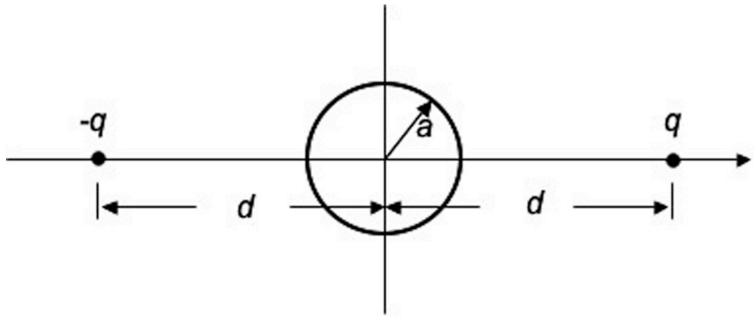
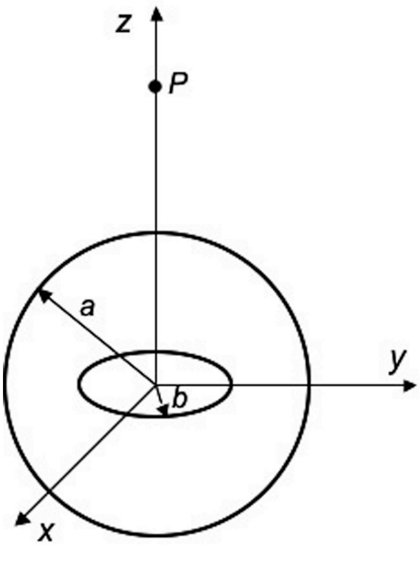
NIU Ph.D. qualifier examination 2006 fall (9/23/2006)
Electricity and Magnetism
Solve 3 out of 4 problems.

1. a. Two point charges, $+q$ and $-q$, are separated by a distance 2d. An isolated conducting spherical shell of outer radius a is inserted at the center point of the line joining the two charges. Determine the force acting on the charges for the case $d >> a$.



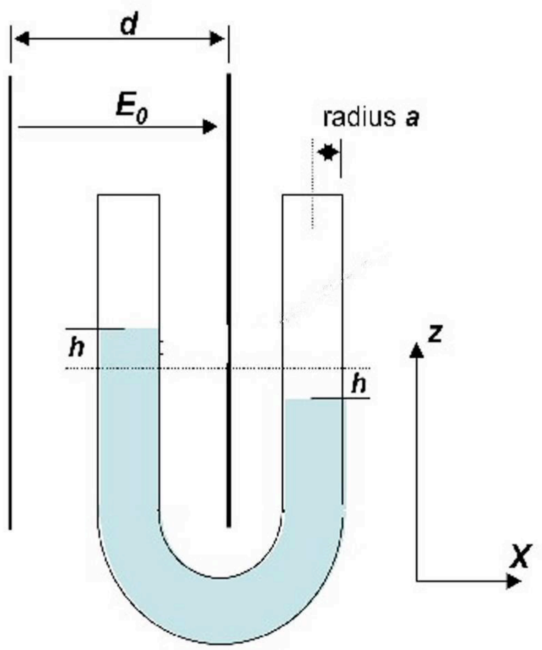
b. An isolated conducting spherical shell of inner radius a is centered on the origin of a Cartesian coordinate system. The shell contains a thin ring of radius b located in the xy-plane and centered on the origin. The ring carries a line charge density λ. Find the electric potential outside the shell at points P on the z-axis.



2. A non-conducting liquid of dielectric constant ε and mass density ρ is contained in a U-tube of circular cross section with radius a. The dielectric constant of the tube material is $\varepsilon'=1$. The tube is inserted in the middle between the plates of a parallel-plate capacitor (see Figure) which produces a fixed uniform electric field $\boldsymbol{E}_0$ in the region far from the dielectric liquid. Separation d and width (along the y-axis) of the plates are both much larger than a.

a) Neglecting all edge effects, determine the electric field inside and outside the liquid.

b) Find the height h by which the liquid rises in terms of $\boldsymbol{E}_0$.

