NIU Physics PhD Candidacy Exam – Fall 2014 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. Consider a galaxy consisting of a very large number N of identical stars, each of mass m, distributed uniformly as a very thin disk with radius R. The galaxy also has a concentric spherical halo of dark matter, with a finite but very large radius and uniform mass density ρ. The stars only interact gravitationally with the dark matter and each other.

(a) If the star orbits are perfectly circular, find the speed v and the angular momentum ℓ (about the center of the galaxy) for each star, as a function of the star's distance from the center r and N, m, R, ρ, and G (Newton's constant). [14 points]

(b) Now consider star orbits that can be non-circular. The equation of motion of r for each star with angular momentum ℓ is the same as that of an equivalent 1-dimensional problem. For that problem, find the effective potential as a function of r. Your answer may depend on N, m, R, ρ, G, and ℓ. [13 points]

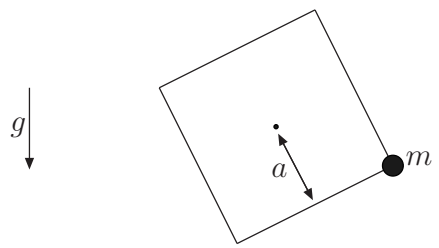
(c) Consider a star with a nearly circular orbit $r(t) = r_0 + \epsilon(t)$, where r_0 is constant and $\epsilon(t)$ is very small. Find an expression for the frequency of oscillations for ϵ. [13 points]

Problem 2. A solid cube with side $2a$ has mass M uniformly distributed throughout its volume. It rotates frictionlessly on a fixed horizontal axis which passes through the centers of two opposite sides. A point mass m is attached to one corner of the cube. There is a constant downward gravitational field with acceleration g.

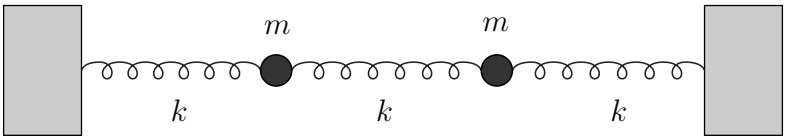
(a) Find the Lagrangian and the corresponding equation of motion for the system. [16 points]

(b) Find the frequency of small oscillations when the mass m is near its lowest point. [10 points]

(c) If the mass m is held at the same height as the axis of rotation, and then released, how much time will it take to reach its lowest point? You may leave your answer in terms of a definite integral over a single variable. [14 points]

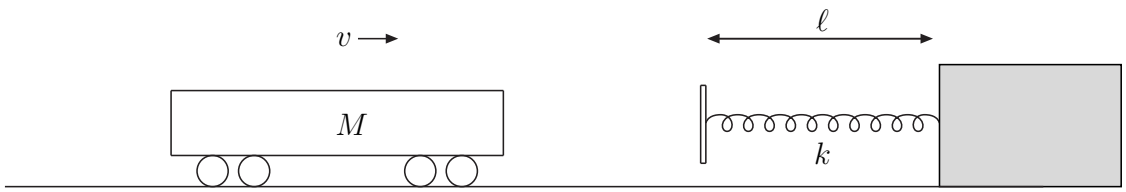


Problem 3. Consider a system of two masses m and three identical springs with spring constant k between two stationary walls as shown. At equilibrium, the lengths of the springs are a, and if they were unstretched their lengths would be b. Consider only longitudinal motions (along the axis of the springs).



(a) Find the Lagrangian and Lagrange's equations for the system. [14 points]
(b) Find the normal-mode frequencies of vibration and the eigenvectors. [13 points]
(c) Suppose that at time $t = 0$ the mass on the left is displaced from equilibrium by a distance X to the right, while the mass on the right is not displaced, and both masses are at rest. Compute the motion of the left mass for $t > 0$. [13 points]

