NIU Physics PhD Qualifying Exam – September 2006 – Classical Mechanics

Do ONLY THREE out of four problems

Problem 1 A small planet of mass m revolves around a star of mass M in a nearly circular orbit with slightly varying separation, r. You may neglect m in comparison to M. Centered on the star is a spherical dust cloud of uniform density ρ, with radius large enough to contain the orbit of the planet. (The planet does not lose energy due to collisions with the dust cloud.) Let $G =$ the gravitational constant.

(a) [14 points] If the orbit were exactly circular with $r = R$, what would be the angular frequency $\omega \equiv d\theta/dt$ and the angular momentum L? (Give your answers in terms of m, M, R, ρ, and G.)

(b) [14 points] The equation of motion for r is the same as that of an equivalent one-dimensional problem. For that problem, find the effective potential. (Give your answer in terms of m, M, ρ, G, the angular momentum L, and r.)

(c) [12 points] Consider the case of small ρ, and small deviations from the circular orbit, $r(t) = R + \epsilon(t)$. Find the angular velocity of the precession of the perihelion, $\omega_p = \omega_\epsilon - \omega$, and show that it can be written as

$$a(GR^3/M)^c\rho,$$

where a and c are non-zero quantities that you will determine.

Problem 2 [40 points] A particle of mass m slides from the very top of a sphere of radius R, with initial horizontal velocity v, under the influence of a uniform gravitational field. The coefficient of friction between the particle and the sphere is μ. (This means that the force of friction opposing the motion is μN, where N is the force normal to the surface of the sphere.) The constant acceleration of gravity downward is g. What is the minimum value of v for which the particle will fall off of the sphere?

Problem 3 A projectile is launched from the ground at an angle of 45 degrees and with an initial kinetic energy E_0. At the top of its trajectory, the projectile explodes into two fragments with masses m_1 and m_2. The explosion imparts an additional mechanical energy E_0 to the system. The fragment of mass m_1 is then observed to travel straight down. Assume the motion is in the xy plane, with x horizontal and y vertical, and the acceleration of gravity downward is g.

(a) [20 points] Find both of the components v_{2x} and v_{2y} of the velocity vector of the mass m_2, and the magnitude v_1 of the velocity of the mass m_1, immediately after the explosion. What is the maximum possible value of m_1/m_2?

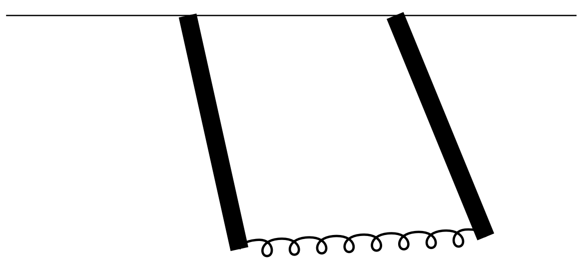
(b) [20 points] Find an expression for the horizontal range of m_2, measured from the initial launch position, in terms of the given quantities.

Problem 4 Two rigid thin rods of uniform mass density each have mass M and length L. They are pivoted freely at their top ends, and joined by a spring at their bottom ends, as shown in the figure below. The rods are constrained to move in the plane of the page. The spring has spring constant k and length b when unstretched, and the distance between the pivot points is also b. The acceleration due to gravity is g.

(a) [16 points] Find the Lagrangian for this system.

(b) [12 points] Find the equations of motion for *small oscillations* of this system.

(c) [12 points] Sketch the motions corresponding to the normal modes. Find the corresponding normal frequencies for small oscillations.



NIU Physics PhD Qualifying Exam – February 2006 – Classical Mechanics

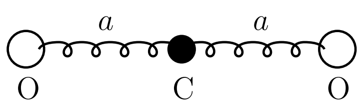
Do any THREE out of four problems

Problem 1 A spherical star of a definite radius is constructed from an incompressible fluid of constant mass density ρ. The star is held together by its own gravitational attraction. The total mass of the star is M. Find the pressure $P(r)$ within the star as a function of the distance from the center, assuming that the star is not rotating.

Problem 2 A particle of mass m is confined to move on the frictionless surface of a right circular cone whose axis is vertical, with a half opening angle α. The vertex of the cone is at the origin and the axis of symmetry is the z axis. For a given non-zero angular momentum L about the z-axis, find:
(a) the height z_0 at which one can have a uniform circular motion in a horizontal plane.
(b) the frequency of small oscillations about the solution found in part (a).
Give your answer in terms of only m, α, L, and the acceleration due to gravity g.

