Ph. D. Qualifying Exam Jan. 2018

Modern and Statistical Physics

You can do all the problems—the best 3 count towards your total score

Problem 1: (40 points)

10 liters of an ideal gas at atmospheric pressure is compressed isothermally to a volume of 1 liter and then allowed to expand adiabatically to 10 liters.

(a) Sketch the process in a pV diagram for a monoatomic gas. Label the end points of your graphs quantitatively. [12 points]

(b) Make a similar sketch for a diatomic gas. [10 points]

(c) Is net work done on or by the system? [10 points]

(d) Is the net work greater or less for the diatomic gas? [8 points]

Problem 2: (40 points)

The radius of a neutron star of mass equal to a solar mass is 12.4 km. Find the radius of a neutron star of mass equal to two solar masses. Show all steps needed to derive the answer. Assume no interaction between the neutrons other than the gravitational force and treat the problem non-relativistically.

Hints:

(1) The gravitational energy of a sphere of mass M and radius R is

$$E_{grav} = -\frac{3}{5} G \frac{M^2}{R}$$

(2) The kinetic energy of N spin ½ fermions is

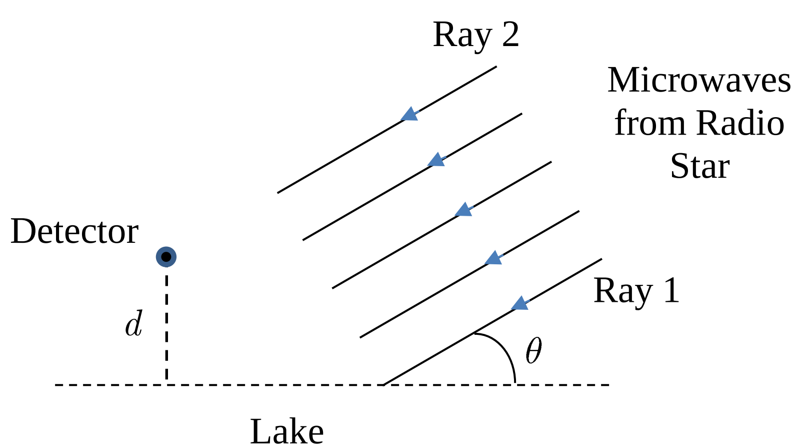
$$E_{kin} = \frac{3}{5} N \varepsilon_F$$

where ε_F is the Fermi energy.

Problem 3: (40 points)
Two particles, each of mass m, collide. The result of the collision is two other particles, each of mass M. In the center-of-momentum frame, each of the two initial-state particles have a very high (relativistic) energy E, and the final-state particles are produced at an angle θ with respect to the collision axis. Now consider what the collision looks like in the "lab" frame, in which one of the initial-state particles is at rest.

(a) What are the energies of the final state particles in the lab frame? [15 points]
(b) At what angles with respect to the collision axis are the final state particles produced, in the lab frame? [15 points]
(c) Now assume that $M = m$, and that $\theta = 60°$, and that the mean lifetime of the final-state particles when they are at rest is τ. What is the smallest that E can be in order for the mean lifetime of one of the final-state particles to be 10τ, as measured in the lab frame of this experiment? [10 points].

Problem 4: (40 points)



Microwave Detector
A microwave detector is located at the shore of a lake at a height d above the water level. As a radio star emitting monochromatic microwaves of wavelength λ rises slowly above the horizon, the detector indicates successive maxima and minima of signal intensity.

(a) What is the difference in phase between Ray 1 and Ray 2 arriving at the detector (in terms of d, λ, and θ)? [20 points]
(b) At what angle θ above the horizon is the radio star when the first maximum is received (in terms of d and λ)? [20 points]

Problem 5: (40 points)

Boltzmann Gas

Here we consider a non-interacting gas of N classical particles in three dimensions. The gas is confined in a container of volume V. The energy of the i^{th} particle is given by $\varepsilon_i = \mathbf{p}_i^2 / (2m)$.

(a) Calculate the canonical partition function

$$Z^{(c)}(T,N) = \frac{c_N}{h^f} \int e^{-H\left(\{\mathbf{q}_i, \mathbf{p}_i\}\right)/(k_B T)} d\Gamma$$

where H is the Hamiltonian of the system and $d\Gamma$ is the volume element of the $2f$-dimensional phase-space spanned by the components of the coordinates $\mathbf{q}_i$ and momenta $\mathbf{p}_i$ $(i = 1,\ldots,N)$. What is the number of spatial or momentum degrees of freedom, f, of the gas? What is the value and meaning of the constant c_N? Use the thermal de Broglie wavelength, $\lambda_B = h\big/\sqrt{2\pi m k_B T}$, and the average particle distance, $a = (V/N)^{1/3}$, in your expression. [15 points]

(b) Calculate the free energy F from $Z^{(c)}$. F is the thermodynamic potential associated with the canonical ensemble, i.e., [5 points]

$$F = -k_B T \ln Z^{(c)}(T,N)$$

(c) Now, the particle number is not fixed anymore and we go over to the grand canonical ensemble. Calculate the grand canonical partition function by Laplace transformation

$$Z^{(gc)}(T,\mu) = \sum_{N=0}^{\infty} e^{\frac{\mu}{k_B T} N} Z^{(c)}(T,N)$$

What is the meaning of μ? [10 points]

(d) Show that the relation $J = F - \mu N$ for the thermodynamic potential $J(T,\mu)$ (Planck-Massieu function) of the grand canonical ensemble holds in the thermodynamic limit using the fact that the average particle number in the grand canonical ensemble can be obtained as [10 points]

$$\tilde{N} = \frac{\partial}{\partial(\mu/(k_B T))} \ln Z^{(gc)}(T,\mu)$$

Hint: Use Stirling's formula for large N: $N! \approx N^N e^{-N}$

Do not forget to answer the short questions in (a) and (c)!