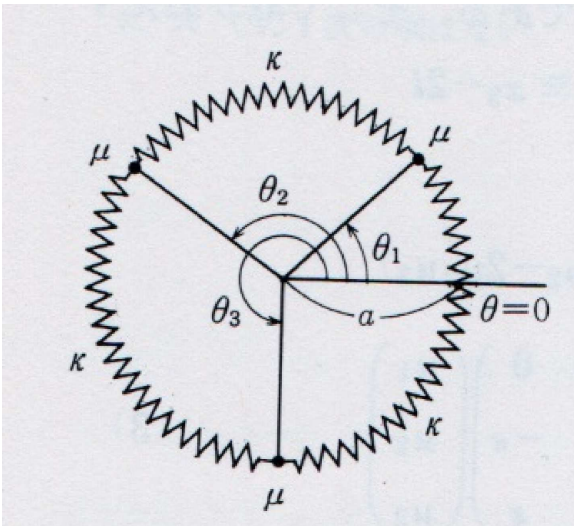
Problem 4. A speeding train car of mass M, moving with speed v, is to be stopped with a coiled-spring buffer of uncompressed length ℓ and spring constant k. If the spring becomes fully compressed, the spring constant will suddenly change to become very large. Assuming that you can choose k to be any fixed constant that you want, what is the minimum value of ℓ needed to assure that the maximum absolute value of the deceleration of the train does not exceed $a_{\max}$? What value of k should you choose to achieve this? [40 points]



NIU Physics PhD Candidacy Exam – Spring 2014 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. Consider three identical springs with spring constant κ and unstretched length L, connecting three identical point masses μ. These masses and springs are constrained to move frictionlessly on a circle of radius a as in the figure below. There is no gravity. The positions of the masses are described by three angles θ_1, θ_2, θ_3 as shown.



(a) What is the kinetic energy of the system? [10 points]
(b) What is the potential energy of the system? Express it using a matrix. [10 points]
(c) Find the normal modes and frequencies of oscillation for the system. [20 points]

Problem 2. A small satellite with mass m is in a circular orbit of radius r around a planet of mass M. The planet's thin atmosphere results in a frictional force $F = Av^\alpha$ on the satellite, where v is the speed of the satellite and A and α are constants. It is observed that with the passage of time, the orbit of the satellite remains circular, but with a radius that decreases very slowly with time according to $dr/dt = -C$, where C is a constant independent of the orbit's radius and the speed. Assume that $m \ll M$.
(a) Show that α can only be a certain integer, and find that integer. [20 points]
(b) Solve for the constant A. Express your answer in terms of C, the masses m and M, and Newton's gravitational constant G. [20 points]

Problem 3. Consider a point mass m moving in the (x, z) plane on the parabola $z = x^2/(2a)$ with $a > 0$ and subject to the constant gravitational field $\mathbf{g} = (0, -g)$.

(a) Are there conserved quantities? If so, what are they? [8 points]

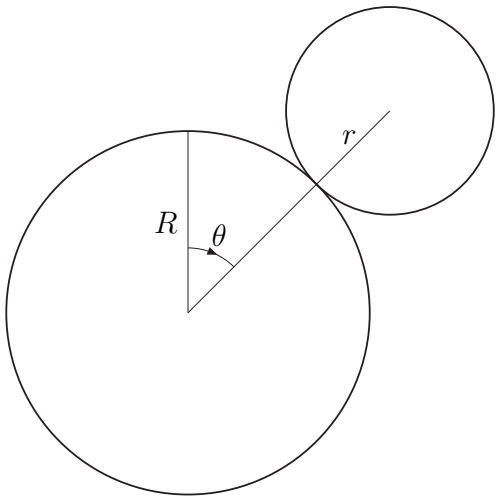
(b) Give the kinetic and potential energies for the point mass. What is the Lagrangian, if one chooses x as the generalized coordinate? [8 points]

(c) Derive the canonical momentum p conjugate to x as well as the Lagrange equation for $x(t)$. [8 points]

(d) Determine the Hamiltonian function H. [8 points]

(e) Assuming $x \ll a$, obtain for H a quadratic form in x and p. Write down the canonical equations of motion for this approximate Hamiltonian function, and solve them assuming the initial condition $x(0) = 0$, $p(0) = p_0$. [8 points]

Problem 4. A uniform solid **sphere** of radius r and mass m rolls off of a fixed **cylinder** of radius R, starting nearly from rest at the top of the cylinder, in the Earth's constant gravitational field with acceleration g. The moment of inertia of a solid sphere of radius r and mass m about its center is $\frac{2}{5}mr^2$.



