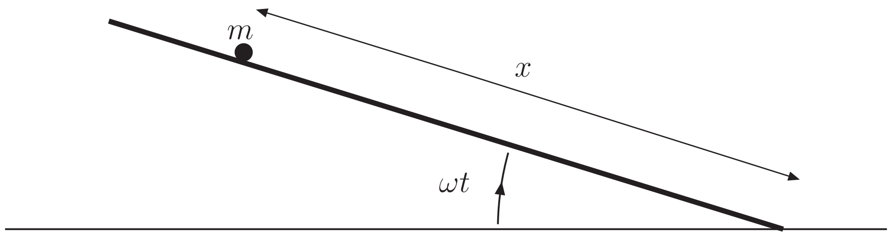
NIU Physics PhD Candidacy Exam – Fall 2018 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

Problem 1. A small mass m is sitting on a frictionless flat board resting on a horizontal table. Starting at time $t = 0$, the edge of the left side of the board is raised so that the board pivots about the fixed opposite end with constant angular velocity ω. The mass then starts to slide towards the fixed downhill end, under the influence of a uniform gravitational field of acceleration g.



(a) Using coordinate x, the distance of the mass from the fixed downhill edge, find the Lagrangian for the mass. [10 points]
(b) Find the equation of motion, and the general solution. [15 points]
(c) Find the particular solution of the equations of motion, assuming that the mass started from rest and was at position $x = x_0$ at $t = 0$. [10 points]
(d) In addition to the above initial conditions from part (c), assume that ω is small, and find a simplified form for $x(t)$ up to second-order terms in ω. [5 points]

Problem 2. A spherical marble of known mass M and radius a rolls without slipping near the bottom of a fixed spherical bowl of radius R, in a uniform gravitational field with acceleration g. The interior mass density profile of the marble is unknown, but its moment of inertia about its center is I. Assume that the motion of the marble takes place within a plane (which includes the equilibrium position at the bottom of the bowl).
(a) Find a Lagrangian for the system in terms of a single dynamical variable. [15 points]
(b) Find the frequency of small oscillations. [15 points]
(c) Now suppose that the density of the marble is constant, except that it has a hollow spherical core of radius b. What is the moment of inertia of the marble? For what limiting value of b is the frequency of small oscillations maximized? [10 points] (Hint: if the marble were completely solid, its moment of inertia about its center would be $\frac{2}{5}Ma^2$.)

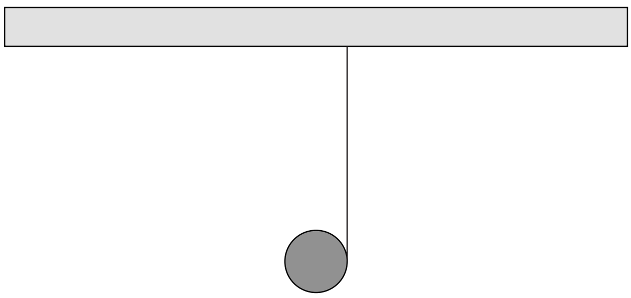
Problem 3 Consider the (planar) motion of a particle of mass m and initial angular momentum L in the central 3-dimensional potential corresponding to a "spring" with a non-zero relaxed length a:

$$V(r) = \frac{1}{2}k(r-a)^2$$

(a) [20 points] Find a condition relating the radius r_0 of circular orbits and the given quantities. (It is not necessary to solve for r_0.)

(b) [20 points] For nearly circular orbits, find an expression for the period of time between successive radial maxima for $a/r_0 \ll 1$, expressing your answer to first order in a/r_0 (with L eliminated from the answer). Use this to find the angular change between successive radial maxima in the limit $a/r_0 \to 0$, and check that the orbits are closed in that limit.

Problem 4 A uniform cylinder of mass M and radius R is rolled up on an unstretchable, massless, very thin string which is attached to the fixed ceiling. The cylinder is released and unrolls the string as it falls (like a yo-yo) in a uniform gravitational field with acceleration g.



Considering only vertical motion of the center of the cylinder, and assuming that the cylinder never reaches the end of the string:

(a) Specify the degrees of freedom and state any equations of constraint. [5 points]

(b) Find the Lagrangian, and use it to solve for the motion as a function of time. [25 points]

(c) Find the tension in the string as a function of time. [10 points]

NIU Physics PhD Candidacy Exam – Spring 2018 – Classical Mechanics

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Problem 1. A charged particle moves in the xy plane, under the influence of a uniform magnetic field pointing along the z direction. The Hamiltonian has the form:

$$H = \frac{1}{2m}(p_x + by)^2 + \frac{1}{2m}p_y^2,$$

where b is a constant proportional to the magnetic field strength, and p_x, p_y are the canonical momenta for the rectangular coordinates x, y. You may assume that no motion takes place along the z direction.

