You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

**Problem 1.** A small bead of mass $M$ moves without friction on a circular wire hoop of radius $R$. The hoop is forced to rotate about its vertical diameter with a constant angular frequency $\Omega$. There is a uniform downward gravitational field with acceleration $g$.

(a) Find a Lagrangian that describes the motion of the bead in terms of the angle $\theta$ shown, and derive the equation of motion. [14 points]
(b) Find all of the equilibrium positions, and find the conditions under which each of these is stable. [12 points]
(c) For each possible stable equilibrium position, find the angular frequency of small oscillations. [14 points]

**Problem 2.** A very small comet with mass $m$ is discovered approaching the Sun (with mass $M \gg m$) along an orbit that is a perfect parabola. At the present time the comet is noted to be moving with speed $v$, and has an impact parameter $b$ with respect to the Sun. For parts (b) and (c) below, you may leave your answer in terms of definite integrals, if you choose.

(a) Find the distance of closest approach of the comet to the Sun, and find its speed when it reaches closest approach. [16 points]
(b) How long does it take the comet to again reach its present distance from the Sun? [14 points]
(c) When the comet again reaches its present distance from the Sun, what angle (as measured from the Sun) has it swept out on its orbit? [10 points]
Problem 3. A thin stick of mass $M$ (with uniform mass density) and length $L$ is placed with one end on a wall and the other end on the floor. Both ends slide frictionlessly without losing contact, and the motion remains within the vertical plane. The angle of the stick with the floor is $\alpha$. There is a uniform gravitational field with acceleration $g$ downwards.

(a) Find a Lagrangian describing the system in terms of $\alpha$ and derive its equation of motion. [20 points]

(b) Suppose that at time $t = 0$ the stick is released from rest with $\alpha = \alpha_0$. At what time does the top end of the stick hit the floor? You may leave your answer in terms of a single definite integral over a dimensionless variable. [20 points]

Problem 4. An infinitely long string with linear mass density $\sigma$ and tension $T$ is attached at one end to a ring with mass $M$. The ring slides frictionlessly on a vertical pole at $x = 0$ as shown. The diameters of the ring and the pole can be ignored. Gravity can also be ignored. Suppose a sinusoidal wave with transverse (vertical) displacement $u_I(x, t) = A \cos(kx - \omega t)$ is incident from the far left ($x = -\infty$). Note that the constant phase angle of the incoming wave has been set to 0 for convenience. The displacement is defined so that $u = 0$ at equilibrium, and the angle $\theta$ as defined in the figure is assumed to remain small.

(a) Derive the relationship between the angular frequency $\omega$, tension $T$, linear mass density $\sigma$, and wavevector $k$ using the equation of motion. [10 points]

(b) Derive the reflected wave transverse displacement $u_R(x, t)$. In particular, find the relationship between the constant phase angle $\phi$ of the reflected wave, the mass of the ring $M$, and the angular frequency $\omega$. [24 points]

(c) Discuss the dependences of $\cos \phi$ from part (b) on $M$ for constant $\omega$, including $M = 0$ and $\infty$, and its dependences on the $\omega$ of the incoming wave, including $\omega = 0$ and $\infty$. [6 points]
Problem 1. Consider an infinitely long 1-dimensional chain of alternating masses $m_1$ and $m_2$ and identical ideal springs with spring constant $k$, as shown below. The equilibrium distance $d$ between neighboring masses is also equal to the unstretched spring lengths. Consider only longitudinal oscillations (with motion along the direction of the chain) with a fixed wavenumber $a = 2\pi/\lambda$, where $\lambda$ is the wavelength. You may want to use a complex wave notation, with the positions of masses $m_1$ given by $Ae^{i(2n\pi d - \omega t)}$ and the positions of masses $m_2$ given by $Be^{i((2n+1)\pi d - \omega t)}$, where $n$ are integers.

(a) Show that there are two collective modes with a given wavenumber $a$, and find their angular frequencies $\omega$. [28 points]
(b) In the special case $m_1 = m_2 = m$ and the long wavelength limit $ad \ll 1$, characterize each of the two modes you found by identifying the motion of neighboring masses as either in phase or out of phase. [12 points]

Problem 2. Consider a binary system, consisting of two stars with masses $m_1$ and $m_2$ in circular orbits at a distance $R$ from each other, in the presence of Newtonian gravity. Ignore all relativistic effects, and take the stars to be point masses.
(a) Derive the expression for the orbital period, $T$. [8 points]
(b) Suppose that $m_1 = 6M_{\text{Sun}}$ and $m_2 = 10M_{\text{Sun}}$ and that $R = 4$ a.u., where 1 a.u. is the distance between the Earth and the Sun. Find the orbital period in years. [4 points]
(c) Suppose that for a brief moment, the laws of physics are temporarily suspended and both stars are brought instantaneously to rest, following which Newtonian physics is restored. Find the amount of time needed for the stars to collide. If you wish, you may leave your answer in terms of a definite integral over a dimensionless variable. [16 points]
(d) Suppose that as in part (c) there is a very brief suspension of the laws of physics, but that in the center-of-mass frame it instantaneously brings $m_2$ to rest while increasing the speed of $m_1$ to $v$ without affecting its direction. How large must $v$ be in order for the stars to completely escape each other, leading to infinite separation? [12 points]
Problem 3. The pendulum shown below consists of a mass $M$ attached to the end of a massless ideal spring with spring constant $k$ and unstretched length $a$, in a uniform downward gravitational field with acceleration $g$. The other end of the spring is fixed, and the spring stretches and contracts but does not curve.

(a) Considering only motion confined to a vertical plane, find the Lagrangian for the system in terms of configuration variables $r$ and $\phi$, and derive the equations of motion. [10 points]

(b) Find the general solution for small oscillations about the equilibrium position, considering only motion confined to a vertical plane as in part (a). [18 points]

(c) Find the Hamiltonian for the system in terms of appropriate phase space variables, this time allowing motions of $M$ that are not confined to the vertical plane. [12 points]

Problem 4. A yo-yo of total mass $M$ and uniform mass density consists of 2 large cylinders of radius $R$ and thickness $d$, joined by an axle of radius $r$ and length also given by $d$. A very thin string is wound around the axle, with one end fixed to the ceiling in a uniform gravitational field with acceleration $g$. The axis of symmetry of the yo-yo remains parallel to the ground and does not rotate.

(a) Find the moment of inertia of the yo-yo about its axis of symmetry. [10 points]

(b) If the yo-yo starts from rest, find the time needed to descend a distance $h$. [15 points]

(c) Find the tension in the string as the yo-yo descends. [15 points]