(a) Write the Lagrangian for the system using the single configuration variable θ shown. [15 points]

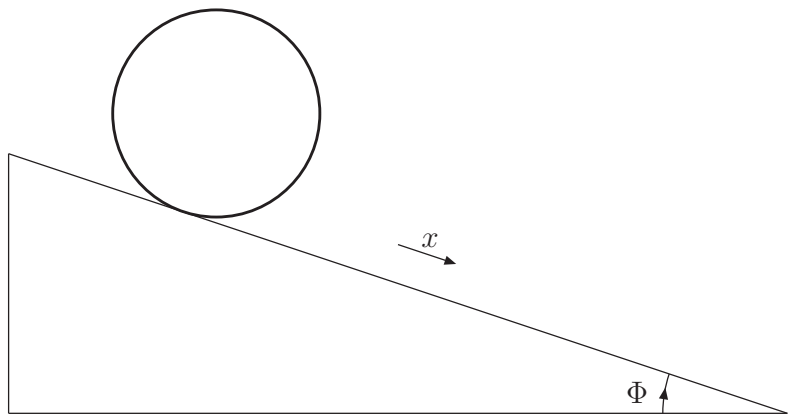
(b) At what angle will the sphere leave the cylinder? [20 points]

(c) If the sphere had the same radius and total mass, but was a thin shell instead of a uniform solid, would the angle be greater or less than in part (b)? Explain your answer. [5 points]

NIU Physics PhD Candidacy Exam – Fall 2013 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. Consider a circular hoop with mass M and radius R rolling down a fixed inclined plane without slipping. The inclined plane of length L is placed in Earth's gravitational field at a fixed angle Φ with respect to the horizontal plane. Use the generalized coordinates x (measured along the inclined plane) and θ (the rotation angle of the hoop) to describe the motion of the hoop. At time $t = 0$, the hoop starts at rest with $x = 0$, $\theta = 0$.

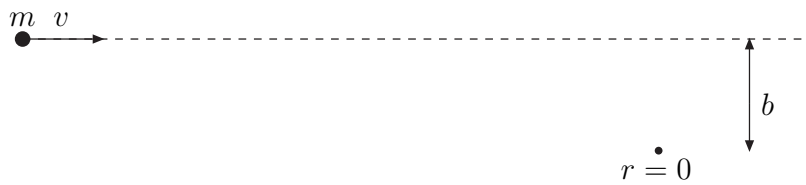


(a) Draw a figure showing all the forces acting on the hoop. [4 points]
(b) Find the Lagrangian in terms of generalized coordinates and generalized velocities. [6 points]
(c) Determine the holonomic constraint $f(x, \theta) = 0$ between x and θ. [2 points]
(d) Eliminate one of the generalized coordinates in the Lagrangian by using $f(x, \theta) = 0$. [2 points]
(e) Find an equation of motion, and solve it to determine the motion of the hoop by finding $x(t)$ and $\theta(t)$. [6 points]
(f) By using the method of undetermined Lagrange multipliers with the holonomic constraint $f(x, \theta)$ find equations of motion, solve them in terms of $x(t)$ and $\theta(t)$, and find the force of constraint. [12 points]
(g) Determine the non-holonomic (semi-holonomic) constraint between generalized velocities $\dot{x}$ and $\dot{\theta}$, of the form $f(\dot{x}, \dot{\theta}) = 0$. [2 points]
(h) Use the method of undetermined Lagrange multipliers with the semi-holonomic constraint $f(\dot{x}, \dot{\theta})$ to find equations of motion, solve them in terms of $x(t)$ and $\theta(t)$, and find the force of constraint. [6 points]

Problem 2. Consider a point particle of mass m moving in three dimensions in a central potential

$$V(r) = -\frac{\alpha}{r} - \frac{\beta}{r^2}$$

where α and β are positive constants and r is the distance from the origin. The particle approaches from very far away with speed v and impact parameter b, as shown. (The dashed line is what the path would be if there were no potential.)

