylindrical shell system with relevant physical parameters, charges, and the electrical current in the cylindrical coordinate system ($\hat{\mathbf{s}}, \hat{\phi}, \hat{\mathbf{z}}$). [4 points]
   (b) *Before* the current I was switched off, find the electric field in the region between the cylinders ($a < s < b$) $\boldsymbol{E}$, a magnetic field inside the solenoid $\boldsymbol{B}$, and the total angular momentum in the fields $\mathbf{L}_{em}$. [16 points]
   (c) *When* the current I was switched off, find induced circumferential electric fields $\boldsymbol{E}$ inside ($s > R$) and outside ($s < R$) of the solenoid, the torque $\boldsymbol{N}_b$ and an angular momentum on the outer cylinder $\mathbf{L}_b$, and the torque $\boldsymbol{N}_a$ and an angular momentum on the inner cylinder $\mathbf{L}_a$. What is the total angular momentum $\mathbf{L}_{\mathrm{em}}$? [16 points]
   (d) Summarizing the above, is the total angular momentum (fields plus matter) $\mathbf{L}_{em}$ conserved before and after the current I was switched off? Yes or No. [4 points]

4. [40 points] An electromagnetic wave moving in vacuum in the $+\hat{z}$ direction with angular frequency ω is normally incident at the plane of interface ($z = 0$) with an Ohmic conducting material. The vacuum occupies the region $z < 0$, while the conductor has conductivity σ and occupies the region $z > 0$. The conductor has the same electric permittivity and magnetic permeability as vacuum (so $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$). The electric field amplitude of the wave just inside the conductor is E_0.

   (a) [25 points] In the limit of high conductivity, $\sigma \gg \omega\varepsilon_0$, calculate how deeply the electric field penetrates into the conductor before decreasing to $1/e$ of its amplitude at the surface.

   (b) [15 points] Find the time-averaged power absorbed per unit area of the conductor.

NIU Ph.D. Candidacy Examination Fall 2016 (8/16/2016)

Electricity and Magnetism

You may solve ALL FOUR problems if you choose. The points of the best three problems will be counted towards your final score of this part of the examination. (40 points each. Total of 120 points)

Do not just quote a result. Show your work clearly step by step.

1. [40 points] **Charge-limited emission in an electron gun:** We consider an electron gun diagrammed in the figure below. An electrostatic field E_a is applied between the cathode and anode planes. A thin sheet of electron [with infinitely small length, surface charge density $-\sigma$ (in our convention $\sigma>0$), and total charge Q] is emitted from the cathode plane located at z=0. We consider the sheet to be an infinite plane and assume it has a vanishing velocity at the emission time ($t=0$).

   (a) Compute the electric field $\boldsymbol{E_b}$ generated by the electron sheet at any axial distance z (both magnitude and direction) when the sheet is at $z=d$. Assuming the applied field $\boldsymbol{E_a}$ is small enough to ensure the dynamics of the electron sheet remains non-relativistic, give the kinetic energy of the electron sheet as function of z? [12 points]

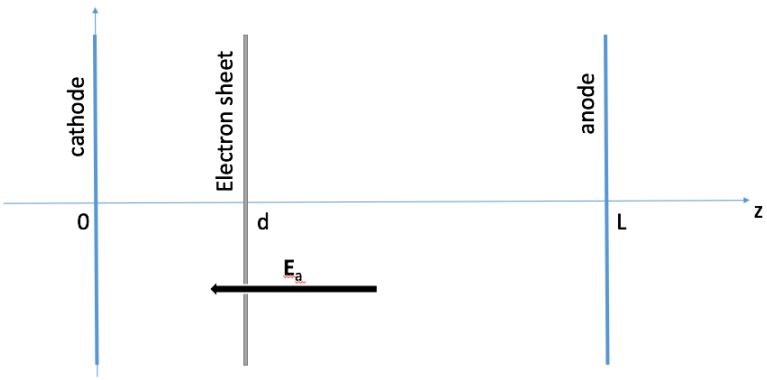
   (b) We now consider a second *identical* thin sheet emitted at a time τ after the first electron sheet (discussed in previous questions)

