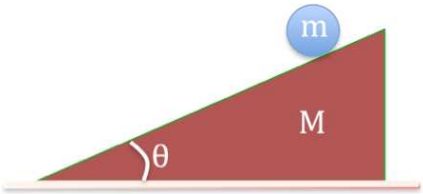
NIU Physics PhD Candidacy Exam – Fall 2010 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

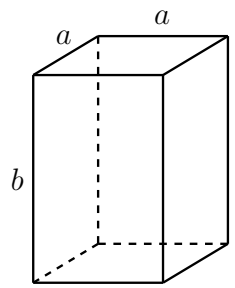
Problem 1. Consider a solid cylinder of mass m and radius r sliding without rolling down the smooth inclined face of a wedge of mass M and angle θ, as shown. The wedge is free to move on a horizontal plane without friction.



(a) [16 points] How far has the wedge moved by the time the cylinder has descended from rest a vertical distance h?
(b) [16 points] Now suppose that the cylinder is free to roll down the wedge without slipping. How far does the wedge move in this case?
(c) [8 points] In which case does the cylinder reach the bottom faster? How does this depend on the radius of the cylinder?

Problem 2. Consider a binary system consisting of two small stars with comparable but unequal masses m_1 and m_2. The stars attract each other according to Newtonian gravity, and orbit each other at a fixed distance, with period T.
(a) [15 points] Find the distance between the stars.
(b) [25 points] Now suppose the motion of both stars is suddenly stopped at a given instant of time, and they are then released and allowed to fall into each other. Find the time that it takes for the stars to collide. (You may leave your answer in terms of a single definite integral over a real dimensionless variable.)

Problem 3. Consider a solid rectangular prism with mass M uniformly distributed, and sides a, a, b, as shown.



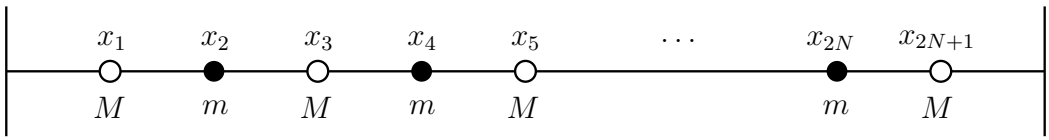
Suppose that the object is suspended in a uniform gravitational field g from one of its edges of length a, with that edge kept fixed and horizontal, and is free to swing as a pendulum.

(a) [20 points] Find a Lagrangian (with one degree of freedom) describing the motion.

(b) [10 points] Suppose that the pendulum starts from rest with its square side horizontal. Find the maximum speed reached by any part of the pendulum.

(c) [10 points] Find the angular frequency of small oscillations.

Problem 4. A mass-spring chain is formed by alternately connecting two kinds of masses, m and M, by identical springs with spring constant k, as shown below. Both ends of this mass-spring chain are fixed, and the springs are unstretched in equilibrium. There is no gravity. Assume that this system vibrates only longitudinally along the length of the chain. The masses are numbered from one end to the other, and the displacement of the nth mass from its equilibrium position is x_n.



(a) [5 points] Find the equations of motion for the x_n describing the system, for $n = 1, 2, \ldots, 2N+1$, and write the boundary conditions for x_0 and x_{2N+2}.

(b) [20 points] Assume that x_n may be expressed as:

$$\begin{aligned} x_{2j} &= a_j e^{i\omega t} \qquad (j = 0, 1, \ldots, N+1) \\ \text{and} \\ x_{2j+1} &= b_j e^{i\omega t} \qquad (j = 0, 1, \ldots, N) \end{aligned}$$

where the a_j and b_j are time-independent. Express b_j in terms of a_j and a_{j+1}, using the equations of motion obtained above. Also, what are a_0 and a_{N+1}?

(c) [15 points] Using the results of part (b), find a_j in terms of a_{j-1} and a_{j+1}. Now, assume that a_j can be expressed as $a_j = A\sin(j\alpha + \phi)$, where A, α and ϕ are constants independent of j. Find the angular frequency ω. Also, determine the possible values of α and ϕ.