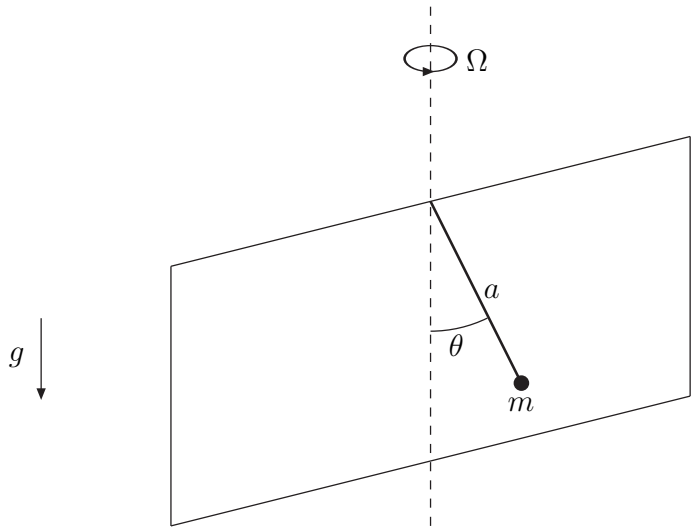


(a) Find $r_{\min}$, the distance of closest approach of the particle to $r = 0$. Show that the particle will go through the origin if $\beta > \beta_c$, where β_c is a critical value that you will determine in terms of the other given quantities. [12 points]

In the remainder of this problem, you should assume $\beta < \beta_c$, and you may leave your answers in terms of $r_{\min}$ and the other given quantities.

(b) What is the maximum speed reached by the particle on its trajectory? [8 points]
(c) What is the maximum acceleration reached by the particle on its trajectory? [8 points]
(d) When the particle is very far from the origin again, find the angle by which it has been scattered from its original direction. You may leave your answer in terms of a single definite integral. [12 points]

Problem 3. A pendulum of fixed length a is constrained to move in a vertical plane. This plane is forced to rotate at fixed angular velocity, Ω, about a vertical axis passing through the pendulum's pivot. The mass of the pendulum bob is m and there is a uniform gravitational field with acceleration g pointing down.

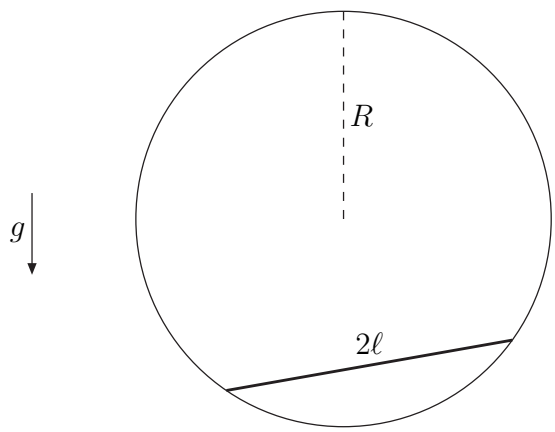


(a) Find all of the equilibrium points of the pendulum, and find whether they are stable or unstable. [25 points]
(b) Find the frequency of small oscillations about the point or points of stable equilibrium. [15 points]

Some possibly useful identities:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta \\
\sin(\cos^{-1}(x)) &= \sqrt{1-x^2} \\
\cos(\sin^{-1}(x)) &= \sqrt{1-x^2}
\end{aligned}$$

Problem 4. A thin straight rod of fixed length 2ℓ and linear mass density ρ is constrained to move with its ends on a circle of radius R, where $R > \ell$. The whole circle is held absolutely fixed in a vertical plane here on the Earth's surface, and the contacts between the circle and the rod are frictionless.



(a) Find the Lagrangian for the motion of the rod. [18 points]

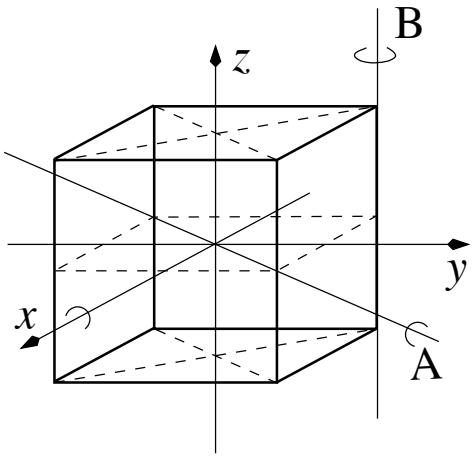
(b) Obtain the equations of motion for the rod. [10 points]

(c) Find the frequency of small oscillations, assuming that the departures from equilibrium are small. [12 points]

NIU Physics PhD Candidacy Exam – Spring 2013 – Classical Mechanics

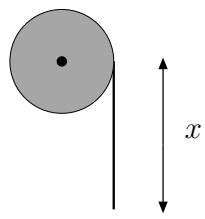
Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. Consider a homogeneous cube with edge length a and mass m. We study the rotational motion (angular velocity ω) of the cube about different axes of rotation.