      i. What is the total force on this second sheet and how does it affects its motion. Especially compare the kinetic energy of the second sheet with respect to the first one at the time each reaches the anode at $z=L$?[7 points]

      ii. Give the axial separation between the two sheets when the first sheet is at $z=d$ in the limit when $\tau=0$. [6 points]

      iii. Using your answer from (a) and (b), explain how does the first sheet affects the dynamics of the second sheet and especially consider the case when the field from the first sheet E_b become comparable or larger than the applied field E_a [7 points]

      iv. Consider that the setup is used to emit a *continuous* electron beam. Based on the previous analysis what can you conclude on the maximum possible surface charge density of this beam. What would happen if one tries to produce higher surface charge densities? [8 points]



2. [40 points] A parallel-plate capacitor is made of circular plates as shown is Fig. 1. The voltage across the plates (supplied by long resistanceless lead wires) has the time dependence $V = V_0 \cos\omega t$. Assume $d << a << c/\omega$, so that fringing of the electric field and retardation may be ignored (Region (I): between two plates, Region (II): right above the upper plate. These two regions are sufficiently distant from the edges of the plates.).

   (a) Use Maxwell's equation and symmetry arguments to determine the electric and magnetic fields in region I as function of time. [20 points]

   (b) What current flows in the lead wires and what is the current density in the plates as a function of time? [20 points]

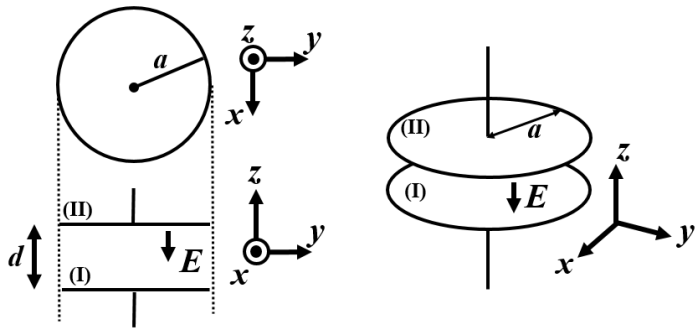


Fig. 1

3. [40 points] Wave equations.

   (a) Write down the integral and differential forms of Maxwell's equations. [10 points]

   (b) Set the source terms in the differential forms to zero and from the resulting equations derive the wave equations for the electric and magnetic fields. [15 points]

   (c) Assume the electric field solution to the wave equation is $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = E_0 \hat{x} cos(kz - \omega t + \delta)$ and hence show $c = 1/(\mu_0 \varepsilon_0)^{1/2}$. [15 points]

4. [40 points] A small source of electromagnetic radiation is located near the origin. The potentials for large r are given in spherical coordinates by:

$$V(\vec{\boldsymbol{r}}, t) = -V_0 cos\big(\omega(t - r/c)\big)\frac{sin\theta}{r}$$
$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}, t) = A_0 cos\big(\omega(t - r/c)\big)\frac{sin\theta}{r}\big(\hat{\phi} + \hat{r}\big),$$

   where V_0 and A_0 are constants.

   (a) Find the electric and magnetic fields for large r (consistently neglecting contributions that fall off faster than $1/r$). [13 points]

   (b) Solve for V_0 in terms of other quantities, and use the result to simplify your answers for the fields. [13 points]

   (c) Is this electric dipole radiation, magnetic dipole radiation, both, or neither? [5 points]

   (d) A gauge transformation is performed, so that $V = 0$ in the new gauge. What is $\vec{\boldsymbol{A}}$ in the new gauge? Again, neglect contributions that fall off faster than $1/r$. [9 points]

NIU Ph.D. Candidacy Examination Spring 2015 (2/20/2016)

Electricity and Magnetism

Solve 3 out of 4 problems. (40 points each. Total of 120 points)

Do not just quote a result, show your work clearly step by step.