Hint:

$$\begin{aligned} \cos(x+y) - \cos(x-y) &= -2\sin x \, \sin y \\ \sin(x+y) + \sin(x-y) &= 2\sin x \, \cos y \\ \sin(x+y) - \sin(x-y) &= 2\cos x \, \sin y \\ \cos(x+y) + \cos(x-y) &= 2\cos x \, \cos y \end{aligned}$$

NIU Physics PhD Candidacy Exam – Spring 2010 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. **Rocket Motion.** A rocket of initial total mass $m(0)$ at time $t = 0$ moves by ejecting gas at a variable rate $-dm/dt = \alpha(t)$. The gas is ejected at a constant velocity u with respect to the rocket. The rocket moves vertically in a constant uniform gravitational field g starting from rest on the ground at $t = 0$.

(a) Find the velocity of the rocket at time t, in terms of u, g, $m(0)$ and $\alpha(t)$. (Your answer may involve an integral.) [25 points]

(b) Find a condition on $\alpha(t)$ that is necessary and sufficient to guarantee that the rocket leaves the ground. [5 points]

(c) Now suppose that α is a constant. Find a set of specific algebraic equations whose solution determines the maximum height reached by the rocket. You do not need to solve the equations. [10 points]

Problem 2. **Spherical Pendulum.** A particle of mass m is constrained to move on the surface of a sphere of radius R in the presence of a uniform gravitational field g.

(a) Find the Lagrangian and the equations of motion in terms of spherical coordinates, with one of your coordinates being θ, the polar angle measured from the bottom of the sphere. [15 points]

(b) Reduce the problem to an effective 1-dimensional problem in terms of θ, for fixed angular momentum L_z, with energy $E = \frac{1}{2}mR^2\dot{\theta}^2 + V_{\text{eff}}$, and find V_{eff}. Sketch the corresponding effective potential V_{eff}, and use it to discuss the qualitative features of the motion. [10 points]

(c) As a special case, find the period for a horizontal circular orbit with the mass at a fixed polar angle θ_0. [15 points]

Problem 3. **Scattering by a central potential.** Consider a particle of mass m scattering from a central potential $V = -k/r^n$, where k and n are positive constants. The particle approaches from very far away with a non-zero impact parameter b and initial velocity v_0.

(a) Show that for the particle to have a chance at hitting the origin ($r = 0$), it is necessary that n is greater than or equal to a certain number that you will determine. [15 points]

(b) Now taking $n = 4$, show that a necessary and sufficient condition for the particle to hit the origin is that $b < b_{\text{crit}}$, where b_{crit} is a quantity that you will determine in terms of k, v_0, and m. [20 points]

(c) Still taking $n = 4$, what is the cross section for particles to hit the origin? [5 points]

Problem 4. **Transverse motion of a string.**

A flexible, elastic string is initially stretched along the z axis with tension T and mass per unit length μ. The string has length L and is fixed at both ends to solid walls.

(a) Derive the second-order differential equation for small transverse displacements $\psi(z,t)$ of the string in a fixed plane containing the z axis. Assume that each point on the string moves strictly at right angles to the z axis and that the tension at all points in the string remains close to its equilibrium value T. [10 points]

(b) What is the general solution for $\psi(z,t)$? Express it in the form of separation of variables with sinusoidal functions of z and of t. [10 points]

(c) Now suppose that the middle point of the string is pulled a small amount a perpendicular to the z-axis and held until released at $t = 0$. What are the initial conditions for ψ? [10 points]

(d) Apply the initial conditions of part (c) to the general solution of part (b) to obtain the complete solution $\psi(z,t)$. [10 points]