(a) Evaluate the inertia tensor assuming that the origin of the coordinate system is at the center of the cube. [10 points]
(b) Evaluate the kinetic energy for a rotation with angular velocity ω about the x axis of the coordinate system, i.e., $\vec{\omega} = (\omega, 0, 0)$. [10 points]
(c) Does the kinetic energy change as compared with (b) if the cube rotates about the axis A defined by $x = y$ with $z = 0$ (see figure)? Justify your answer. [10 points]
(d) Evaluate the kinetic energy for a rotation about the axis B, defined by $-x = y = a/2$. [10 points]

Problem 2. A very long, perfectly flexible material is rolled up on a fixed, horizontal, massless, very thin axle. The rolled-up portion rotates freely on the axle. The material has a (small) thickness s, width w, and, when completely unrolled, length ℓ. The mass density per unit volume of the material is ρ. At time $t = 0$, a length x_0 hangs from the roll and the system is at rest. For $t > 0$, the material unrolls under the influence of a constant gravitational field g (downward). [Useful information: the moment of inertia of a solid cylinder of mass M and radius R about its axis of symmetry is $I = MR^2/2$.]

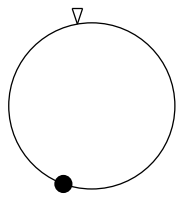


(a) Using the length x of the hanging part of the material as your configuration variable, find the Lagrangian of the system. [15 points]

(b) Find the canonical momentum conjugate to x, and the Hamiltonian of the system. [9 points]

(c) What is the velocity of the free end of the material at the time when half of it is off the roll ($x = \ell/2$)? [16 points]

Problem 3. A small bead of mass M is free to slide on a frictionless, uniform wire, also of mass M, which is formed into a circle of radius R. The circular wire is suspended at one point from a fixed pivot, so that it is free to swing under gravity, but only within its own plane (the plane of the page in the figure below). The gravitational field is a constant g (downward), and the small triangle in the figure represents the pivot point.



(a) Find the Lagrangian of the system using appropriate configuration variables, and the equations of motion. [20 points]
(b) Find the angular frequencies of *small* oscillation modes. [20 points]

Problem 4 Consider the (planar) motion of a particle of mass m and initial angular momentum L in the central 3-dimensional potential corresponding to a "spring" with a non-zero relaxed length a:

$$V(r) = \frac{1}{2}k(r-a)^2$$

(a) [20 points] Find a condition relating the radius r_0 of circular orbits and the given quantities. (It is not necessary to solve for r_0.)
(b) [20 points] For nearly circular orbits, find an expression for the period of time between successive radial maxima for $a/r_0 \ll 1$, expressing your answer to first order in a/r_0 (with L eliminated from the answer). Use this to find the angular change between successive radial maxima in the limit $a/r_0 \to 0$, and check that the orbits are closed in that limit.

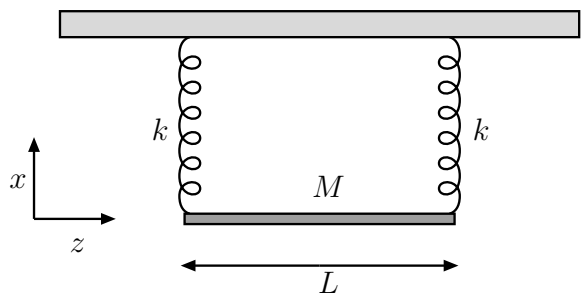
NIU Physics PhD Candidacy Exam – Fall 2012 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. A thin rigid uniform bar of mass M and length L is supported in equilibrium in a horizontal position by two massless springs attached to each end, as shown. The springs have the same force constant k, and are suspended from a horizontal ceiling in a uniform gravitational field. The motion of the bar is constrained to the x, z plane, and the center of gravity of the bar is further constrained to move parallel to the vertical x-axis.

(a) Write a Lagrangian describing the dynamics of this system for small deviations from equilibrium, using as configuration variables the heights of the endpoints of the bar.
[15 points]

(b) Find the normal modes and angular frequencies of small vibrations of the system.
[25 points]



Problem 2. A particle of mass m moves in one dimension in a harmonic oscillator potential, and is also subject to a time-dependent force $F(t)$ so that the Lagrangian is:

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 + xF(t).$$

Let

$$F(t) = \begin{cases} 0 & (\text{for } t \leq 0), \\ \frac{t}{T}F_0 & (\text{for } 0 < t < T), \\ F_0 & (\text{for } t \geq T), \end{cases}$$

Suppose that the particle is at rest in equilibrium at $t = 0$.