Problem 3: A thin uniform disk of radius a and mass m is rotating with a constant angular velocity ω about a fixed axis passing through its center but inclined at an angle α with respect to the axis of symmetry of the disk.
(a) Find the magnitude of the angular momentum vector about the center of the disk. [Hint: the moments of inertia of the disk about its principal axes satisfy $I_1 = I_2 = \frac{1}{2}I_3$.]
(b) Find the torque that is exerted about the center of the disk to make this happen. Give your answers in terms of a, m, ω, α only.

Problem 4: A CO_2 gas molecule is linear, as shown in the figure below, with its long molecular axis in parallel with the x-axis. The equilibrium distance between the carbon atom and each of the two oxygen atoms is a. The spring constant of each CO bond is k. The mass of an oxygen atom is M, and the mass of a carbon atom is m. Using the Lagrangian method, find all of the normal modes of the molecule that have motion only along the x-axis, and describe their motions. Ignore the sizes of the atoms.



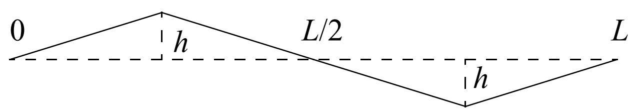
Ph.D. Qualifying Exam - Classical Mechanics Sep. 2005
Complete 3 out of 4

1. Consider a projectile fired from the origin with a velocity $\overrightarrow{v_0} = \hat{i}v_{0x} + \hat{k}v_{0z}$ where z is the vertical coordinate. There is a wind with velocity $\overrightarrow{v_w} = \hat{j}v_w$ and air resistance is proportional to velocity $\overrightarrow{F_d} = -b\vec{v}$.
   (a) Find the position as a function of time for all three coordinates.
   (b) Find the displacement in the x-position due to air resistance at z=0 keeping only the first order terms in b.
   (c) Find the displacement in the y-position due to the wind at z=0 keeping only the first order terms in b.

2. Suppose at a latitude $\lambda = 20^{\circ}$N the atmospheric pressure is $P = 10^5$ N/m^2 and the air density is $\rho = 1.3$ kg/m^3.
   (a) Determine an expression for the velocity v as a function of the pressure gradient and the radius r.
   (b) Use the result in part (a) to find the wind speed for a low pressure region with a pressure gradient of 3 millibar/m at 100 km from the center of low pressure.

3. Consider the motion of a particle of mass m moving on the outer surface of a hoop of radius R. The particle is subject to the force of gravity on the Earth's surface mg. Use polar coordinates (r, θ) as generalized coordinates to describe the motion of the particle.
   (a) Find the Lagrangian for the particle in terms of the generalized coordinates.
   (b) Use Lagrange's equations to find an expression for the force of the constraint.
   (c) Determine the angle at which the particle leaves the surface of the hoop.

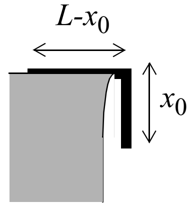
4. A uniform string of length L and linear mass density μ under tension T is displaced initially at rest as shown below.



   (a) Write expressions for the initial conditions of the string.
   (b) Find a general solution of the vibrating string as a Fourier series.
   (c) Use the initial conditions to find the coefficients of the Fourier series.

Ph.D. Qualifying Exam - Classical Mechanics Feb. 2005
Complete 3 out of 4

1. A flexible but unstretchable rope has a length L and has a mass-per-unit-length λ. It is held at rest on a smooth frictionless horizontal surface with a length x_0 hanging vertically over the edge. The rope is released at time t_0 and slides off the surface.

   (a) Find the speed of the rope at any time t after it is released, but before it leaves the surface.

   (b) Find the time at which the rope leaves the surface.

2. A comet is observed at a distance of 10^8 km from the center of the sun. At that point the comet is traveling with a total velocity of 56.6 km/s with equal components v_x and v_y where x is the direction toward the sun.

   (a) Find the angular momentum per unit mass of the comet, and the total energy per unit mass of the comet ($GM_{sun} = 1.33 \times 10^{11}$ km^3/s^2).

   (b) Find the eccentricity of the orbit of the comet, and identify if the orbit is open or closed.

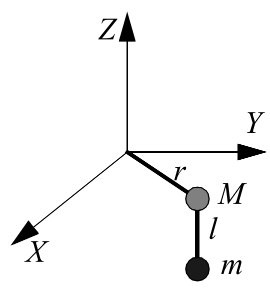
   (c) Find the distance of closest approach between the comet and the center of the sun.

3. Consider the motion of a particle of mass m moving in a plane. The particle is subject to a force $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$. Use cylindrical coordinates (ρ, θ, z) as generalized coordinates to describe the motion of the particle.