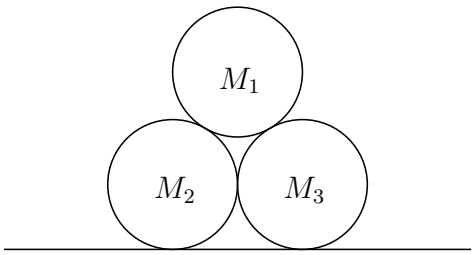
Possibly useful integrals:

$$\begin{aligned}
\int x\sin(x)\,dx &= \sin(x) - x\cos(x) \\
\int x\cos(x)\,dx &= \cos(x) + x\sin(x) \\
\int x\sin^2(x)\,dx &= \frac{x^2}{4} - \frac{\cos(2x)}{8} - \frac{x\sin(2x)}{4} \\
\int x\cos^2(x)\,dx &= \frac{x^2}{4} + \frac{\cos(2x)}{8} + \frac{x\sin(2x)}{4} \\
\int x\sin(x)\cos(x)\,dx &= \frac{\sin(2x)}{8} - \frac{x\cos(2x)}{4}
\end{aligned}$$

NIU Physics PhD Candidacy Exam – Fall 2009 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. Three rough cylinders of equal radius are stacked as shown on a rough level surface. They have unequal masses M_1, M_2, M_3, but the coefficient of static friction is everywhere equal to μ. How large must μ be so that the arrangement is stable? Assume that cylinders 2 and 3 do not quite touch.)



Problem 2. Suppose that the gravitational interaction between two objects with masses M and m is modified by replacing the potential term in the Lagrangian according to:

$$\frac{GMm}{r} \rightarrow \frac{GMm}{r}\left(1 + \frac{\dot{r}^2}{C^2}\right)$$

where $r = |\vec{r}|$ with $\vec{r}$ the relative position vector, a dot means a time derivative, and $C =$ a new constant. In the following, assume that $m \ll M$ and treat the heavier object as stationary.

(a) Find the equations of motion for orbital motion in appropriate variables. [12 points]

(b) Does the radial vector of the small orbiting object sweep out equal areas in equal times? (Explain briefly.) [4 points]

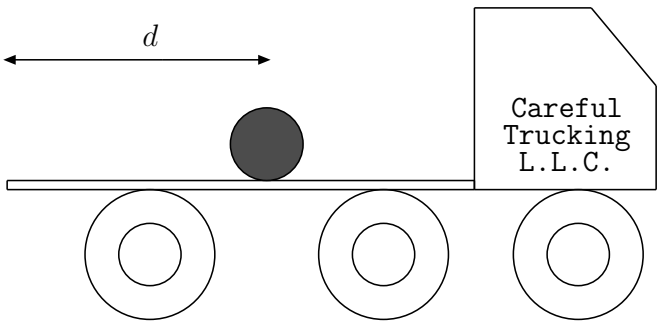
(c) Find the angular frequency of small oscillations about a circular orbit of radius R. Write your answer in terms of only G, M, m, R, and C. (Do not assume that C is small.) [12 points]

(d) Let M_E and R_E be the mass and the radius of the Earth (assumed to be a non-rotating perfect sphere). Compute the escape velocity for an object initially on the surface of the Earth, as a function of G, M_E, R_E, C, and $\alpha =$ the angle between the initial velocity at the Earth's surface and the vertical. [12 points]

Problem 3. A very heavy flat-bed truck starts at rest on a level surface. It has a ball which is a solid sphere of radius R and mass M (with uniform mass density) on the bed as shown, with its center a distance d from the back end. When the truck starts moving, the truck accelerates uniformly with acceleration a_T, and the ball rolls without slipping on the truck bed. (Note: moment of inertia of a uniform sphere about the center is $I = \frac{2}{5}MR^2$.)

(a) How long does it take for the ball to roll off of the truck? [30 points]

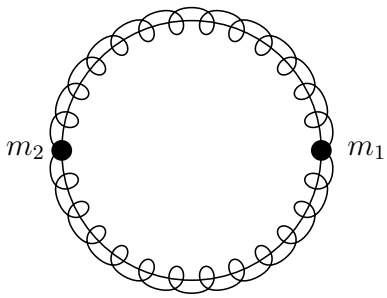
(b) What density distribution in the ball (with same total mass M and radius R) would minimize the time to fall off? What is this minimum time? [10 points]



Problem 4. Two beads, with unequal masses m_1 and m_2, are constrained to slide frictionlessly on a stationary hoop of radius R. The beads are connected as shown by two identical springs with spring constant k and unstretched length d. There is no gravity.