(a) Derive the equation of motion and find the solution for $x(t)$ when $0 < t < T$. [20 points]

(b) Find the solution for $x(t)$ for all times $t > T$. [20 points]

Problem 3. A slowly moving spinning planet of radius R, with no atmosphere, encounters a region with many tiny meteors moving in random directions. A thin layer of dust of thickness h then forms from the fall of meteors hitting the planet from all directions. Let the original uniform density of the planet be ρ, and the density of the layer of meteor dust be ρ_d. Assuming that $h \ll R$, show that the fractional change in the length of the day is

$$N_1 \left(\frac{h}{R}\right)^{N_2} \left(\frac{\rho_d}{\rho}\right)^{N_3},$$

where N_1, N_2, and N_3 are each certain non-zero integers that you will determine. (The moment of inertia of a sphere about an axis through its center is $\frac{2}{5}MR^2$.) [40 points]

Problem 4. Express your answers for the questions below in terms of the gravitational constant G, the masses, and the distance L. Assume that the stars move on circular orbits.

(a) What is the rotational period T of an equal-mass ($M_1 = M_2 = M$) double star of separation L? [10 points]

(b) What is the rotational period T of an unequal-mass ($M_1 \neq M_2$) double star of separation L? [15 points]

(c) What is the rotational period T of an equal-mass ($M_1 = M_2 = M_3 = M$) triple star, where the stars are located at the corners of an equilateral triangle (side L)? [15 points]

NIU Physics PhD Candidacy Exam – Spring 2012 – Classical Mechanics

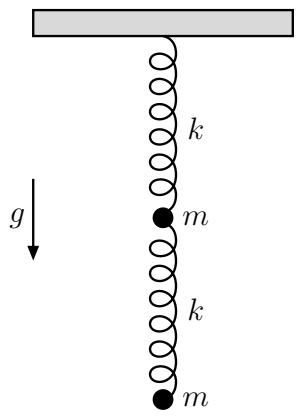
Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. A coupled oscillator consists of two springs with equal spring constants k and two equal masses m, which hang from a fixed ceiling in a uniform gravitational field with acceleration g. The system oscillates in the vertical direction only.

(a) Find the Lagrangian for the system. [10 points]

(b) Find the angular frequencies of the normal modes of oscillation. [20 points]

(c) In the slower mode, find the ratio of the amplitude of oscillation of the upper mass to that of the lower mass. [10 points]



Problem 2. A particle of mass m moves along the x-axis under the influence of a time-dependent force given by $F(x,t) = -kxe^{-t/\tau}$, where k and τ are positive constants.

(a) Compute the Lagrangian function. [7 points]

(b) Find the Lagrangian equation of motion explicitly. [7 points]

(c) Compute the Hamiltonian function in terms of the generalized coordinate and generalized momentum. (Show clearly how you got this.) [7 points]

(d) Determine Hamilton's equations of motion explicitly for this particular problem (not just general formulae). [7 points]

(e) Does the Hamiltonian equal the total energy of the mass? Explain. [6 points]

(f) Is the total energy of the mass conserved? Explain. [6 points]

Problem 3. Consider a particle of mass m moving in a three-dimensional central potential

$$V(r) = -K/(r - R),$$

where K and R are fixed positive constants. Let the angular momentum of the particle be ℓ.

(a) Find the differential equation of motion for r, the radial coordinate of the particle. [10 points]

(b) Derive an algebraic equation for the possible values of the radius r_c of a circular orbit. (You do not need to solve the equation.) [10 points]

(c) Consider a small perturbation of a circular orbit, $r = r_c + \epsilon$. Find a linearized differential equation for ϵ, written in terms of r_c, R, and K (with ℓ eliminated). Use it to show that circular orbits are stable if either $r_c < R$ or $r_c > nR$, where n is a certain integer that you will find. [10 points]

(d) Using graphical methods, show that the equation for a circular orbit that you found in part (b) has either one or three solutions, depending on whether ℓ^2/mRK is smaller or larger than a certain rational number that you will find. [10 points]

Problem 4. A homogeneous solid cube with mass M and sides of length a is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a tiny displacement and allowed to fall. (This is done in a uniform gravitational field with acceleration g. The moment of inertia of a cube about an axis through its center and parallel to an edge is $I = Ma^2/6$.)