(a) Derive Hamilton's equations of motion from this Hamiltonian. [10 points]

(b) Find three independent constants of the motion (not counting p_z, since you should be ignoring motion in the z direction). [8 points]

(c) Solve the Hamilton's equations of motion found in part (a), taking as given the initial value data at time $t = 0$: $x(0) = x_0$, $y(0) = 0$, $p_x(0) = 0$, and $p_y(0) = p_0$. [10 points]

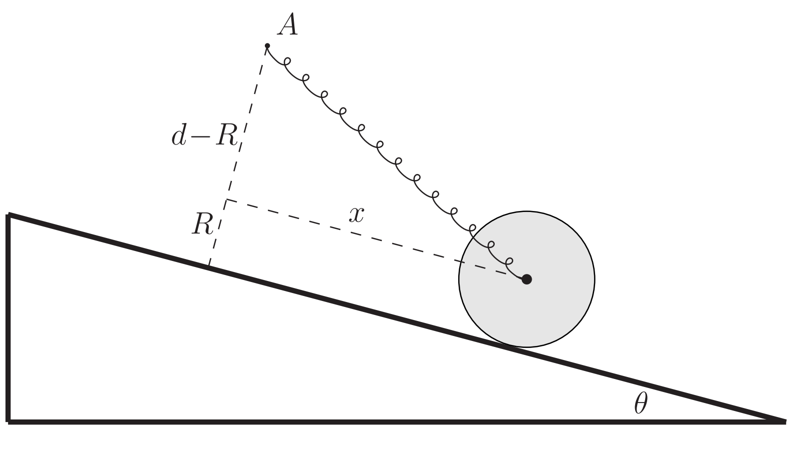
(d) Find the Lagrangian corresponding to this Hamiltonian. Derive the Euler-Lagrange equations of motion, and check that your solutions from part (c) satisfy them. [12 points]

Problem 2. A solid homogeneous flat disk of radius R and mass M rolls without slipping on a stationary inclined plane, which makes an angle θ with respect to the horizontal, as shown. The disk is in contact with the inclined plane at all times. A spring provides an attractive force on the disk towards a fixed point A, given in magnitude by $F = kr$, where k is a constant and r is the distance from the point A to the center of the disk. The point A is a distance d from the inclined plane. The disk is also acted on by the Earth's gravitational field with constant acceleration g downwards.

(a) Find the equilibrium position x, as measured in the picture. [12 points]

(b) Find the Lagrangian of the system, with x as the only dynamical variable. [12 points]

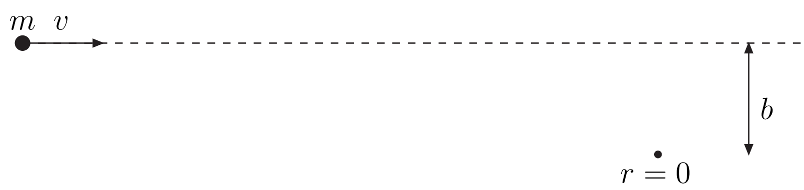
(c) Find the frequency of small oscillations around the equilibrium position. [16 points]



Problem 3. Consider a point particle of mass m moving in three dimensions in a central potential

$$V(r) = -\frac{g}{r} - \frac{k}{r^2}$$

where g and k are positive constants and r is the distance from the origin. The particle approaches from very far away with speed v and impact parameter b, as shown. (The dashed line is what the path would be if there were no potential.)

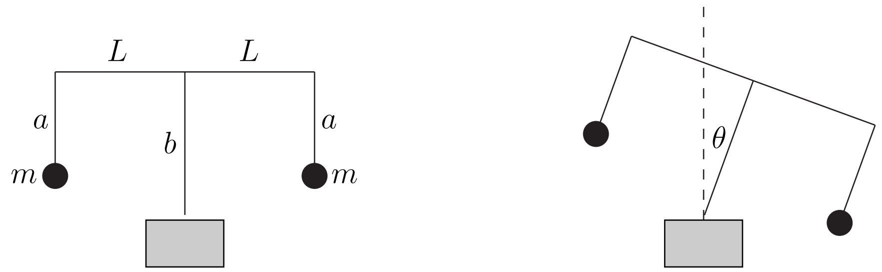


(a) Find r_{min}, the distance of closest approach of the particle to $r = 0$. Show that the particle will go through the origin if $k > k_{\text{crit}}$, where k_{crit} is a critical value that you will determine in terms of the other given quantities. [12 points]

In the remainder of this problem, you should assume $k < k_{\text{crit}}$, and you may leave your answers in terms of r_{min} and the other given quantities.

(b) What is the maximum speed reached by the particle on its trajectory? [8 points]
(c) What is the maximum acceleration reached by the particle on its trajectory? [8 points]
(d) When the particle is very far from the origin again, find the angle by which it has been scattered from its original direction. You may leave your answer in terms of a single definite integral. [12 points]

Problem 4. A symmetric balancing structure below consists of two equal point masses m attached with rigid massless supports of lengths a, b, and L, connected at right angles as shown on the left below in its equilibrium balanced position. The structure is allowed to pivot on its base in one dimension (so that it always remains within the plane of the page), as measured by the angle θ from the vertical as shown on the right below. The system is in a constant gravitational field with acceleration g downwards. The base does not move.