(a) Find the normal modes of the system and their angular frequencies. [25 points]

(b) Now suppose that at time $t = 0$ the beads are on opposite sides of the hoop, with bead m_1 having an instantaneous speed v and bead m_2 at rest. Solve for the subsequent motion of the first bead. [15 points]

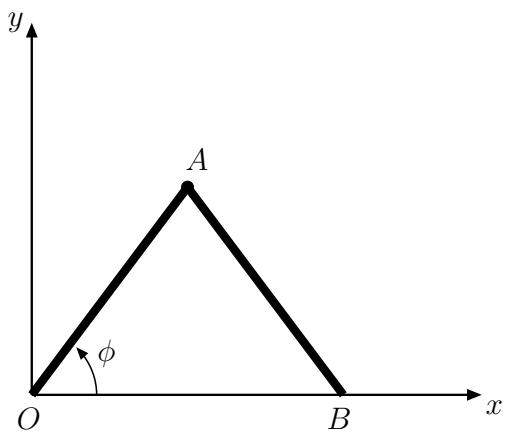


NIU Physics PhD Candidacy Exam – Spring 2009 – Classical Mechanics

Do ONLY THREE out of four problems. Total points on each problem = 40.

Problem 1 A uniform chain of length L and total mass M contains many links. It is held vertically by one end over a table with the other end just touching the table top.
(a) The chain is released and falls freely. What is the speed of the falling section of the chain at time t after release? What is the force between the links? [10 points]
(b) Work out the increment of mass dm that hits the table in an increment of time dt. Find the corresponding change in momentum and hence the instantaneous impulsive force on the table. [15 points]
(c) What is the total normal force acting on the table as a function of time? Show that the maximum value of the total force is kMg, where k is a constant. Determine the value of k. [15 points]

Problem 2 Two identical thin uniform rods, each of length d and mass m, are hinged together at the point A. The rod on the left has one end hinged at the fixed point O, while the end B of the other rod slides freely along the horizontal x axis. The system is in a uniform vertical gravitational field with acceleration g. All motion is frictionless.



(a) Find the total kinetic energy of the system. You should find the result:

$$T = md^2\dot{\phi}^2(a + b\sin\phi + c\sin^2\phi)$$

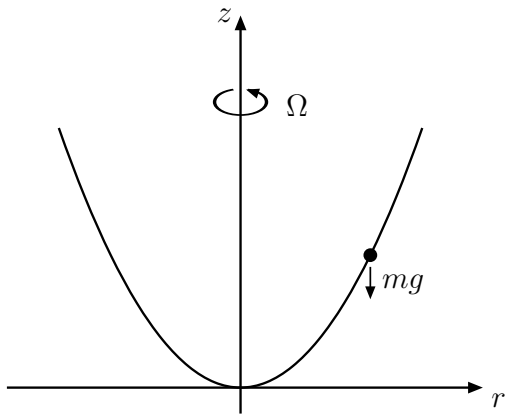
where a, b and c are constant numbers that you will determine. (Exactly one of a, b, and c is 0.) [25 points]
(b) Suppose the system is at rest at time $t = 0$ with $\phi = \phi_0$. What is the velocity of the hinge A when it hits the horizontal x axis? [15 points]

Problem 3 A small spacecraft with mass m and energy $E > 0$ approaches a star with mass M and radius R from far away.

(a) Find an expression for the effective total cross-section for the spacecraft to hit the star. [30 points]

(b) Sketch a graph of the cross-section as a function of E, and give simple physical explanations for the low and high energy limits. [10 points]

Problem 4 A wire in the shape of a parabola $z = \dfrac{r^2}{2a}$ rotates about the vertical z axis with constant angular frequency Ω. Here r is the distance from the z axis, and a is a constant. A constant gravitational acceleration g is directed in the negative z direction. A small bead of mass m slides on the wire without friction.