(a) Find the angular velocity of the cube when one face strikes the plane, assuming that the edge cannot slide due to friction. [15 points]

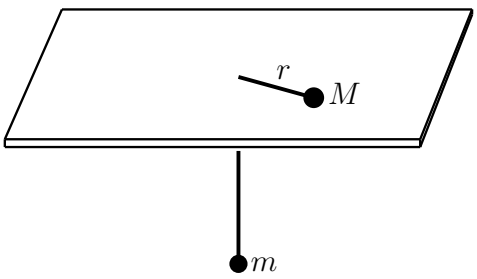
(b) Same question as (a), but now assuming that the edge slides without friction on the plane. [15 points]

(c) For the frictionless case in part (b), what is the force exerted by the surface on the cube just before the face strikes the plane? [10 points]

NIU Physics PhD Candidacy Exam – Fall 2011 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. A particle of mass M is constrained to move on a smooth horizontal plane. A second particle of mass m is attached to it by hanging from a string passing through a hole in the plane as shown, and is constrained to move in a vertical line in a uniform gravitational field of acceleration g. All motion is frictionless and the string is massless.
(a) Find the Lagrangian for the system and derive the equations of motion. [15 points]
(b) Consider solutions in which M moves in a circle with a constant speed v_0. Find the radius of the circle r_0 in terms of the other quantities. [12 points]
(c) Show that the solution in part (b) is stable and find the angular frequency of small oscillations about the stable circular orbit. [13 points]

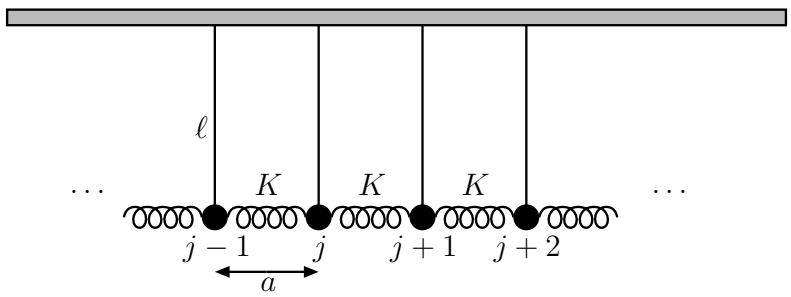


Problem 2. Consider pointlike particles of mass m which approach a sphere of mass M and radius R. The particles are attracted to the sphere in accordance with Newton's law. When they are very far away, the particles have velocity v_∞. You may assume that $m \ll M$. Find the effective cross-section (with units of area) for the particles to strike the sphere. [40 points]

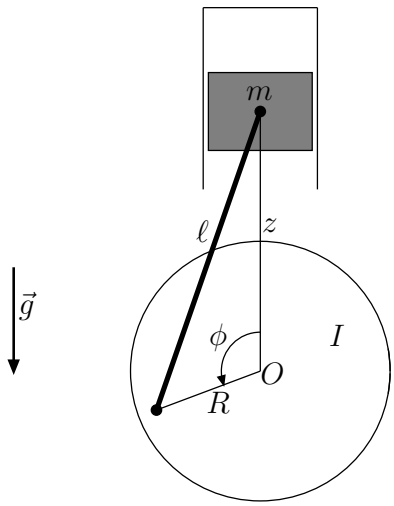
Problem 3. Consider an infinite number of identical pendulums of mass M in a uniform gravitational field with acceleration g, each hanging by a massless string of length ℓ, and coupled to each other with massless springs of spring constant K as shown. In the equilibrium position, the springs are at their natural length, a. The masses move only in the plane of the page, and with only a small displacement from equilibrium.

(a) Denote the small horizontal displacement of the jth mass from equilibrium as $u_j(t) = u(x,t)$, where $x = ja$ is the equilibrium position. Derive a wave equation of motion for $u(x,t)$ for this system as a second-order differential equation in x and t, in the long wavelength approximation. [25 points]

(b) Find the dispersion relation (a relation between the angular frequency and the wavenumber). What is the minimum angular frequency? [15 points]



Problem 4. A lawn-mower engine contains a piston of mass m that moves along $\hat{z}$ in a field of constant gravitational acceleration $\vec{g} = g\hat{\mathbf{z}}$. The center of mass of the piston is connected to a flywheel of moment of intertia I at a distance R from its center by a rigid and massless rod of length ℓ, as shown. The system has only one degree of freedom but two natural coordinates, ϕ and z.



