

Qualifying Examination E
Numerical Analysis
June 2016

Instructions: Write your answers to Problems A1-A4 and Problems B1-B4 in separate blue books.

Part A: Work 3 of the following 4 problems. **Clearly indicate which problem is not to be graded.**

A1. Suppose that we use Newton's method to generate a sequence of approximations x_n to a zero α of a function $f(x) \in C^2(\mathbb{R})$. Let $e_n = \alpha - x_n$ denote the error in the approximation x_n .

(a) Prove that

$$e_{n+1} = \frac{-f''(\xi_n)}{2f'(x_n)}e_n^2$$

for some ξ_n between x_n and α .

(b) Further assume that $f'(x) < 0$ and $f''(x) < 0$ for all $x \in \mathbb{R}$. Prove that Newton's method will converge to the root α for any choice of the starting point x_0 .

A2. Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix. Show that A has an LU factorization if and only if every leading principal submatrix of A is nonsingular. (Hint: Consider a partitioned form of the LU factorization.)

A3. Let $I = \int_a^b f(x)dx$, where $f^{(4)}(x)$ is continuous on the interval $[a, b]$, and let S_1 denote the (basic) Simpson's approximation to I .

(a) Derive the formula for S_1 . Hint: $S_1 = \alpha * f(a) + \beta * f(\frac{a+b}{2}) + \gamma * f(b)$ you need to find the coefficients α, β, γ .

(b) Use the error formula for interpolating polynomials to show that

$$I - S_1 = \frac{-f^{(4)}(\xi)}{90} \left(\frac{b-a}{2} \right)^5$$

for some $\xi \in [a, b]$. You may use, without proof, the fact that $f[x_0, x_1, \dots, x_n] = \frac{1}{n!}f^{(n)}(\eta)$ for some η in the smallest interval containing the nodes x_0, \dots, x_n , and that this formula is valid for arbitrary (possibly multiple) nodes.

(c) Use (b) to show that Simpson's formula is exact for polynomials of degree 3.

A4. Consider the inner product of the form

$$\langle f(x), g(x) \rangle = \int_a^b f(x)g(x)w(x)dx,$$

where $w(x)$ is a positive weight function on $[a, b]$, and let $\{P_j(x)\}_{j=0}^{\infty}$ denote the family of orthogonal polynomials with respect to this inner product, where $P_j(x)$ is monic and of degree j .

- (a) Prove that the orthogonal polynomials satisfy a recurrence relation of the form $P_{n+1}(x) = (x - a_n)P_n(x) - b_nP_{n-1}(x)$ for $n \geq 1$.
- (b) Derive formulas for a_n and b_n in terms of the inner product.

Part B. Work 3 of the following 4 problems. **Clearly indicate which problem is not to be graded.** In the absence of an indication, problems 1-3 will be graded.

B1. Let $f(x)$ be a continuous function and consider the boundary value problem

$$-u'' + 3u = f \quad \text{for } x \in (0, 1); \quad u'(0) = u'(1) = 0.$$

- (a) Suggest a numerical method for obtaining an approximation, which we will denote by u^* , of the solution u to this boundary value problem.
- (b) Does the method you suggested in a) always yield a unique u^* ? Justify your answer.
- (c) What can you say about the error $|u(x) - u^*(x)|$?

B2. (a) Define the region of absolute stability of a multisteps method for initial value problems.

- (b) Prove that the region of absolute stability for any explicit method is bounded.

B3. We intend to solve numerically the initial boundary value problem

$$\begin{aligned} u_t &= u_{xx} + f(x, t) && \text{on } (0, 1) \times (0, T) \\ u(x, 0) &= u_0(x) && \text{in } (0, 1) \\ u(0, t) &= u(1, t) = 0 && t \leq T. \end{aligned}$$

After discretization in space (using central differences) the partial differential equation above yields:

$$\frac{dU}{dt} = AU + F, \tag{1}$$

where A is an $N \times N$ matrix and U is a vector with N components.

- (a) Describe the matrix A .
- (b) To solve equation (1) we could use the explicit method or the implicit method. Explain (describe) each method.
- (c) State (without proving) the conditions which ensure the stability of the implicit method.
- (d) State (without proving) the conditions which ensure the stability of the explicit method.
- (e) Justify your answer to either c or d. (State which).

B4. For solving the initial value problem

$$u' = f(t, u); \quad u(0) = u_0$$

we use a method of the type:

$$u(t_{k+1}) + \frac{4}{3}u(t_k) + \alpha_1 u(t_{k-1}) = h\alpha_2 f(t_{k+1}, u(t_{k+1})).$$

- (a) Find α_1, α_2 which minimize the truncation error.
- (b) Is the method stable for these values of α_1, α_2 ?
- (c) What is the region of absolute stability?