(a) Find the Lagrangian and obtain Lagrange's equations of motion for the bead, using r as the coordinate. [14 points]

(b) There exists a solution of Lagrange's equations for which $\dot{r}$ = constant. What is this constant? From where does the energy come to permit such unbounded motion? [8 points]

(c) Obtain the equations of motion for small deviations of the bead from rest at the bottom of the parabola. Give a condition for stable oscillation about this position. [8 points]

(d) Find the canonical momentum, the Hamiltonian, and Hamilton's equations of motion for the bead. Is the Hamiltonian conserved? [10 points]

NIU Physics PhD Candidacy Exam – Fall 2008 – Classical Mechanics

Do ONLY THREE out of four problems. Total points on each problem = 40.

<u>Problem 1</u> A spacecraft is placed in a nearly circular orbit about the Sun in the *opposite* direction but at nearly the same radius R (and in the same plane) as an orbiting planet. Assume that $M_{\text{spacecraft}} \ll M_{\text{planet}} \ll M_{\text{Sun}}$, where the masses of the spacecraft, planet and sun are $M_{\text{spacecraft}}$, M_{planet} and M_{Sun}. You should also assume that all motions are non-relativistic.

(a) What is the velocity of the spacecraft relative to the Sun when it is very far away from the planet? [7 points]

(b) What is the escape velocity of the spacecraft from the solar system? [7 points]

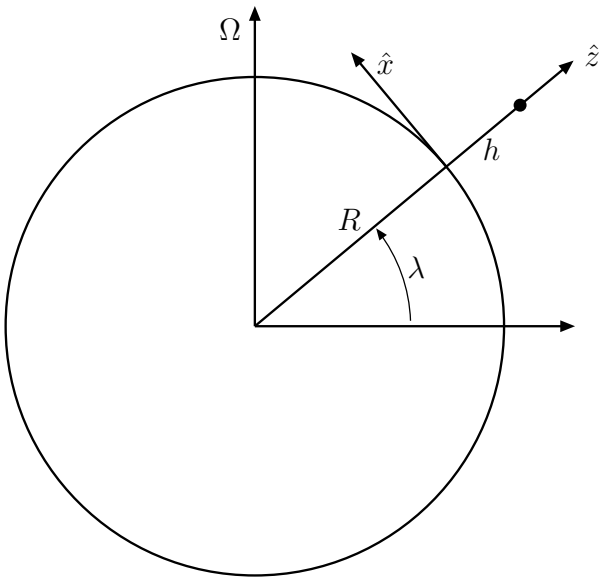
(c) Because the spacecraft and planet are nearly the same distance from the Sun and in the same plane, they will eventually have a close encounter. Suppose that the spacecraft encounters the planet (without colliding) and scatters at an angle θ in the rest frame of the planet. How large must θ be in order for the spacecraft to escape the solar system?
[10 points]

(d) Use conservation of energy and angular momentum to find the total cross-section (with units of area) for the spacecraft to collide with the planet during a close encounter, if the planet's radius is a. (Your answer should depend on other given quantities besides a.)
[16 points]

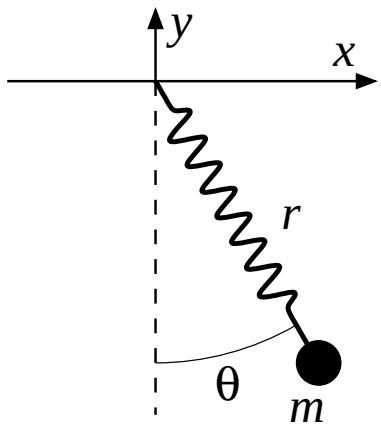
Problem 2 A particle is dropped from a tower attached to the Earth at latitude λ from a height h above the surface of the Earth. The Earth is rotating at constant angular velocity Ω counterclockwise looking down from the North Pole (the vertical line in the figure). The motion is to be described in a non-inertial set of coordinates fixed with respect to the Earth's surface, in which the z axis is an extension of the radius from the center, and the x axis points along a meridian. (The y axis is straight out of the page in the figure.) Assume the Earth is spherical and that h is much smaller than the radius R.