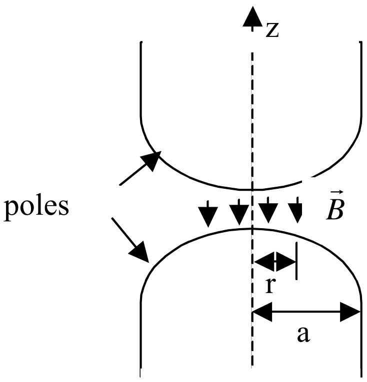


3. By shaping the pole faces of a cylindrical magnet, as shown in the figure in cross section, and applying a time varying current to the coils, a magnetic field

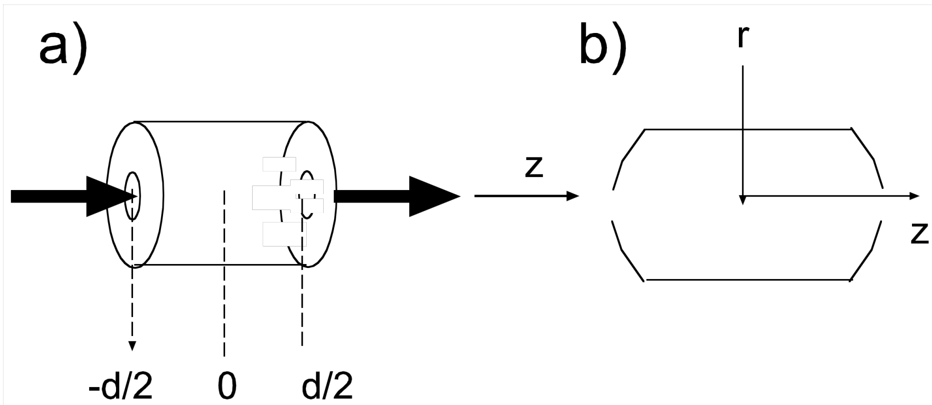
$$\vec{B} = -\hat{\mathbf{z}} B_0 \frac{a}{\sqrt{a^2 + r^2}} t, \qquad (r < a)$$

$$\vec{B} = 0, \qquad (r > a)$$

can be produced for times t > 0. For t < 0, the field is B = 0. Find the electric field induced for 1) r < a and 2) r > a and all t.



4. Resonant cavities operated on the TM_{010} mode are commonly used to accelerate charged-particles. The idealized right-circular cylinder cavity is not useful for accelerating charged particle since it does not include entrance/exit apertures to inject/extract the charged particles. In the following, we seek the form of the fields associated to the TM_{010} mode of a realistic resonant cavity that incorporates entrance and exit apertures (see Fig. 1a). We will work in cylindrical coordinate (r, ϕ, z). The TM_{mnp} mode corresponds to an electromagnetic fields configuration where the longitudinal magnetic field, B_z, vanishes. The subscripts *mnp* represent the number of "field antinodes" along respectively the ϕ, r, and z directions.



a) From the cylindrical symmetry and the definition of the TM_{010} mode, what is the expected ϕ-dependence of the fields? What are the non-zero **E** and **B**-field components?

b) Assume the axial E-field [$E_z(r=0,z)$] is zero at $z = d/2$ and $z = -d/2$ (see Fig. 1a). What is the general z-dependence for the axial electric field $E_z(r=0,z)$?

c) From now we take $d = \lambda/2$ where λ is the resonant wavelength of the cavity and let $L(z) \equiv E_z(r=0,z)$. The z-component of the electric field at any location (r, ϕ, z) can be expressed in the form $E_z(r,\phi,z) = T(r,\phi)L(z)$, where $T(r,\phi)$ represent the transverse coordinate dependence of E_z. Find the differential equation satisfied by T and the corresponding solution for T.

d) Write down the expressions for $E_z(r,\phi,z)$ and the other non-zero components of the **E** and **B**-fields. Expand these solutions to 1st order in r.

e) Although Fig. 1b seems to suggest the cavity with aperture is still a right circular cylinder this is not the case in practice: the ends of the cavity are not flat. Find the equation of the end plate $r(z)$ in the vicinity of the aperture from the boundary equation satisfied by the E-field. Use the 1st order expansion for the fields.

<u>Hints:</u> You may need **some** of the following.

The equation $r^2 \dfrac{d^2Z}{dr^2} + r\dfrac{dZ}{dr} + (k^2r^2 - n^2)Z = 0$, has the solution $Z = J_n(kr)$ where J stands for the 1st kind Bessel function. The equation $r^2 \dfrac{d^2Z}{dr^2} + r\dfrac{dZ}{dr} - (k^2r^2 - n^2)Z = 0$, has the solution $Z = I_n(kr)$ where I stands for the modified Bessel function.