   (a) Find the changes in the Cartesian coordinates δx, δy, δz in terms of the generalized coordinates.

   (b) Find the generalized forces Q_ρ, Q_θ, and Q_z associated with the generalized coordinates.

4. A light rod of length r is fixed at the origin, and a mass M is attached to the other end, as shown at right. The rod is constrained to move in the XY-plane. A pendulum of length l and mass m attached at A can oscillate in the YZ-plane. Use θ for the angle of the rod in the XY-plane, and ϕ for the angle of the pendulum in the YZ-plane.

   (a) Find Lagrange's equations for the system of the rod and pendulum in terms of θ and ϕ.

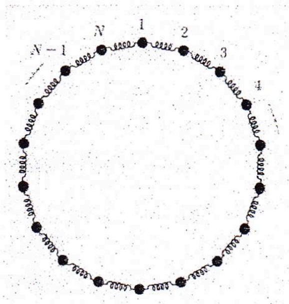
   (b) Find the normal frequencies and normal modes of vibration for small oscillations.

Ph.D. Qualifying Exam - Classical Mechanics Sep. 2004
Complete 3 out of 4

1. A mass-spring ring system as shown at right consists of N identical masses, m, and N springs with spring constant k.

   (a) Show the equation of motion of this system when the masses move along the circle of this mass-spring system.

   (b) Find the amplitude mode angular frequency of the mode in part (a). mode and



2. Consider two pendula of equal length b and equal mass $m_1 = m_2 = m$ connected by a spring of force constant k and both constrained to move in the same plane. The spring is unstretched when the system is in its static equilibrium configuration, and the pivot points are separated by a distance L.

   (a) Write down the Lagrangian for the system. Do not assume small-amplitude motion. How would you determine the equations of motion.

   (b) Now, in the limit of small-amplitude motion, the equations of motion simplify to:

$$\frac{d^2}{dt^2}(\theta_1) + \left(\frac{g}{b} + \frac{k}{m}\right)\theta_1 - \frac{k}{m}\theta_2 = 0$$

$$\frac{d^2}{dt^2}(\theta_2) + \left(\frac{g}{b} + \frac{k}{m}\right)\theta_2 - \frac{k}{m}\theta_1 = 0$$

   Using normal coordinates $\theta_1+\theta_2$ and $\theta_1-\theta_2$, derive the corresponding eigenfrequencies ω_1 and ω_2.

3. A fluid of density ρ and viscosity η flows at a constant rate between two plates separated by a distance L. The total fluid flow per unit length between the walls perpendicular to the direction of current is I. Assume that the pressure varies only in the direction of flow.

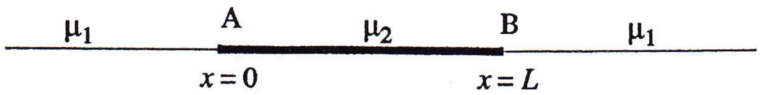
   (a) Find the component of velocity parallel to the walls as a function of the distance from the midpoint between the plates and the pressure gradient parallel to the walls.

   (b) Find a relationship between the pressure gradient and the fluid flow per unit length.

4. A cable is hung at equilibrium with the end points at (x_A, y_A) and (x_B, y_B). The cable is supporting a load that is uniformly distributed in the horizontal direction where w is the weight per unit length. The tension of the cable at the lowest point is T_0. Find an expression for the span $L = x_B - x_A$ as a function of w, T_0, y_A and y_B.

Ph.D. Qualifying Exam - Classical Mechanics Jan. 2004
Complete 3 out of 4

1. Consider an infinitely long string, as shown below. For $x < 0$ and $x > L$, the linear mass density of the string is μ_1, and for $0 < x < L$, the linear mass density is $\mu_2\,(> \mu_1)$. A wave of amplitude A_0 and frequency ω is incident from the left side. Find the reflected and transmitted intensities at A and B.



2. A door on frictionless hinges is hung at a slight angle θ with respect to the vertical. Calculate the moment of inertia for the door and use it to write the Lagrangian for the door. Find Lagrange's equation of motion for the door and use it to determine the period of small oscillations. If it takes 1 s for the door to close when slightly ajar, what is the angle of the door with respect to the vertical?

3. Consider two wells filled with water dug entirely to the center of the earth. One well is directly under the moon at the equator and the other well is at the equator but perpendicular to the first well. Assume that the water in the well is incompressible and that the pressures at the top of the two wells are equal and also the pressures at the bottom of both wells are equal. Write an expression for the pressure at the bottom of the well as a function of the weight of the water. Use this expression to determine the difference in the height of the water in the two wells. Newton used this method to estimate the tides.