(a) Find differential equations describing the motion of the particle, to lowest non-vanishing order in the angular velocity Ω. [20 points]

(b) Compute the trajectory of the particle, again to lowest non-vanishing order in the angular velocity Ω. What is the impact point where the particle hits the Earth? [20 points]



Problem 3 A spring pendulum consists of a mass m attached to one end of a massless spring with spring constant k. The other end of the spring is tied to a fixed support. When no weight is on the spring, its length is l. Assume that the motion of the system is confined to a vertical plane in the gravitational field of the Earth.



(a) Give the Lagrangian for this system. [14 points]

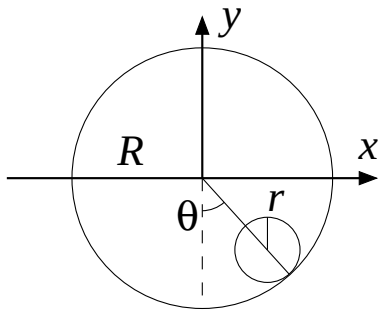
(b) Derive the equations of motion. [13 points]

(c) Find the general solution to the equations of motion in the approximation of small angular and radial displacements from equilibrium. [13 points]

Problem 4 A solid homogeneous cylinder of radius r and mass m rolls without slipping on the inside of a stationary larger cylinder of radius R in the gravitational field of the Earth as shown in the figure.

(a) How is the center-of-mass motion of the inner cylinder related with its rotational motion? [10 points]

(b) If the small cylinder starts at rest from an angle θ_0 from the vertical, what is the total downward force it exerts on the outer cylinder as it passes through the lowest point?
[10 points]



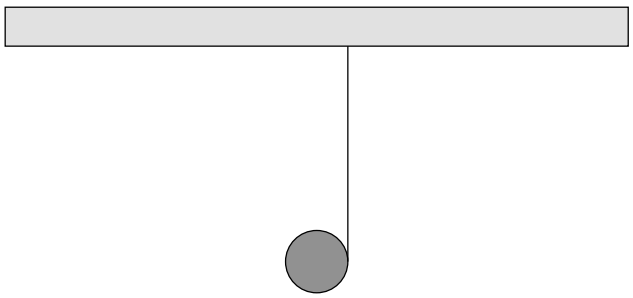
(c) Determine the equation of motion of the inside cylinder using Lagrangian techniques. [10 points]

(d) Find the period of small oscillations about the stable equilibrium position. [10 points]

NIU Physics PhD Candidacy Exam – Spring 2008 – Classical Mechanics

Do ONLY THREE out of four problems. Total points on each problem = 40.

Problem 1 A uniform cylinder of mass M and radius R is rolled up on an unstretchable massless string which is attached to the ceiling. The cylinder is released and unrolls the string as it falls (like a yo-yo).



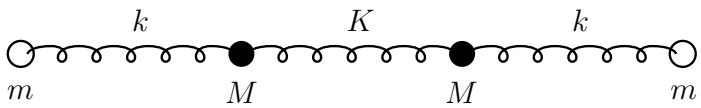
Considering only vertical motion of the center of the cylinder, and assuming that the cylinder never reaches the end of the string:

(a) Specify the degrees of freedom and state any equations of constraint. [5 points]

(b) Find the Lagrangian, and use it to solve for the motion as a function of time. [25 points]

(c) Find the tension in the string as a function of time. [10 points]

Problem 2 The extremely rare compound di-Cryptonite dioxide is composed of molecules that can be pictured as four particles (with two different masses m and M) attached by springs (with two different spring constants k and K) as shown in the diagram:



Find all of the normal mode frequencies of the molecule under the assumption that motion is only along the line joining the particles. For each normal mode, indicate by a set of four arrows the directions of motion of the particles at one instant in time.