(a) Find the Lagrangian of the system. [12 points]

(b) Under what conditions (on a, b, and/or L) is the balanced position stable with respect to small displacements? [6 points]

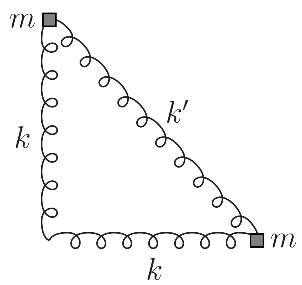
(c) Find the period of small oscillations. [6 points]

(d) Suppose that the condition found in part (b) is not satisfied, and the structure is displaced very slightly from its equilibrium balanced position at time $t = 0$. Find an expression for the time at which the supports of length a and b become horizontal (so that $\theta = 90$ degrees). You may leave your answer in terms of a definite integral. [16 points]

NIU Physics PhD Candidacy Exam – Fall 2017 – Classical Mechanics

You may solve ALL FOUR problems, if you choose. Only the THREE BEST PROBLEM GRADES count towards your score.
Total points on each problem = 40. Total possible score = 120.

Problem 1. Two equal masses (m) are constrained to move without friction, the first on the positive x axis and the second on the positive y axis. They are attached to two identical springs (force constant k and natural length L) whose other ends are attached to the origin. In addition, the two masses are connected to each other by a third spring of force constant k'. The length of the third spring is chosen so that the system is in equilibrium with all three springs at their relaxed natural lengths. Use coordinate x as the displacement of the first mass from equilibrium and coordinate y as the displacement of the second mass from equilibrium.



(a) Find the Lagrangian of the system. [12 points]
(b) Find the equations of motion of the system, assuming small displacements. [10 points]
(c) Solve for the normal frequencies, and find and describe the corresponding normal modes. [18 points]

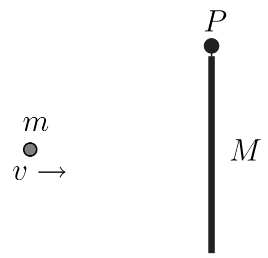
Problem 2. Consider a rocket that has mass $m(t)$ and velocity $v(t)$ at time t. It gains speed by expelling propellant at constant velocity v_{ex} relative to the rocket. The amount of propellant expelled between times t and $t + dt$ can therefore be written as $-dm(t)$. The rocket starts with initial velocity v_0 and initial mass m_0 (including the propellant).
(a) Ignoring gravity, write down the equation relating the momenta of the rocket+propellant system at times t and $t + dt$. [5 points]
(b) Use the result from part (a) to find the velocity of the rocket v as a function of v_0, v_{ex}, m_0, and m. [10 points]
(c) Redo part (a) but now assuming that the rocket is rising vertically in a uniform gravitational field with acceleration g, and that the rate at which the rocket loses mass is a constant value $k = -dm/dt$. [10 points]
(d) Use the result of part (c) to find both the velocity and height of the rocket as a function of t, v_0, v_{ex}, m_0, and k. [15 points]

Problem 3. Consider a particle of mass m moving in a three-dimensional central potential

$$V(r) = -k/(r-a),$$

where k and a are fixed positive constants. Let the angular momentum of the particle be ℓ.
(a) Find the differential equation of motion for r, the radial coordinate of the particle. [8 points]
(b) Derive an algebraic equation for the possible values of the radius r_c of a circular orbit. (You do not need to solve the equation.) [12 points]
(c) Consider a small perturbation of a circular orbit, $r = r_c + \epsilon$. Find a linearized differential equation for ϵ, written in terms of r_c, a, and k (with ℓ eliminated). Use it to show that circular orbits are stable if either $r_c < a$ or $r_c > Na$, where N is a certain integer that you will find. [12 points]
(d) Using graphical methods, show that the equation for a circular orbit that you found in part (b) has either one or three solutions, depending on whether ℓ^2/mak is smaller or larger than a certain rational number that you will find. [8 points]

Problem 4. A thin rod of length L and mass M is suspended in a uniform gravitational field of acceleration g from a fixed frictionless pivot point P. The rod is struck midway along its length by a small lump of clay with mass m moving horizontally at a high speed v. The lump sticks to the middle of the rod after the collision.