1. [40 points] As in Fig. 1, you are given the not-so-parallel plate capacitor.
   (a) Neglecting edge effects, when a voltage difference V is placed across the two conductors, find the potential everywhere between the plates. [20 points]
   (b) In case of this wedge filled with a medium of dielectric constant ε, calculate the capacitance of the system in terms of the constants given. [20 points]

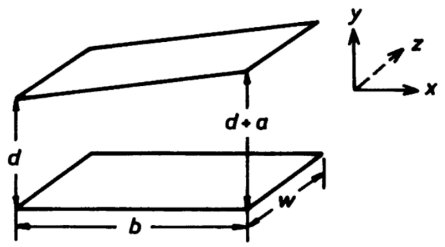


Fig. 1

2. [40 points] Consider the Earth as a sphere with its mean radius of R = 6371 *k*meters, and magnetically as a dipole situated at its center, with dipole moment magnitude $m = |\vec{m}| = 7.9 \times 10^{22}\, A\text{meters}^2$. [*In this problem, the units of meters are spelt out in order to distinguish it from the symbol *m* for the magnetic moment.]
   (a) The dipolar magnetic fields have the following components:

$$B_r = \frac{\mu_0}{2\pi}\frac{\cos\theta}{r^k} m, \qquad (1)$$

$$B_\theta = \frac{\mu_0}{4\pi}\frac{\sin\theta}{r^k} m, \qquad (2)$$

$$B_\phi = 0, \qquad (3)$$

   in spherical coordinates (r,θ,ϕ). What is the value of k ? Justify your answer with mathematical details. [15 points]
   (b) The Maxwell tensor can be represented in component form as follows:

$$T_{ij} = E_i D_j + H_i B_j - \frac{1}{2}\delta_{ij} U_V \qquad (4)$$

   What is the rank of the tensor? Define every symbol that appears in (4), and give the explicit definition of U_V [15 points]
   (c) Derive the expression of the mechanical stress in the radial direction at the Earth's magnetic North pole, and then estimate the order of magnitude in N/meters^2 . [10 points]

3. [40 points] Consider a small circular conducting wire ring of radius a and resistance R, which lies in the xy plane with its center at the origin. In the following, neglect the self-inductance of the ring; this turns out to be a good approximation in the limit of large resistance R. Suppose the ring is placed in a uniform but time-dependent magnetic field $\vec{B} = B_0 \cos(\omega t)\hat{z}$.

   (a) What is the current induced in the ring as a function of t? How much total energy is lost to heating the ring, during a very long time T? [15 points]
   (b) Assuming that ω is not too large, find the total energy lost by the ring to electromagnetic radiation, in a very long time T. [20 points]
   (c) Find the value of the resistance R such that the energy lost to radiation in part (b) is equal to the energy lost to heat in part (a). [5 points]

   Possibly useful information: far from an oscillating magnetic dipole moment, $m_0 \cos(\omega t)\hat{z}$, the magnetic field expressed in spherical coordinates is given by

$$\vec{B} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\frac{\sin\theta}{r}\cos\left[\omega\left(t - r/c\right)\right]\hat{\theta} \qquad (1)$$

4. [40 points] Suppose velocity $\vec{v}$ and acceleration $\vec{a}$ of a moving charged particle are collinear (at time of t_r, as in the regarded time), as for example, in straight-line motion. Here, velocity $\vec{v}$ is along the z axis as in Fig. 2.

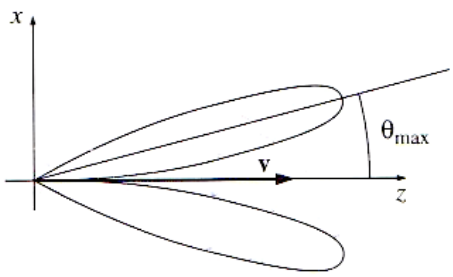


Fig.2

   (a) Show that the angular distribution of the radiation is expressed as the power P radiated by the particle into the solid angle Ω,

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{sin^2\theta}{(1-\beta cos\theta)^5}$$

   where $\beta \equiv v/c$, q is the charge of the particle. [20 points]
   (b) Find the angle θ_{max} at which the maximum radiation is emitted. [10 points]
   (c) Find the total power emitted. [10 points]

NIU Ph.D. Candidacy Examination Fall 2015 (9/19/2015)

Electricity and Magnetism

Solve 3 out of 4 problems. (40 points each. Total of 120 points)

Do not just quote a result, show your work clearly step by step.