Problem 3 Suppose the force of attraction between a point object of mass M and a planet of mass m (where $m \ll M$) is:

$$F = \frac{a}{r^2} + \frac{3b\ell^2}{r^4}$$

where ℓ is the angular momentum of the planet and a, b are both positive constants. [Note: this does mimic the force of attraction between a planet and a black hole, in the non-relativistic limit, with $a = GMm$.]

(a) Under what conditions is a stable circular orbit possible? Give the radius in terms of the given parameters. [15 points]
(b) What is the smallest radius possible for any circular orbit as a function of a and b, allowing for arbitrary ℓ? (Hint: this occurs in the limit of very large ℓ.) Is this circular orbit stable or unstable? [15 points]
(c) Find an expression for the angular frequency of small radial oscillations for the planet if it travels in a slightly non-circular orbit about the stable radius. [10 points]

Problem 4 A particle of unit mass is constrained to move under gravity (acting in the negative z direction with constant acceleration g) on a smooth surface with height:

$$z = \frac{x^2}{2a} + \frac{y^2}{2b},$$

where a and b are positive constants. The surface is forced to rotate with constant angular velocity Ω about the z axis.

(a) Find the equations describing **small amplitude** oscillations near the bottom of the surface, and show that they have the form:

$$\ddot{x} - c_1\dot{y} + c_2 x = 0,$$
$$\ddot{y} + c_1\dot{x} + c_3 y = 0,$$

where c_1, c_2, and c_3 are distinct non-zero constants that you will find in terms of Ω, g, a, and b. [25 points]

(b) Find the frequencies of oscillatory solutions for these equations. [15 points]

NIU Physics PhD Candidacy Exam – September 2007 – Classical Mechanics

Do ONLY THREE out of four problems. Total points on each problem = 40.

Problem 1 A non-relativistic particle of mass m moves in one dimension x under the force

$$F(x) = ax^3 - bx,$$

where a and b are positive constants.

(a) Does a potential energy function exist for this force? Why? Is the total energy conserved? [6 points]
(b) Find an expression for the potential energy $V(x)$, and sketch its graph. Choose the arbitrary constant so that the potential energy vanishes at $x = 0$. [6 points]
(c) Find all of the turning points of the motion if the total energy is E. [8 points]
(d) For what values of the total energy E is the motion bounded? [8 points]
(e) For positive values of E less than that found in part (d), find the frequency for periodic oscillations. Do not assume that the oscillations are small. You may leave your answer in terms of a definite integral. [12 points]

Problem 2 Consider a pendulum hanging from the top of a car. The pendulum consists of a mass m hanging from a massless rod of length ℓ. Initially, the car is at rest and the pendulum is in its equilibrium position (aligned with the vertical). Now assume the car suddenly accelerates with constant horizontal acceleration A_0.
(a) Find the new equilibrium positions's angle with respect to the vertical. [10 points]
(b) Find the differential equation governing the time dependence of the angle ϕ from the vertical. [10 points]
(c) What is the maximum angle through which the pendulum swings with respect to the vertical? (HINT: this is NOT the same as part (a).) [20 points]

Problem 3 The minimum distance of a comet from the sun is observed to be exactly half the radius of the earth's orbit, and its speed at that point is twice the orbital speed of the earth. Ignore the effects of the earth and other planets on the comet, assume the mass of the comet is negligible compared to the mass of the sun M, and assume the earth's orbit is circular with radius R and in the same plane as the comet's orbit.