(a) Find the angular velocity ω_0 of the rod immediately after the collision. [Hint: at the moment of the collision, the total energy of the rod and the clay lump is not conserved, but there is another conserved quantity.] [14 points]
(b) Find an expression for the minimum speed $v = v_{\text{crit}}$ that will result in the rod making a complete circle around the pivot point. [13 points]
(c) For a given $v > v_{\text{crit}}$, find an expression for the time needed for for the rod to make the complete circle around the pivot point. You should leave your answer in terms of a definite integral. [13 points]

NIU Physics PhD Candidacy Exam – Spring 2017 – Classical Mechanics

You may solve ALL FOUR problems, if you choose. Only the THREE BEST PROBLEM GRADES count towards your score.
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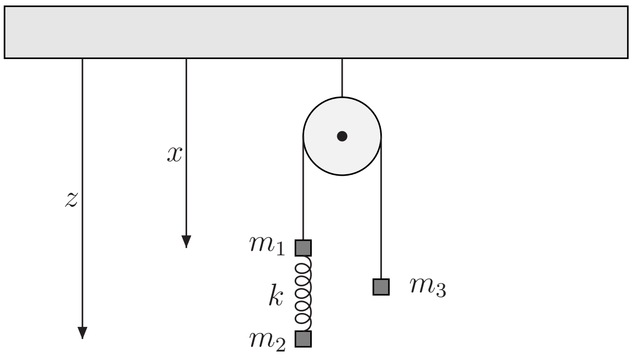
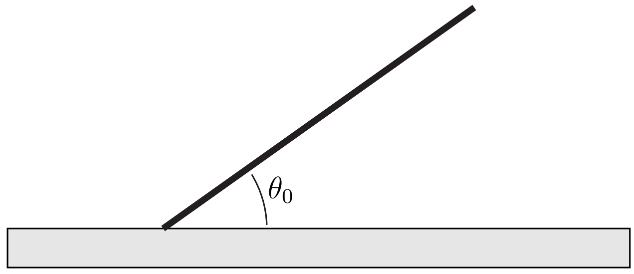
Problem 1. Three masses m_1, m_2, and m_3 are suspended as shown in a uniform gravitational field with acceleration g. Masses m_1 and m_3 are connected by an unstretchable string which moves over a massless pully whose center does not move. Mass m_2 is suspended from mass m_1 by a spring with constant k and unstretched length L. Call x and z the heights of masses m_1 and m_2, respectively, as shown.



(a) Find the Lagrangian of the system, using x and $y = z - x - L$ as your configuration variables. [10 points]
(b) Find the equations of motion of the system. [10 points]
(c) Solve for the relative motion of the masses m_1 and m_2, and in particular find the angular frequency of oscillation [10 points]
(d) What is the oscillation frequency if $m_1 \gg m_2, m_3$? Show using your answer above and/or physical arguments. [5 points]
(e) What is the oscillation frequency if $m_2 \gg m_1, m_3$? Show using your answer above and/or physical arguments. [5 points]

Problem 2. A point particle of mass M_1 approaches a massive sphere of radius R and mass M_2. The two masses attract each other according to Newton's law of gravitation. When very far away from each other, M_1 has initial speed v_0 while the sphere M_2 is initially at rest.
(a) If the initial impact parameter distance is b, find the distance of closest approach of the centers of the two objects. [20 points]
(b) Find the cross-section for the two objects to collide. [12 points]
(c) Give your answers and physical explanations for the limits $v_0 \to 0$ and $v_0 \to \infty$. [8 points]

Problem 3. A thin rod of length ℓ and mass M is supported at one end by a smooth floor on which it slides without friction. The rod falls in a uniform gravitational field with acceleration g, starting from rest with an initial angle θ_0 relative to the horizontal, as shown.



(a) Find the Lagrangian for the system. [10 points]

(b) Find the Hamiltonian for the system. [6 points]

(c) Find the time needed for the rod to fall to the floor. You may leave your answer in terms of a definite integral. [18 points]

(d) How far does the lower end of the rod move during this time? [6 points]

Problem 4. A water drop falls vertically in a uniform gravitational field g, growing by accretion of much smaller micro-droplets of water in the atmosphere. The micro-droplets have negligible velocity and air resistance is neglected. As the falling drop grows, it maintains a perfectly spherical shape, with radius we will call R. The mass density of water is a constant ρ, and the density of the micro-droplets (per volume of atmosphere) is $\epsilon\rho$, so that the mass of the falling drop increases at a rate given by its speed multiplied by $(\epsilon\rho)(\pi R^2)$.

(a) Find a relation between the speed of the falling water drop and the time derivative of its size, and use it to show that $R(t)$ obeys the differential equation

$$\frac{d^2R}{dt^2} + \frac{n}{R}\left(\frac{dR}{dt}\right)^2 = C,$$

where n is a certain integer that you will determine, and C is a constant that you will find in terms of the given quantities. [20 points]

(b) Find a general relation for the downward acceleration of the drop in terms of g, ϵ, and the instantaneous values of its speed v and radius R. [8 points]

(c) Starting from the results found in part (a), try a power law solution of the form $R = At^\alpha$, and solve for the constants A and α, to show that for large t the downward acceleration of the drop approaches the constant g/N, where N is another integer that you will determine. [12 points]

NIU Physics PhD Candidacy Exam – Fall 2016 – Classical Mechanics

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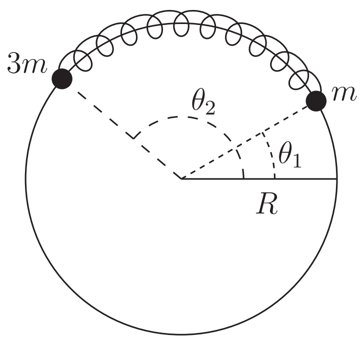
<u>Problem 1.</u> Object 1 (mass m) is attached to object 2 (mass $3m$) by a spring of unstretched length L and spring constant k. As shown in the figure, the two masses are constrained to move on a circle with radius R. The spring is also constrained to be on the circle. Ignore gravity and friction.



