1. (16 pts.) Determine the solution $y(x)$ of the initial value problem $xy' - 3y = x^2$, $y(1) = 2$.
2. (16 pts.) Find the solution $y(x)$ of the initial value problem $(x^2+1)y' = xy^2$, $y(0) = 4$.
3. (16 pts. each) Find the general solution of each of the following differential equations.
   (a) $xyy' + y^2 = x^2$
   (b) $(6xy + 2e^{2x} + \ln y) + (1 + (x/y) + 3x^2)\dfrac{dy}{dx} = 0$
4. (16 pts.) Use the method of undetermined coefficients to find the general solution $y(x)$ of $y^{(4)} + 2y^{(3)} + y'' = x$.
5. (16 pts.) Use the variation of parameters method to find the general solution $y(x)$ of $y'' + 16y = \dfrac{1}{\sin 4x}$
6. (18 pts.) Use Laplace transforms to find the solution $x(t)$ of the initial value problem

$$x'' + x' - 6x = e^{2t}, \; x(0) = 0, \; x'(0) = 2.$$

7. (10 pts.) **Use <u>the definition</u>** of the Laplace transform to determine $F(s) = \mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} e^{3t} & \text{if } 0 \le t < 2, \\ 0 & \text{if } t \ge 2. \end{cases}$
8. (12 pts.) Use Euler's method with a step size of $h = 1/2$ to approximate $y(2.5)$, where $y(x)$ is the solution of the initial value problem $y' = 4x^2 + y$, $y(1) = 2$.
9. (10 pts.) A mass of 40 kg is attached to a spring that is stretched 0.01 m by a force of 80 N. A periodic external force of $30\sin(\omega t)$ N acts on the mass, where ω is constant. Determine the frequency of the external force in Hertz (cycles per second) for which resonance occurs.
10. (18 pts.) Find the general solution of the following system of differential equations. ($x = x(t)$ and $y = y(t)$ are both functions of the independent variable t.)

$$x' = 4x + 2y, \quad y' = 3x - y.$$

11. (18 pts.) A rumor is spreading in a population of N people. Assume that the rate of change in the number of people who have heard the rumor is proportional to the number of people who have not yet heard the rumor. If 1% of the population has heard the rumor initially, and after one day 3% of the population has heard it, how long will it be until half of the population has heard the rumor?
12. (18 pts.) At time $t = 0$, a tank with a 1000 gallon capacity contains 600 gallons of pure water. A brine solution containing 5 pounds of salt per gallon is pumped into the tank at a rate of 4 gallons per minute and the well-mixed solution is pumped out of the tank at a rate of 2 gallons per minute. How much salt is in the tank when it is full of brine?

Answers are on the other side of this sheet.

1. $y(x) = -x^2 + 3x^3$
2. $y(x) = \dfrac{4}{1 - 2\ln(x^2+1)}$
3. (a) $y(x) = x\sqrt{\dfrac{1 - Ax^{-4}}{2}}$

   (b) $3x^2y + e^{2x} + x\ln y + y = C$
4. $y(x) = c_1 + c_2x + c_3e^{-x} + c_4xe^{-x} - x^2 + \frac{1}{6}x^3$
5. $y(x) = c_1\cos 4x + c_2\sin 4x - \dfrac{x}{4}\cos 4x + \dfrac{1}{16}(\ln|\sin 4x|)\sin 4x$
6. $x(t) = \dfrac{-9}{25}e^{-3t} + \dfrac{9}{25}e^{2t} + \dfrac{1}{5}te^{2t}$
7. $F(s) = \displaystyle\int_0^2 e^{-st}e^{3t}dt = \frac{1}{s-3}(1 - e^{2(3-s)})$
8. Three steps of Euler's method yields $y(2.5) \approx y_3 = 26$.
9. $\dfrac{10\sqrt{2}}{2\pi}$ Hz.
10. $x(t) = c_1e^{5t} + c_2e^{-2t}$, $y(t) = \dfrac{1}{2}c_1e^{5t} - 3c_2e^{-2t}$
11. $\dfrac{\ln(0.5/0.99)}{\ln(0.97/0.99)} \approx 33.47$ days.
12. $x(200) = 3200$ lbs.