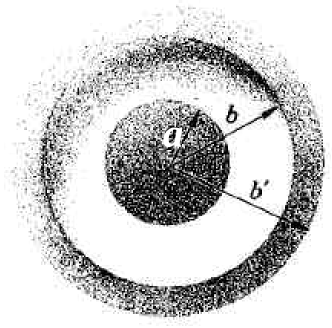
The Taylor expansions of J and I functions are:

$I_n(x) = \left(\dfrac{1}{2}x\right)^n \left[1 + \left(\dfrac{x^2}{4}\right)^n + \ldots\right]$, and $J_n(x) = \left(\dfrac{1}{2}x\right)^n \left[1 + \left(-\dfrac{x^2}{4}\right)^n + \ldots\right]$.

We also remind the identities $\dfrac{d}{dx}(xI_1) = xI_0$, $\dfrac{d}{dx}(xJ_1) = xJ_0$, $\dfrac{d}{dx}J_0(x) = J_1(x)$ and $\dfrac{d}{dx}I_0(x) = I_1(x)$. The Laplacian in (r, ϕ, z) is $\nabla^2 = \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{d}{dr}\right) + \dfrac{1}{r^2}\dfrac{d^2}{d^2\phi} + \dfrac{d^2}{dz^2}$.

NIU Ph.D. qualifier examination 2006 Spring (2/11/2006)
Electricity and Magnetism
Solve 3 out of 4 problems.

1. An isolated conducting sphere of radius a is contained in an isolated spherical metallic shell of inner and outer radius b and b', respectively. The centers coincide.

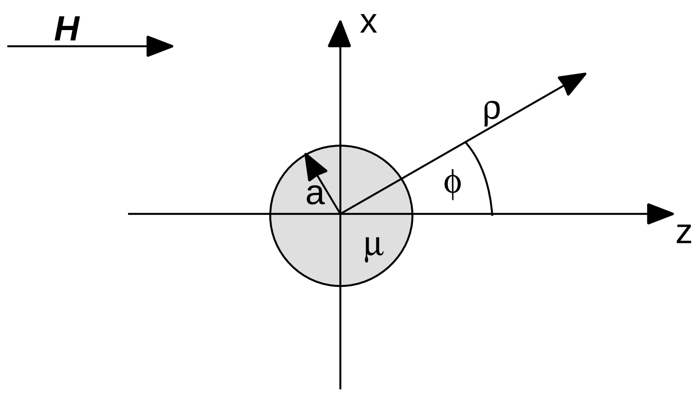


With the boundary condition that at infinity the potential is zero find the electrostatic potential everywhere for

a) A charge Q placed on the sphere

b) A charge Q placed on the shell

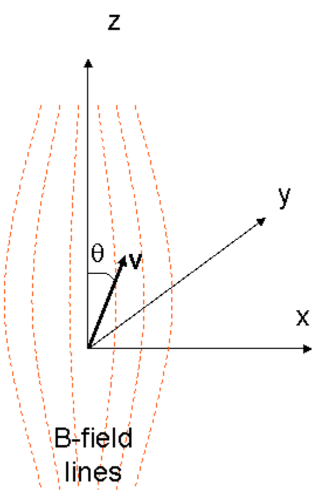
2. An infinitely long cylinder with radius a and permeability μ is placed with its axis into an initially uniform magnetic field $\boldsymbol{H}_0$ with its axis perpendicular to $\boldsymbol{H}_0$. Find the resultant field inside and outside of the cylinder.



3. Consider a region of empty space with a static magnetic field whose z component has the form $B_z = B_0\left(1+\left(\frac{z}{l}\right)^2\right)$ where B_0 and l are constants. The field is cylindrical-symmetric about the z-axis. The transverse components, B_x and B_y, are zero at the origin (z=0). A particle of charge q and mass m is initially started at the origin with speed $v << c$. For the value $l = \infty$, the magnetic field is uniform and the trajectory of the particle assumes a periodic motion in x-y. For large values of l, B changes by a small fraction over a cycle of motion. Under such conditions, the quantity $S \equiv \int(dxP_x + dyP_y)$ is constant (so-called adiabatic invariant). Here P_x and P_y are the transverse components of the canonical momentum and the integral is performed over one cycle of the *x*-*y* motion.

a) Find the flux of B through the particle's orbit projected onto the *x*-*y* plane in term of *S*.

b) Under the conditions described, the particle is confined within an interval $|z| < z_{MAX}$. Find the value of z_{MAX} (as a function of initial angle q and l) for a particle whose initial velocity is directed at an angle q from the *z*-axis; see Figure below.