1. [40 points]

   (a) Find the force on a square loop placed as shown in Fig 1, near an infinite straight wire. Both the loop and the wire carry a steady current I. [20 points]

   (b) Suppose the current $I(t)$ in the long straight wire in Fig 1 is changing slowly according to $I(t) = I_0 \cos(\omega t)$. Find the current induced in the square loop as a function of time, if it has resistance R. [20 points]

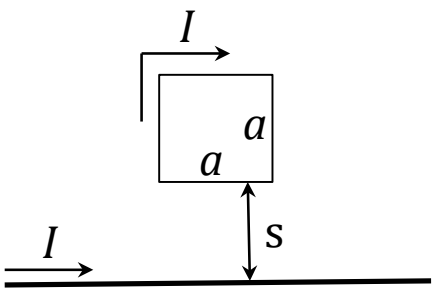


Fig 1

2. [40 points] A capacitor is made of two plane parallel plates of width a and length b separated by a distance d ($d << a$, b), as in Fig. 2. The capacitor has a dielectric slab of relative dielectric constant K between the two plates.

   (a) The capacitance is connected to a battery of emf V. The dielectric slab is partially pulled out of the plates such that only a length x remains between the plates. Calculate the force on the dielectric slab which tends to pull it back into the plates. [20 points]

   (b) With the dielectric slab fully inside, the capacitor plates are charged to a potential difference V and the battery is disconnected. Again, the dielectric slab is pulled out such that only a length x remains inside the plates. Calculate the force on the dielectric slab which tends to pull it back into the plates. [20 points]

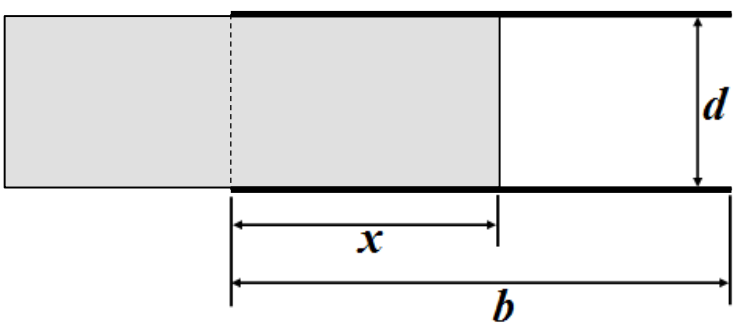


Fig. 2. A capacitor made of two plane parallel plates with a dielectric slab.

3. [40 points] A soap film of thickness a and permittivity ε is suspended in empty space (Fig. 3). The permittivity of the soap film is very large compared to that of vacuum, $\varepsilon >> \varepsilon_0$. Two charged ions, each of charge $+Q$, are located a distance R apart, in the midplane of the film. Find the force between the ions in the three limiting cases:

   (a) if $R \ll a$ [13 points]

   (b) if $a \ll R \ll \varepsilon a$ [14 points]

   (c) if $\varepsilon a \ll R$. [13 points]

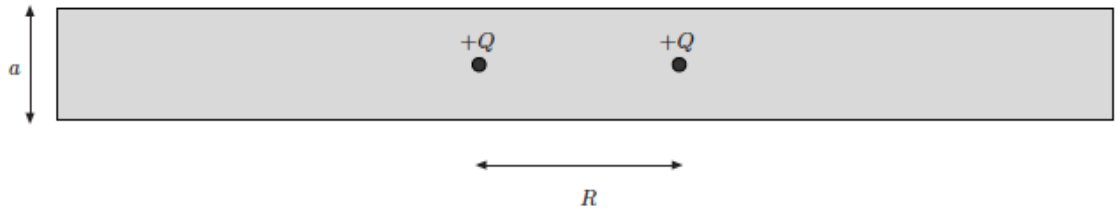


Fig. 3

4. [40 points] Consider the following idealized situation with an infinitely long, thin, conducting wire along the z axis. For $t < 0$ it is current-free, but at $t = 0$ a constant current density $\vec{J}$ is applied simultaneously over the entire length of the wire. Consequently, the wire carries the current

$$\vec{J} = \begin{cases} 0, & t < 0; \\ |\vec{J}|\hat{z}, & t \geq 0. \end{cases} \tag{1}$$

It is assumed that the conductor can be kept uncharged, i.e. $\rho = 0$.