(a) Find the kinetic energy of the system in terms of the coordinates θ_1 and θ_2. [6 points]
(b) Find the potential energy of the system in terms of the coordinates θ_1 and θ_2. [6 points]
(c) Find the Lagrangian of the system. [2 points]
(d) Find the two Lagrange equations of motion for coordinates θ_1 and θ_2. [9 points]
(e) Use the equations of motion to find the general solution for the motion of the two objects. [9 points]
(f) At time $t = 0$, both masses are at rest, $\theta_2 = 0$, and the spring is at twice its natural, unstretched length. Find the subsequent motion. [8 points]

Problem 2. A point particle of mass m moves subject to a 3-dimensional central potential:

$$V(r) = -\frac{k}{r^n}$$

where k and n are positive constants.

(a) If the particle has angular momentum L, what is the radius R for which the orbit is circular? [10 points]

(b) Suppose the motion is close to the circular orbit mentioned in part (a). Writing $r(t) = R + \delta r(t)$ and assuming that $\delta r(t)$ is small, find an equation of motion for $\delta r(t)$. Write your equation in a form that does not involve the angular momentum L. [15 points]

(c) Solve this equation for δr. For what values of n are the circular orbits stable? [15 points]

Problem 3. A point particle of mass m is fixed to the bottom end of a thin wire suspended from a fixed point on the ceiling. The thin wire has total mass M and length L. The acceleration due to gravity is g. At time $t = 0$, the point m is given a very small tap.

(a) Find the tension in the wire and the speed of waves in the wire as a function of y, the distance from m. [16 points]

(b) Find the total time needed for the perturbation to reach the top end of the wire (the ceiling). [24 points]

Problem 4. A uniform solid spherical ball of mass M and radius R rests on a horizontal surface. Assume a constant coefficient of friction μ (this means that the frictional force is equal to the normal force multiplied by μ). The acceleration due to gravity is g. At time $t = 0$, the ball is struck impulsively on center, causing it to go instantaneously from rest to a horizontal speed v_0 with no initial rotation.

(a) Find the horizontal speed, and the angular velocity of the ball about its center, as a function of t. [16 points]

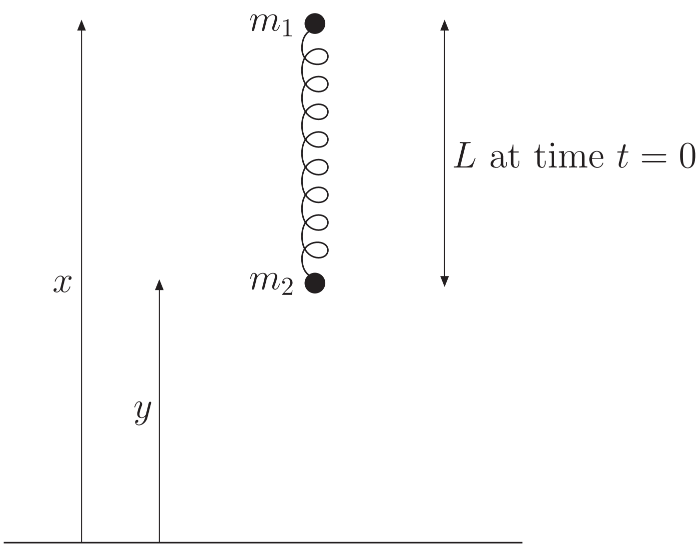
(b) Find the distance travelled by the ball until it begins to roll without slipping. [24 points]

[Hint: the moment of inertia of the sphere about its center is $\frac{2}{5}MR^2$.]

NIU Physics PhD Candidacy Exam – Spring 2016 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. Two masses m_1 and m_2 are attached via a massless spring of natural (unstretched) length L and spring constant k, as shown in the figure. The masses are freely falling in a uniform gravitational field with acceleration g. The mass m_1 is located vertically above the mass m_2, with heights x and y respectively, and all motion is vertical. At time t=0, mass m_1 is at height x_0 and traveling vertically downward with velocity v_0, and mass m_2 is at height y_0 and is at rest. The masses are initially a distance L apart ($x_0 - y_0 = L$), so that the spring is neither compressed nor stretched.