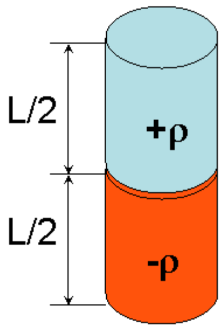
4. A solid cylinder rod of radius R and length L is divided in half. One half of the rod is filled with a positive charge density $(+\,r\,)$ and the other half with a negative charge density $(-\,r\,)$; see Figure A (the charge density r is a constant).

a) What is the dipole moment $\vec{p}$?

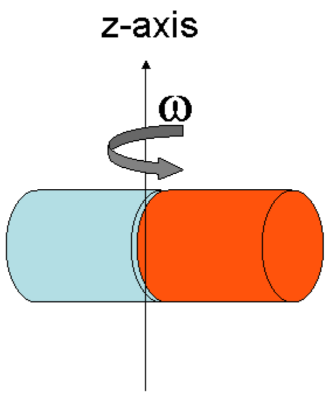
The cylinder is spun about an axis perpendicular to its length and passing through its center, with angular velocity ω (such that relativistic effects are ignorable); see Figure B.

b) How does the radiated power per unit solid angle far from the dipole, $\left\langle \frac{dP}{d\Omega} \right\rangle$ as a function of the observation angles. [**Hint:** recall that $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{4pc^3}\left\langle \ddot{\vec{p}} \times \hat{n} \right\rangle^2$; where $\hat{n}$ is the direction of observation, and $\ddot{\vec{p}} \equiv \frac{d^2 \vec{p}}{dt^2}$. The observation angle is defined as $q \equiv \angle(\hat{n}, \hat{z})$].

c) What is the total radiated power?



A



B

NIU Physics PhD Exam – Electricity and Magnetism September 2005

Do any THREE out of the four problems.

Problem 1: An infinitely long line carries charge per unit length λ. It is placed parallel to and at a distance d from the center of an infinitely long grounded conducting cylinder of radius a, where $a < d$.

(a) Calculate the force per unit length on the line charge.

(b) Calculate the surface charge density on the cylinder as a function of position on the cylinder.

Problem 2: Consider the propagation of an electromagnetic wave in vacuum between two infinite parallel conducting plates spaced a distance d apart, occupying the planes $z = 0$ and $z = d$. For the case in which the electric field is everywhere parallel to the plates, and the wave propagates in the x direction:

(a) Give an expression for the electric and magnetic fields between the plates.

(b) Show that there is a minimum angular frequency ω_c below which waves will not propagate, and derive an expression for ω_c.

Problem 3: A very long solenoid has n turns of wire per unit length, and carries a slowly time-varying current $I(t)$. The radius of the solenoid is a, and the axis of symmetry is the z axis.

(a) Find the magnetic field $\vec{\mathbf{B}}$ inside the solenoid.

(b) Use Faraday's Law in integral form to find the electric field $\vec{\mathbf{E}}$ both inside and outside of the solenoid.

(c) Consider an imaginary cylinder, coaxial with the solenoid, and with length d and radius r. Find the rate at which electromagnetic energy is flowing into this cylinder from the outside, for the two cases $r < a$ and $r > a$.

Problem 4: A coaxial cable consists of two infinitely long coaxial perfectly conducting cylinders with radii a and b, with $a < b$, and with the common axis being the z axis. The inner conductor carries a current I in the $+\hat{z}$ direction, and the outer conductor carries the current back in the opposite direction. The region between the conductors has magnetic permeability μ (different from the permeability μ_0 of empty space).

(a) Find the magnetic fields $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$ everywhere.

(b) Find the self-inductance per unit length of the cable.