(a) Determine scalar and vector potentials induced everywhere in space, $\phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ as functions of time. [10 points]

(b) Determine the magnetic and electric fields induced everywhere in space, $\vec{B}(\vec{x},t)$ and $\vec{E}(\vec{x},t)$ as functions of time. [20 points]

(c) Calculate the total power radiated per unit wire length. Comment on the unphysical behavior at $t = 0$ and its explanation for realistic systems. [10 points]

NIU Ph.D. candidacy examination Spring 2015 (2/21/2015)

Electricity and Magnetism

Solve 3 out of 4 problems. (40 points each. Total of 120 points)

Do not just quote a result, show your work clearly step by step.

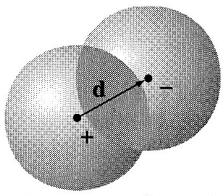
1. [40 points]

   (a) For a uniformly charged sphere (radius R, charge density ρ),

      i. Find the electric field everywhere. [15 points]

      ii. Find the energy stored in this configuration. [15 points]

   (b) Two spheres, each of radius R and carrying uniform charge densities $+\rho$ and $-\rho$, respectively, are placed so that they partially overlay as in the figure below. The vector $\vec{d}$ is the vector from the positive center to the negative center. Find the value of the field in the region of overlap using $\vec{d}$, [10 points]



2. [40 points] A toroidal coil has a rectangular cross section, with inner radius a, outer radius $a + w$, and height h. It carries a total of N tightly wound turns, and *the current is increasing at a constant rate* ($dI/dt = k$).

   (a) In the quasistatic approximation, find the direction and the magnitude of the magnetic field $\vec{B}$ everywhere (for points inside and outside of the toroidal coil). [6 points]

   (b) Calculate the flux of $\vec{B}$ through the cross section the toroid. [14 points]

   (c) Assume w and h are both much less than a. Find the induced electric field $\vec{E}$ at a point z above the center of the toroid. [20 points]

3. [40 points] An electric dipole moment $\vec{p}(t) = p_0 \cos(\omega t)\hat{z}$ oscillating along the z axis generates radiating electric and magnetic fields. Far away from the dipole, the scalar and vector potentials due to this dipole are:

$$V = -\frac{p_0 \omega}{4\pi\varepsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - r/c\right)\right]$$

$$\vec{A} = \frac{X}{r}\sin\left[\omega\left(t - r/c\right)\right]\hat{z}$$

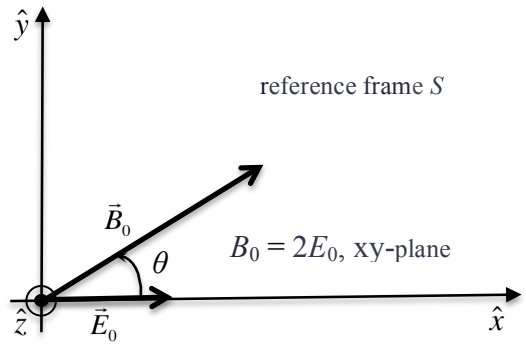
in SI units where $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light, and (r, θ, φ) are the usual spherical coordinates, and X is a constant.

(a) Use Maxwell's equations to derive the constant X in terms of the other quantities. (Do not just quote a result.) [15 points]

(b) Derive, from the results above, the total time-averaged power emitted from this dipole. (Do not just quote a result.) [15 points]

(c) What is the ratio of the time-averaged power per unit area received by two detectors, one at (x, y, z) = (D, 0, D) and the other at (D, 0, 0), where D is a very large distance? [10 points]

4. [40 points]
In a certain reference frame S, a static uniform electric field $\vec{E}_0$ is parallel to the x-axis, and a static uniform magnetic field $\vec{B}_0$ with magnitude $cB_0 = 2E_0$ lies in the xy-plane, making an angle $\theta = 30°$ ($\neq 0$) with respect to the x-axis as shown in the figure below. Here, c is the speed of light in vacuum. Determine the scaled relative velocity (relativistic β) $\beta = v/c$ of another reference frame S' moving in the z-direction relative to S, $\vec{v} = v\hat{z}$, in which the electric and magnetic fields are parallel.



The reference frame's S' velocity relative to S is $\vec{v} = v\hat{z}$