(a) Find the velocity of the comet when it crosses the earth's orbit and the angle at which the orbits cross. [16 points]
(b) Will the comet subsequently escape the solar system? Why? (No credit without valid reasoning.) [8 points]
(c) Calculate how long the comet remains inside the earth's orbit. Give the answer in years. [16 points]

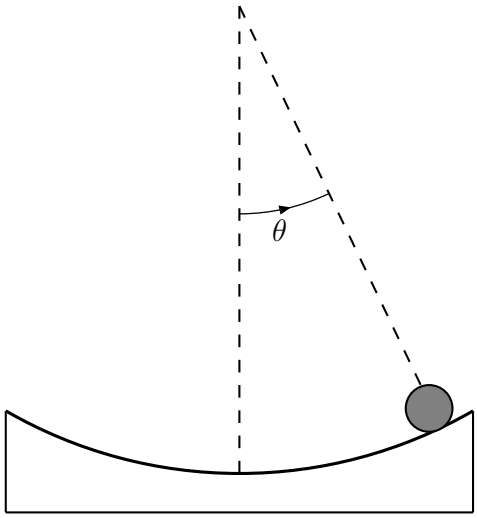
Problem 4 A thin, flat square of mass M and side a has a uniform mass density.
(a) Find the moments of inertia and write down the inertia tensor for a coordinate system with origin at the center of the square and x, y axes parallel to the sides of the square and z axis perpendicular to the square. [10 points]
(b) Suppose that the square is suspended frictionlessly by one fixed corner and allowed to oscillate as a pendulum in a constant gravitational field (with acceleration due to gravity g). The motion takes place within the same plane as the square. Find an equation of motion for the system, and use it to find the angular frequency for small oscillations. [15 points]

(c) Repeat part (b), but now assuming the oscillating motion is along a direction normal to the plane of the square. [15 points]

NIU Physics PhD Candidacy Exam – February 2007 – Classical Mechanics

Do ONLY THREE out of four problems

<u>Problem 1</u> A ball of mass m and radius R (moment of inertia $I = \frac{2}{5}mR^2$ about its center) is released from rest at an angle θ_0 in a stationary spherical bowl of radius of curvature $d + R$. The ball rolls on the bowl without slipping (in the plane of the diagram).



(a) [25 points] Derive an equation of motion for the angle θ through the center of the ball, as defined in the figure. Solve this equation of motion for the case of $\theta_0 \ll 1$.

(b) [15 points] What is the velocity of the center of the ball when $\theta = 0$? (Do not assume $\theta_0 \ll 1$ here.)

Problem 2 An isotropic oscillator potential $V(r) = \frac{1}{2}kr^2$ leads, like the gravitational $-1/r$ potential, to closed planar orbits. Suppose, however, that the "spring" has a non-zero relaxed length a, so that:

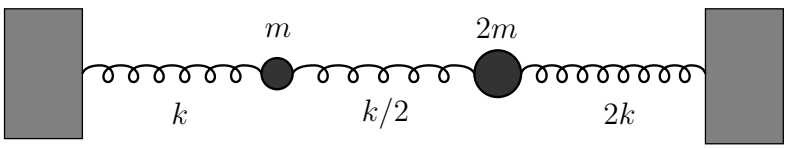
$$V(r) = \frac{1}{2}k(r-a)^2$$

and consider the (planar) motions of a particle of mass m and initial angular momentum L in this potential.

(a) [20 points] Find a condition relating the radius r_0 of circular orbits and the given constants. (It is not necessary to solve for r_0.)

(b) [20 points] For nearly circular orbits, find the angular change between successive radial maxima. For $a/r_0 \ll 1$, express your answer to first order in a/r_0 (eliminate L from the answer). Check that the orbits are indeed closed for $a/r_0 \to 0$

Problem 3 [40 points] Consider a system of masses m and $2m$ and springs with spring constants k, $k/2$, and $2k$, attached between two stationary walls as shown. For motions along the axis of the springs, describe the normal modes of vibration and find their angular frequencies.



<u>Problem 4</u> A thin, flexible, unstretchable loop of string has mass per length μ. It is set spinning with angular frequency Ω in a gravitationless vacuum in such a way that it forms a stable circle of radius R.

(a) [20 points] Find the tension in the string.

(b) [20 points] Now suppose the string is given a small tap so that infinitesimal vibrations travel along the loop in the plane of the string in both directions. Compute the speeds of those waves relative to a non-rotating coordinate system.