NIU Physics PhD Exam – Electricity and Magnetism – February 2005

***** Do ONLY 3 out of the 4 problems. *****

Problem 1. The half space $x > 0$ is filled with a constant magnetic field $\vec{\mathbf{B}} = (0, 0, B_0)$, and the half space $x < 0$ is filled with a field in the opposite direction $\vec{\mathbf{B}} = (0, 0, -B_0)$. An electron is shot out of the origin with initial velocity $\vec{\mathbf{v}} = (-v_0/\sqrt{2}, -v_0/\sqrt{2}, 0)$. Describe its subsequent motion as quantitatively as possible.

Problem 2. A plane electromagnetic wave of angular frequency ω and peak electric field E_0 is moving in the $\hat{z}$ direction, and is polarized in the $\hat{x}$ direction. The wave is incident on a charge Q of mass m. The charge is bound to the origin by a restoring potential

$$V(r) = m\omega_0^2 r^2/2,$$

where r is the distance to the origin and ω_0 is a constant. It is also acted on by a viscous damping force given by

$$\vec{\mathbf{F}}_d = -m\omega_d \vec{\mathbf{v}},$$

where $\vec{\mathbf{v}}$ is the velocity of the charge, and ω_d is another constant.

(a) Find the position $\vec{\mathbf{r}}$ of the charge as a function of time, ignoring magnetic effects. What is the phase angle between the electric field of the wave $\vec{\mathbf{E}}(t)$ and $\vec{\mathbf{r}}(t)$?

(b) Find the intensity of the scattered radiation as a function of direction and of ω. (You do not need to evaluate the overall multiplicative factor.)

Problem 3. An azimuthally symmetric perfect conductor is approximately spherical, with mean radius R_0, in the sense that the radius R as a function of the polar angle θ is

$$R(\theta) = [1 + \Delta(\theta)]R_0$$

with $\Delta(\theta) \ll 1$ and

$$\int_{-1}^{1} \Delta(\theta)\, d(\cos\theta) = 0.$$

This conductor is given a total charge Q. To first order in the function Δ, find the potential outside the conductor.

(You may leave your result in terms of the Legendre polynomials $P_n(x)$, which satisfy:

$$\int_{-1}^{1} P_n(x)P_{n'}(x)dx \quad = \quad \delta_{nn'}\frac{2}{2n+1} \tag{1}$$

for non-negative integers n, n'.)

Problem 4. The angular distribution of radiation emitted by an accelerated charged particle with charge q and velocity $\vec{\mathbf{v}} = c\vec{\beta}$ may be determined by

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c}\frac{|\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \vec{\beta}) \times (d\vec{\beta}/dt')]|^2}{(1 - \vec{\beta}\cdot\hat{\mathbf{n}})^5}$$

where $\hat{\mathbf{n}}$ is the unit vector in the direction to the observer.

(a) What interpretation does t' have?

(b) What approximations, if any, are inherent to this expression?

(c) Suppose the charge undergoes linear acceleration, so that $d\vec{\beta}/dt$ is parallel to $\vec{\beta}$. Let θ be the angle between $\vec{\beta}$ and $\hat{\mathbf{n}}$. Find the locations θ_{max} of the peak radiated intensity.

(d) How do these θ_{max} values depend on the particle's total energy E if the particle is highly relativistic?

NIU Physics PhD Exam – Electricity and Magnetism – September 2004

*** Do any 3 of the 4 problems. ***

Problem EM1. In a vacuum diode, electrons with mass m and charge q are "boiled" off a hot cathode, at potential $V = 0$, and accelerated across a gap to the anode, which is held at positive potential $V = V_0$. The cloud of moving electrons within the gap (called space charge) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation (area $A \gg d^2$ in the figure below), so that edge effects can be neglected. Then the potential V, the charge density ρ, and the speed of the electrons v are all functions of x only.

(a) Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$?

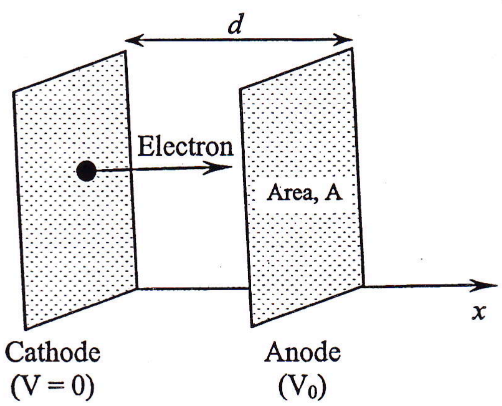
(b) In the steady state, the current I is independent of x. What, then, is the relation between ρ and v?