(a) Express the Lagrangian in terms of $q_1 = z$, $q_2 = \phi$. [5 points]

(b) Write the constraint equation that connects the two coordinates. [5 points]

(c) From the above results, write down the two coupled equations of motion using the method of "undetermined multipliers". Then eliminate the undetermined multiplier to obtain a single equation of motion (it can still involve both coordinates). [15 points]

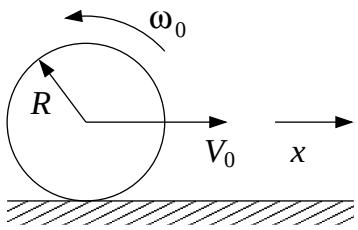
(d) Find $p_\phi(z, \phi, \dot{\phi})$. [15 points]

Hint: A constraint equation of the form $C(q_i) = 0$ leads to Lagrange equation(s) of motion for the configuration variables q_i that include additional terms $-\lambda \frac{\partial C}{\partial q_i}$, where $\lambda(q_i, t)$ is the undetermined multiplier.

NIU Physics PhD Candidacy Exam – Spring 2011 – Classical Mechanics

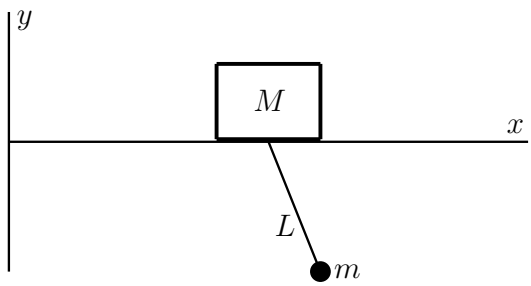
Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. A thin hollow cylinder of radius R and mass M slides across a rough horizontal surface with an initial linear velocity V_0. As it slides, it also has an initial angular velocity ω_0 as shown in the figure. (Note that positive ω_0 tends to produce rolling corresponding to motion in the direction opposite to V_0.) Let the coefficient of friction between the cylinder and the surface be μ.



(a) How long will it take till the sliding stops? [16 points]
(b) What is the velocity of the center of mass of the cylinder at the time the sliding stops? [16 points]
(c) How do the results in (a) and (b) change for ω_0 in the opposite direction from that shown in the figure? [8 points]

Problem 2. A block of mass M is constrained to move without friction along a horizontal line (the x axis in the figure). A simple pendulum of length L and mass m hangs from the center of the block. The pendulum moves in the xy plane.
(a) [20 points] Find the Lagrangian and the equations of motion for the system.
(b) [20 points] Find the normal modes and normal frequencies of the system, assuming that the pendulum always makes a small angle with the vertical.



Problem 3. A uniform circular thin membrane with radius R and mass/area μ is attached to a rigid support along its circumference, like a drumhead. Points on the membrane in the equilibrium position are labeled by polar coordinates (r, θ). The membrane has constant tension C, so that under a small displacement $f(r, \theta, t)$ from equilibrium, each area element da contributes potential energy $\frac{1}{2}C(\vec{\nabla} f)^2 da$.

(a) Find a wave equation satisfied by small time-dependent displacements $f(r, \theta, t)$ of the membrane from the equilibrium position. [16 points]

(b) Show that solutions can be found of the form $f = R(r)S(\theta)T(t)$, where:

$$\begin{aligned} \frac{d^2 S}{d\theta^2} + k^2 S &= 0, \\ \frac{d^2 T}{dt^2} + n^2 T &= 0, \end{aligned}$$

where k and n are constants, and $R(r)$ satisfies a second-order ordinary differential equation that you will determine. [16 points]

(c) Briefly describe how you would determine the vibrational frequencies of the membrane. [8 points]

Problem 4. A particle of mass m moves subject to a central potential:

$$V(r) = -\frac{k}{r^n}$$

where k and n are positive constants.

(a) If the particle has angular momentum L, what is the radius R for which the orbit is circular? [10 points]

(b) Suppose the motion is close to the circular orbit mentioned in part (a). Writing $r(t) = R + \delta r(t)$ and assuming that $\delta r(t)$ is small, find an equation for $\delta r(t)$. [15 points]

(c) Solve this equation for δr and discuss the stability of circular orbits, for different values of n. [15 points]