(a) Write down the Lagrangian for the system in terms of x and y. [8 points]

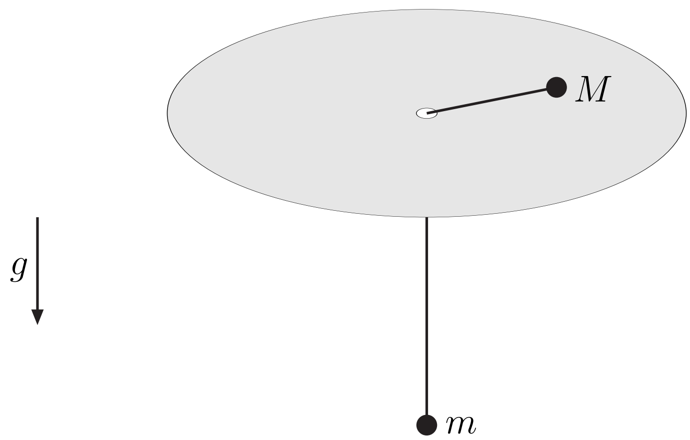
(b) Redefine the coordinates in terms of the center of mass of the system and the relative coordinate between the masses. [8 points]

(c) Rewrite the Lagrangian in terms of the new coordinates and find the equations of motion. [8 points]

(d) Solve the equations of motion for the new coordinates for all times $t > 0$. [8 points]

(e) Rewrite the solution of the equations of motion in terms of the original coordinates x and y for all times $t > 0$. [8 points]

Problem 2. A massless string of length L passes through a hole in a horizontal table. A point pass M at one end of the string moves frictionlessly on the table (i.e. with two degrees of freedom), and another point mass m hangs vertically from the other end. The system is in a uniform gravitational field with acceleration g.



(a) Write the Lagrangian for the system. [13 points]

(b) Suppose the mass M on the table initially has a speed v. Under what conditions will the hanging mass remain stationary? [13 points]

(c) Starting from the situation in part (b), the hanging mass is pulled down very slightly and then released. Compute the angular frequency of the subsequent oscillatory motion of the hanging mass. [14 points]

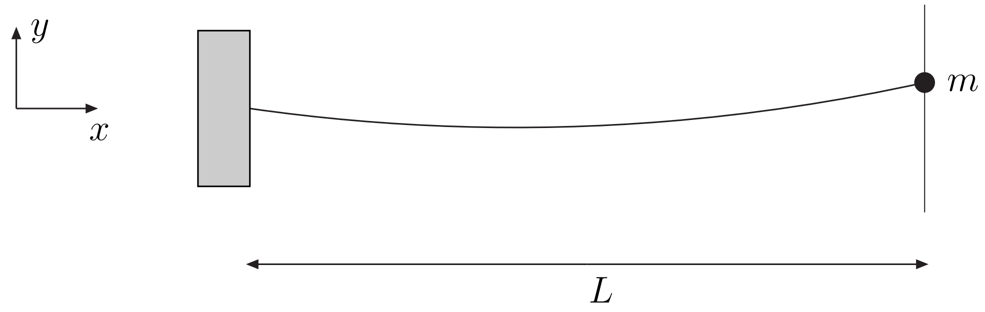
Problem 3. A table consists of a horizontal thin uniform circular disk of radius R and mass M, supported on its rim by three thin vertical legs that are equidistant from each other. The acceleration due to gravity is g.

(a) Suppose two of the legs are suddenly removed. Find the force exerted by the third leg on the table top immediately after. [20 points]

(b) Suppose instead that just one of the legs is suddenly removed. Find the force exerted by each of the remaining two legs immediately after. [20 points]

Possibly useful information: for a thin solid disk of mass M and radius R, the three principal moments of inertia about its center are: $\frac{1}{4}MR^2$, $\frac{1}{4}MR^2$, and $\frac{1}{2}MR^2$.

Problem 4. A uniform string of length L undergoes small transverse oscillations. The tension of the string is T and the mass per unit length is μ. The equilibrium position of the string lies along the x axis and the transverse displacement at time t is restricted to the y direction and is denoted by $y(x,t)$. One end of the string, at $x = 0$, is fixed, so that $y(0,t) = 0$ for all t. The other end of the string is attached to a point particle of mass m that is free to move frictionlessly in the y direction at fixed $x = L$. Neglect gravity.



(a) Write down the wave equation for small amplitude displacements $y(x,t)$, and express the velocity of propagation of transverse waves in terms of T and μ. (For this part only, you may obtain the answers just by using dimensional analysis to supplement your memory, if you wish.) [6 points]

(b) Show that the boundary condition for the mass m at $x = L$ for *small* displacements has the form:

$$-C\frac{\partial y}{\partial x} = \frac{\partial^2 y}{\partial t^2}, \qquad \text{(at } x = L\text{)}$$

where C is a constant that you will determine in terms of the given quantities. [12 points]

(c) Use the boundary condition above to obtain a transcendental equation that implicitly determines the wavenumbers k of the normal modes of the system. Show graphically that there are an infinite number of solutions. [16 points]