(c) Use the results above and Poisson's equation to obtain a differential equation for the potential $V(x)$, not involving ρ or v, and solve it. (Hint: try a power law form.)

(d) Show that a non-linear relation holds between I and V_0, of the form

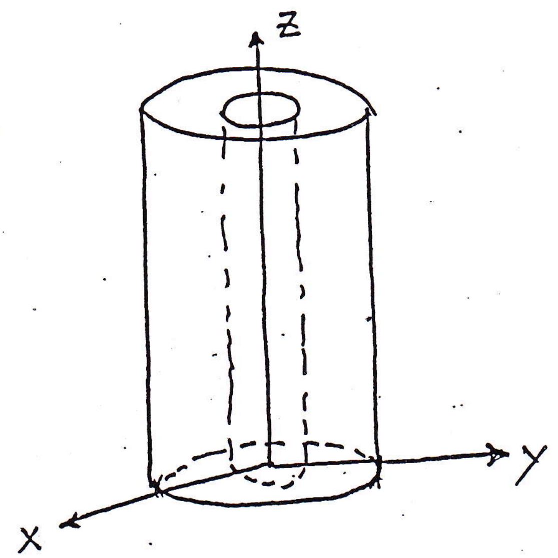
$$I = KV_0^n$$

where K is a constant that you will find in terms of the given quantities, and n is a rational number.



Problem EM2. Consider the right circular cylindrical cavity pictured below. The inner and outer walls of the cavity, and the ends, are perfect conductors. The inner conductor has radius a, the outer conductor has radius b, and they are coaxial with the z-axis. The height of the cavity is h.

Consider the TEM modes of the cavity (the modes that have an electric field with no component in the z direction), in the vacuum region $a \leq r \leq b$ and $0 \leq z \leq h$, with peak electric field E_0. Find expressions for the electromagnetic fields and the resonant frequencies and wavelengths of all such modes in terms of the given quantities.



Problem EM3. A long solenoid consists of N turns of wire with resistance R wound on a right circular cylinder of length d and radius a, with $d \gg a$.

(a) Suppose that a slowly time-varying current $I(t) = I_0 \cos(\omega t)$ flows in the solenoid. Find the magnetic and electric fields everywhere inside and outside of the solenoid.

(b) A circular ring of wire with radius b is coaxial with the solenoid and outside of it ($b > a$). Suppose now that a slowly varying current $I(t) = I_0 \cos(\omega t)$ flows in the ring. What current is induced in the solenoid?

Problem EM4. An electromagnetic wave is propagating in a non-conducting linear medium that has the same permeability as vacuum, $\mu = \mu_0$. The magnetic field of the wave is given by

$$\vec{\mathbf{B}} = B_0 \left(\frac{\hat{y} + \hat{x}}{\sqrt{2}} \right) \cos\left(\frac{\sqrt{7}\omega}{2c} x + ay - \frac{\omega}{2c} z + \omega t\right)$$

where a is a positive constant, and c is the speed of light.

(a) Find a. What is the direction of propagation of the wave?

(b) What are the index of refraction and the permittivity of the material?

(c) Find the electric field $\vec{\mathbf{E}}$ of the wave.

(d) The wave is incident on a square with side d of some perfectly absorbing material that lies in the $y = 0$ plane. How much energy is absorbed by the material in time T? (Assume $T \gg 1/\omega$.)

NIU Physics PhD Exam – Electricity and Magnetism – January 2004

***** Do any 3 of the 4 problems. *****

Problem EM1. Two conducting coaxial cylinders are placed in a uniform magnetic field B, which is parallel to the axes of the cylinders. The inner cylinder has radius R_1 and is grounded. The outer cylinder has radius R_2 and is held at a positive potential V. Electrons are released from the inner cylinder with negligible velocity, and follow curved paths toward the outer cylinder.

(a) Find the system of differential equations governing the coordinates (r, ϕ, z) of an electron as a function of time.