4. A tippie top is a nearly spherical top of mass M, that will flip when spinning such that the heavy end is aligned upwards. Assume that the center of mass is only slightly below the center of curvature of the sphere, and that all three moments of inertia are equal to the moment of inertia for a sphere.

   (a) Consider the body-centered motion of the top with the z-axis along the spindle and draw the forces, torques, angular momentum and angular velocity on the top.

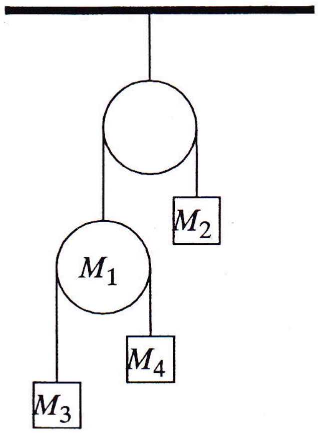
   (b) Write an expression for the force of friction on the table, and the angular equation of motion.

   (c) If θ is the angle between the angular momentum and the z-axis of the top, find an expression for $d\theta/dt$. If the top is 1 cm in radius, spun at 300 rad/s and the coefficient of friction is 0.1, find the time it takes to flip.

Ph.D. Qualifying Exam - Classical Mechanics Sep. 2003

Do 3 out of 4

1. A mass M_2 hangs at one end of a string which passes over a fixed frictionless, non-rotating pulley. At the other end of the string there is a frictionless non-rotating pulley of mass M_1 over which there is a string carrying massses M_3 and M_4.



(a) Set up the Lagrangian of the system.

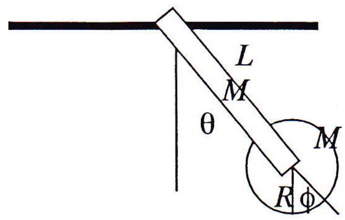
(b) Use Lagrange's equation to find the acceleration of mass M_2.

2. A rocket of initial mass m_0 moves by ejecting gas at a constant rate α. The gas is ejected at a constant velocity u with respect to the rocket. The rocket is moving vertically upward in a constant gravitational field of magnitude g.

(a) Write Newton's law of motion for the rocket.

(b) Find the velocity of the rocket as a function of time in terms of the constants given above.

3. A disk of mass M and radius R is attached to the end of a rod of mass M and length L as shown below. The rod is free to swing without friction from the opposite end that holds the disk, and the disk is free to rotate without friction at its center.



(a) Find the moments of inertia of the rod, disk, and combined system at the pivot of the rod.

(b) Find the Lagrangian of the system in terms of the two angles θ and ϕ.

(c) Find Lagrange's equations of motion, and compare the motion to that of a simple pendulum.

4. Consider an isotropic harmonic oscillator whose potential is given by $V(r) = 1/2\, kr^2$.

(a) Derive the effective potential $V_{eff}(L, r)$ for a particle of mass m, and make a plot of $V_{eff}(r)$ versus r.

(b) Find the values of the energy E and angular momentum L for a circular orbit and identify that point on the graph from part (a).

(c) Find the frequency of revolution for the circular orbit of part (b) and the frequency of small radial oscillations.

NIU Ph.D. qualifier examination 2003 Spring (1/25/2003)
Classical Mechanics

Solve 3 out of 4 problems.

I. A marble of mass m is sliding (not rolling) down the side of a hemispherical dish with a radius b. Find Hamilton's equation of motion for the marble.

II. The differential equation of motion describing the displacement from equilibrium for damped harmonic motion is:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

a. State conditions and describe the motion for overdamping, critical damping, and underdamping.

b. Show that the ratio of two successive maxima in the displacement x is constant.

III. Consider a point particle which moves in a central potential subject to a radial force $F_r = -\frac{a}{r^3} - \frac{b}{r^4}$, with a, b, both positive. At the origin there is a particle eater which swallows any particle that reaches $r = 0$. The point particles starts out at $r = R$ with angular momentum L and initial radial velocity $dr/dt = v_r$. Enumerate the conditions on R, v_r, and L that will result in the particle being eaten.

IV. A homogeneous cube of mass M and edge length L is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a tiny displacement and allowed to fall.

a. Find the angular velocity of the cube when one face strikes the plane, assuming the edge slides without friction.

b. Same question as (a), but now assuming the edge cannot slide because of friction.

c. For case (a), find the force exerted by the surface on the cube just before the face strikes the plane. (The moment of inertia of a cube about an axis through its center and parallel to an edge is $I = ML^2/6$.)