(d) For each of the extreme cases $m = 0$ and $m = \infty$, use your answer to part (c) to solve for the allowed wavenumbers k for the normal modes. [6 points]

NIU Physics PhD Candidacy Exam – Fall 2015 – Classical Mechanics

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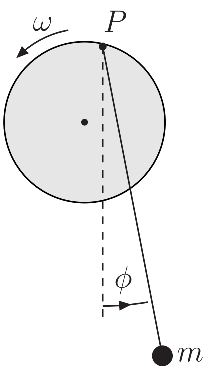
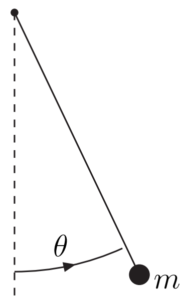


Problem 1. Consider a circular disc of radius R as shown above. The disc is forced to rotate counterclockwise about its center at a constant angular velocity ω. A pendulum with mass m and length L hangs from a point P on the edge of the disc. There is a uniform gravitation field with acceleration g pointing down. At time $t = 0$, the point P is at its highest point.
(a) Find the position and velocity of the point P as a function of time. (4 points)
(b) Find the Lagrangian for the system in terms of the configuration variable ϕ, which is the angle made by the pendulum with the vertical. Simplify the result so that each term has at most one trigonometric function. [The following identities might be useful: $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and $\cos(A + B) = \cos A \cos B - \sin A \sin B$.] (14 points)
(c) Find the equation of motion for ϕ. (12 points)
(d) If the pendulum mass m is instantaneously at rest directly below the center of the disc (and P) at time $t = 0$, find the subsequent motion of ϕ for small t, as an expansion up to and including terms of order t^3. [Hints for part (d): try a solution of the form

$$\phi = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots .$$

To find the correct result, sine functions need only be expanded to linear order in their arguments.] (10 points)



Problem 2 A simple pendulum (string length L and mass m) is freely swinging in a plane under the gravitational force, mg, directed downward, as shown above.

(a) Find the equation of motion of this pendulum. Do not assume that θ is small. [4 points]

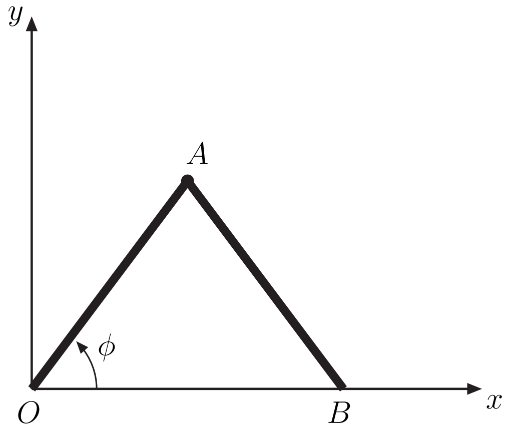
(b) Show that this non-linear differential equation can be approximated by a linear differential equation (simple harmonic oscillator) under a certain condition. What is the condition, and what is the period T_0 of this "approximated" simple harmonic oscillator? [4 points]

(c) For the non-linear differential equation in part (a), suppose that the motion is such that the maximum angular displacement is $\theta = \theta_0$. Find the angular velocity $d\theta/dt$ as a function of θ. Write your answer in terms of g, L, θ_0, and θ. [16 points]

(d) Now suppose that $\theta = 0$ at time $t = 0$. Note that $\theta = \theta_0$ (the maximum angular deviation) at time $t = T/4$, where T is the period of the *non-linear* oscillation. Find an expression for T in terms of θ_0 and T_0 [the period found in part (b)], in the form of an expansion in $\sin(\theta_0/2)$. Keep terms up to and including second order in that expansion. [16 points]

[Hints for part (d): You may find it useful to replace cosines using the formula $\cos(\theta) = 1 - 2\sin^2(\theta/2)$. You may then find it useful to write $\sin(\theta/2) = \sin(\theta_0/2)\sin\phi$, which defines a new variable ϕ. Finally, the expansion $1/\sqrt{1-x^2} = 1 + x^2/2 + \ldots$ may be useful.]

Problem 3 Two identical thin uniform rods, each of length d and mass m, are hinged together at the point A. The rod on the left has one end hinged at the fixed point O, while the end B of the other rod slides freely along the horizontal x axis. The system is in a uniform vertical gravitational field with acceleration g. All motion is frictionless.



(a) Find the total kinetic energy of the system. You should find the result:

$$T = md^2\dot{\phi}^2(a + b\sin\phi + c\sin^2\phi)$$

where a, b and c are constant numbers that you will determine. (Exactly one of a, b, and c is 0.) [25 points]

(b) Suppose the system is at rest at time $t = 0$ with $\phi = \phi_0$. What is the velocity of the hinge A when it hits the horizontal x axis? [15 points]

Problem 4. Consider a particle of mass m scattering from a central potential $V = -k/r^n$, where k and n are positive constants. The particle approaches from very far away with a non-zero impact parameter b and initial velocity v_0.