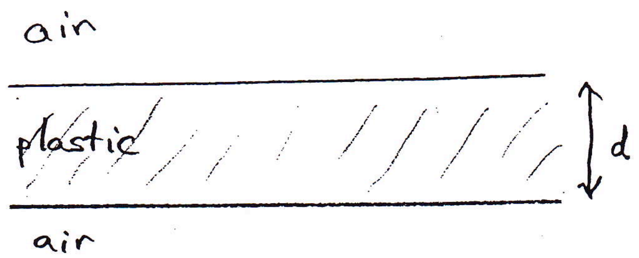
(b) For a given value of B, what is the minimum potential V that will cause an electron to reach the outer cylinder?

Problem EM2. A long straight coaxial cable consists of an inner cylinder of radius a and an outer cylinder of radius b. Both cylinders are perfect conductors, and the volume between them is a vacuum. Consider TEM propagation, i.e. a pure transverse wave traveling down the cable. Take the angular frequency to be ω, and the peak electric field strength to be E_0.

(a) What are the magnetic and electric fields inside the cable, as a function of r, the distance from the center?

(b) What is the time-averaged power transmitted by the cable?

Problem EM3. A microwave antenna radiating at 10 GHz is to be protected from the weather by a flat plastic shield with dielectric constant 2.5 times that of vacuum, no magnetic properties, and a thickness d (to be determined).



(a) Assuming normal incidence, what are the boundary conditions to be satisfied at the air-shield boundaries?

(b) From these results, determine the minimum thickness of the shielding that will allow perfect transmission.

Problem EM4. A point dipole $\vec{p}$ is located at the center of a spherical cavity of radius a. The surface of the sphere is a conductor at zero potential.

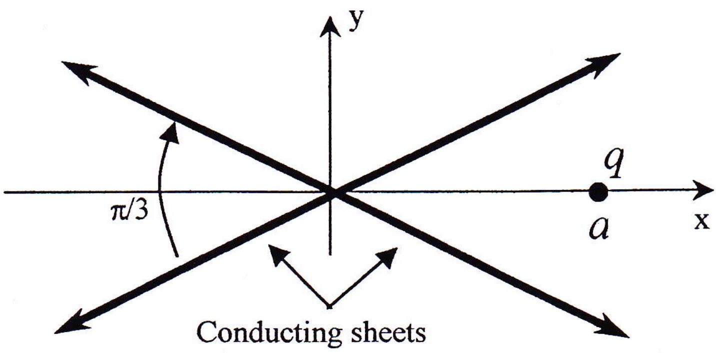
(a) Find the function (a solution to Laplace's equation) that must be added to the dipole potential to satisfy the boundary conditions.

(b) Evaluate the induced charge density on the inside surface of the cavity.

(c) Find the dipole moment of the induced charge density.

NIU Ph.D. qualifier examination 2003 Fall (September 6, 2003)
Electricity and Magnetism
Solve 3 out of 5 problems.

I. Two infinite plane grounded conducting sheets intersect at an angle of $\pi/3$. A charge q is placed equidistant from the conductors and at a distance a from their line of intersection. Sketch the E field in the region around charge q, between its two adjacent conducting sheets. Find the force exerted on the charge by the conducting sheets.



II. A linear dielectric sphere of radius a and dielectric constant κ carries a uniform charge density ρ, surrounded by vacuum.

a) Find $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ inside and outside the sphere.

b) Find the energy W of the system.

III. An infinite flat sheet of charge density per unit area σ, located in the xy-plane, is forced to oscillate along the x-axis. The velocity of charges at time t is given by $\vec{\mathbf{v}} = \hat{x}v_0 \cos(\omega t)$, resulting in electromagnetic radiation.

(a) Solve for all components of the electromagnetic radiation.
(b) How much energy per unit area is radiated away in a time T? (You may assume $T >> 1/\omega$.)

IV. The electric field $\vec{\mathbf{E}}$ and the electric displacement $\vec{\mathbf{D}}$ in a certain linear anisotropic medium are related by an effective dielectric tensor ε_{ij}:

$$D_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j, \qquad \varepsilon_{ij} = \varepsilon_0 \begin{pmatrix} 1 & ia & 0 \\ -ia & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Fall 2003

where a is a positive constant less than 1. The indices 1, 2, 3 correspond to x, y, z respectively. The material is non-conducting and has the same magnetic permeability as vacuum ($\mu = \mu_0$).

(a) Given a plane wave propagating in the z-direction, find the polarization for which the medium has a definite index of refraction N ($N = ck/\omega$), and the corresponding values of N.

