

Name: _____

Student (Z) Number: _____

Signature: _____

There are 200 total points possible. You must **show your work and justify your answers** to receive credit. Clearly indicate your answers by **circling** them.

1. (15 pts.) Determine the solution $y(x)$ of the initial value problem $y' = y^2 e^{3x}$, $y(0) = 1$.

2. (15 pts.) Find the solution $y(x)$ of the initial value problem $(x - 1)y' + 2y = (x - 1)$, $y(2) = 1$.

3. (14 pts. each) Find the general solution of each of the following differential equations.

(a) $4xy^{1/2} + x^2 - 2x^{-1/2}y^{1/2} + (2y + x^2y^{-1/2} - 2x^{1/2}y^{-1/2}) \frac{dy}{dx} = 0.$

(b) $xy \frac{dy}{dx} = x^2 + 3y^2.$

4. (14 pts.) Use the method of undetermined coefficients to find the general solution $y(x)$ of $y^{(4)} + 2y'' - 8y = x^2$.

5. (14 pts.) Use the variation of parameters method to find the general solution $y(x)$ of $y'' + 4y = \frac{1}{\cos 2x}$.

6. (20 pts.) Use Laplace transforms to find the solution $x(t)$ of the initial value problem

$$x'' + 2x' + 5x = e^t, \quad x(0) = 0, \quad x'(0) = 1.$$

7. (14 pts.) Use Euler's method with a step size of $h = 1/2$ to approximate $y(3)$, where $y(x)$ is the solution of the initial value problem $y' = xy^2$, $y(2) = 1$.

8. (20 pts.) Find the general solution of the following system of differential equations. ($x = x(t)$ and $y = y(t)$ are both functions of the independent variable t .)

$$x' = 3x - 2y, \quad y' = 3x - 4y.$$

9. (20 pts.) Let $x(t)$ be the position of the mass in a mass-spring-dashpot system (a mass-spring system with damping) to which a periodic external force is applied. Assume that $x(t)$ satisfies the nonhomogeneous linear differential equation

$$x'' + 4x' + 5x = 4 \sin(3t).$$

- (a) Find the general solution of this differential equation.

- (b) As $t \rightarrow \infty$, the solution tends to a sinusoid $x_p(t) = C \cos(\omega t - \alpha)$. Determine C (the amplitude of this sinusoid).

10. (20 pts.) A motorboat is at rest at time $t = 0$ when its motor instantly turns on. Assume that the motor provides a constant acceleration of 20m/s^2 , and that the water resistance results in a deceleration of $3v \text{ m/s}^2$, where v is the boat's velocity in meters per second.

(a) Set up and solve an initial value problem to determine the boat's velocity $v(t)$ at time t .

(b) Determine how far the boat travels in the first 10 seconds.

11. (20 pts.) At time $t = 0$, a large tank contains 200 pounds of salt dissolved in 100 gallons of water. A brine solution containing 3 pounds of salt per gallon is pumped into the tank at a rate of 2 gallons per minute and the well-mixed solution is pumped out of the tank at a rate of 4 gallons per minute. Set up and solve the initial value problem to determine the amount $x(t)$ of salt in the tank at time t ($0 \leq t \leq 50$).

General Properties

$$\begin{aligned}
\mathcal{L}\{af(t) + bg(t)\} &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}, \\
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0), \\
\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0), \dots \\
\mathcal{L}\{e^{at}f(t)\} &= F(s-a) \quad \text{if } F(s) = \mathcal{L}\{f(t)\}, \\
\mathcal{L}\{u(t-a)f(t-a)\} &= e^{-as}F(s) \quad \text{if } F(s) = \mathcal{L}\{f(t)\}. \\
\mathcal{L}\{f(t) * g(t)\} &= \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.
\end{aligned}$$

Some Special Transforms

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \quad (s > a), \\
\mathcal{L}\{\cos(bt)\} &= \frac{s}{s^2 + b^2} \quad (s > 0), \\
\mathcal{L}\{\sin(bt)\} &= \frac{b}{s^2 + b^2} \quad (s > 0), \\
\mathcal{L}\{\cosh(bt)\} &= \frac{s}{s^2 - b^2} \quad (s > |b|), \\
\mathcal{L}\{\sinh(bt)\} &= \frac{b}{s^2 - b^2} \quad (s > |b|), \\
\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t\} = \frac{1}{s^2}, \quad \mathcal{L}\{t^2\} = \frac{2 \cdot 1}{s^3}, \quad \mathcal{L}\{t^3\} = \frac{3 \cdot 2 \cdot 1}{s^4} \quad (s > 0), \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}} \quad (s > 0) \text{ for any positive integer } n. \\
\mathcal{L}\{u(t-c)\} &= \frac{e^{-cs}}{s} \quad (s > 0).
\end{aligned}$$