(a) Show that for the particle to have a chance at hitting the origin ($r = 0$), it is necessary that n is greater than or equal to a certain number that you will determine. [15 points]

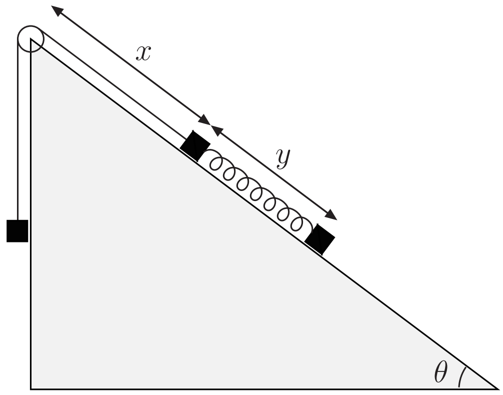
(b) Now taking $n = 4$, show that a necessary and sufficient condition for the particle to hit the origin is that $b < b_{\text{crit}}$, where b_{crit} is a quantity that you will determine in terms of k, v_0, and m. [20 points]

(c) Still taking $n = 4$, what is the cross section for particles to hit the origin? [5 points]

NIU Physics PhD Candidacy Exam – Spring 2015 – Classical Mechanics

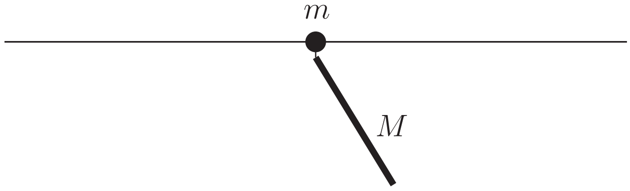
Do ONLY THREE out of the four problems. Total points on each problem = 40.

<u>Problem 1.</u> Consider a stationary inclined plane ramp with a fixed angle θ, with three blocks with equal masses M attached by a spring of constant k and a string over a fixed pulley at the top of the ramp, as shown below. The positions of the blocks are described by the distances $x(t)$ (the distance from the pulley to the higher block on the ramp) and $y(t)$ (the extension of the spring), as shown. The system is in a uniform graviational field of acceleration g. The unstretched length of the spring is 0, and the pulley is massless.



(a) [12 points] Find the Lagrangian and the equations of motion of the system.
(b) [20 points] The two blocks on the ramp are released from rest at the top with $x = y = 0$ at time $t = 0$. Find the motions of the blocks as a function of time.
(c) [8 points] The choice $\theta = 30$ degrees has a special significance, which should be apparent in the results of part (b). What is it?

<u>Problem 2.</u> A bead of mass m moves without friction along a fixed horizontal line. A thin rod of length a and total mass M (uniformly distributed along its length) hangs from the bead, attached by a very short massless string. The rod is constrained to swing in the plane of the page. The system is in a uniform gravitational field with acceleration g downwards.



(a) [20 points] Find the Lagrangian of the system and the equations of motion.
(b) [20 points] Find the normal modes and corresponding frequencies of small oscillations.

Problem 3. Consider a rocket with total initial mass m. Exactly half of m is fuel, which is expelled from the rocket at a constant rate k (mass per unit time) and with speed u with respect to the rocket. There is a uniform gravitational field g downwards, and the motion of the rocket is entirely vertical. Let the height of the rocket as a function of time be $x(t)$. The rocket is at rest on the ground (at $x = 0$) and begins expelling fuel at time $t = 0$.
(a) [16 points] Find a differential equation satisfied by the height of the rocket.
(b) [6 points] Find the condition on m, k, u, g for the rocket to lift off immediately at $t = 0$, and the condition for the rocket to lift off at all (at any t).
(c) [18 points] Assuming that the rocket takes off immediately at $t = 0$, find expressions for the height and speed of the rocket at the moment when it runs out of fuel. Write your answers as algebraic expressions in terms of only the quantities m, k, u, g.
The following indefinite integral may or may not be useful:

$$\int \ln(1 - az)dz \quad = \quad (z - 1/a)\ln(1 - az) - z.$$

Problem 4. The force of attraction between a star of mass M and a planet of mass m (where $m \ll M$) is:

$$F = \frac{a}{r^2} + \frac{3b\ell^2}{r^4}$$

where ℓ is the angular momentum of the planet and a, b are both positive constants. [Note: this does approximate the force of attraction between a planet and a black hole, in the non-relativistic limit, with $a = GMm$.]
(a) [15 points] Under what conditions is a stable circular orbit possible? Give the radius of the stable circular orbit in terms of the given parameters (M, m, a, b, ℓ).
(b) [15 points] What is the smallest radius possible for any circular orbit as a function of a and b, allowing for arbitrary ℓ? (Hint: this occurs in the limit of very large ℓ.) Is this circular orbit stable or unstable?
(c) [10 points] If the planet travels in a slightly non-circular orbit about a stable radius, find an expression for the angular frequency of small radial oscillations.