(b) A semi-infinite slab of the above material fills the region $z > 0$. An electromagnetic plane wave of frequency ω is incident normally on the flat direction of the material, and is therefore propagating in the z-direction. The wave enters the material from vacuum. The incident wave is linearly polarized with the electric field along the x-axis. Find the polarization of the reflected wave.

V. Suppose a photon of wavelength λ strikes a stationary electron and "bounce off" with a wavelength λ' at angle θ. This process is called Compton scattering; photons are neither created nor destroyed.

(a) Derive the Compton scattering formula relating λ', λ and θ.

(b) What part of the electromagnetic spectrum would best be used for experimental verification of Compton scattering, and why?

NIU Ph.D. qualifier examination 2003 Spring (1/25/2003)
Electricity and Magnetism
Solve 3 out of 5 problems.

I. Find the potential Φ and field $\vec{\mathbf{E}}$ for an unchanged conducting sphere placed in an initially uniform electric field, using an expansion in Legendre polynomials. Choose the z axis to be the initially uniform field direction.

   a. Write the most general solution to Laplace's equation in terms of the radial functions $\frac{U(r)}{r} = A_l r^l + B_l / r^{l+1}$ and the Legendre polynomials $P_l(\cos\theta)$. Since the problem has azimuthal symmetry, no spherical harmonics are required.

   b. Use boundary condition at $r = \infty$ to determine all the A_j.

   c. Use boundary condition at $r = a$ to determine all the B_j except B_0. What determines B_0?

   d. With all coefficients determined, write explicit forms for Φ and $\vec{\mathbf{E}}$ for the space outside the sphere.

   e. Determine the induced charge density on an initially uncharged sphere.

   f. Determine the total induced charge on an initially uncharged sphere.

II. A spherical shell of radius R carries a uniform surface charge density σ. Calculate the vector potential $\vec{\mathbf{A}}$ and the magnetic field $\vec{\mathbf{B}}$, which are created when the sphere is rotating with an angular speed ω.

III. Classical model of Zeeman effect.

   a. Consideer an electron that executes 3-dimensional simple harmonic motion (SHM), i.e., it is subject to the potential per unit mass,

$$V(x,y,z) = \frac{1}{2}\omega_0^2\left(x^2 + y^2 + z^2\right).$$

   Now turn on an external magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{z}}$, where B is a constant. Show that the frequency of vibration is modified such that (in mks units)

$$\omega_{\pm} = \sqrt{\omega_0^2 + \left(\frac{eB}{2m}\right)^2} \pm \frac{eB}{2m}; \quad \omega_z = \omega_0;$$

   wherein e = 1.6 x 10^{-19}, e is the electron charge, and m = 9.1 x 10^{-31} kg is the electron rest mass.

b. Suppose SHM is used as a (crude!) classical model of an electron's orbit in an atom, and supposed the frequency shift is small compared to ω_0. Then what is the corresponding splitting $\Delta E_{\pm}$ of the atomic energy level?

c. What percent change does a B = 1T field impart to the ground-state energy E_0 = 13.6 eV of the hydrogen atom? [Note: 1 eV/h = $2.72\text{x}10^{14}$ Hz, where h is Planck's constant.]

d. Is SHM an adequate model of the electron's orbit? If so, why? If not, why not?

IV. Electrodynamic Lagrangian and Hamiltonian

a. Ascertain whether the Lagrangian

$$L(x,v,t) = -mc^2\sqrt{1-\frac{v^2}{c^2}} + e\bar{\mathbf{v}}\cdot\bar{\mathbf{A}} - e\Phi(x),$$

where $\bar{\mathbf{A}}$, $\mathbf{\Phi}$ are the vector and scalar potentials, respectively, yields the Lorentz force equation in mks units.

b. Construct the Hamiltonian associated with the Lagrangian of part (a).

V. Electron Synchrotron

The rate at which a relativistic, accelerating electron (charge e = 1.6 x 10^{-19} C, rest mass m = 9.1 x 10^{-31} kg) radiates energy is, in mks units, $P = \frac{2}{3}\frac{e^2}{4\pi\varepsilon_0 c}\gamma^6[\dot{\beta}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2]$, in which ε_0 = 8.854 x 10^{-12} F/m is the permittivity of free space, and c = 3 x 10^8 m/s is the speed of light. Estimate the maximum beam energy that can viably be achieved using an electron synchrotron (a circular accelerator).