Comprehensive Examination E

AUGUST 2011

Please write your answers to Part A and Part B in separate answer books.

Part A. Answer four of the following five questions. Clearly indicate which problem is not to be graded.

A1. Suppose that we use Newton's method to generate a sequence of approximations x_n to a zero α of a function $f(x) \in C^2(\mathbb{R})$. Let $e_n = \alpha - x_n$ denote the error in the approximation x_n.

(a) Prove that

$$e_{n+1} = \frac{-f''(\xi_n)}{2f'(x_n)} e_n^2$$

for some ξ_n between x_n and α.

(b) Further assume that $f'(x) < 0$ and $f''(x) < 0$ for all $x \in \mathbb{R}$. Prove that Newton's method will converge to the root α for any choice of the starting point x_0.

A2. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Show that A has an LU factorization if and only if every leading principal submatrix of A is nonsingular. (Hint: Consider a partitioned form of the LU factorization.)

A3. Given $n+1$ distinct nodes $x_0, x_1, \dots x_n$, let $f[x_0, x_1, \dots x_n]$ denote the leading coefficient (the coefficient of x^n) of the unique polynomial $p_n(x)$ of degree at most n that interpolates a function $f(x)$ at the specified nodes.

(a) Prove that the leading coefficients satisfy the recursive relationship

$$f[x_0, x_1, \dots x_n] = \frac{f[x_1, x_2, \dots x_n] - f[x_0, x_1, \dots x_{n-1}]}{x_n - x_0}$$

for $n > 0$, where $f[x_j] = f(x_j)$.

(b) Assume that $f \in C^n[a, b]$, where $[a, b]$ is an interval that contains all $n+1$ interpolation nodes. Prove that

$$f[x_0, x_1, \dots x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for some $\xi \in (a, b)$.

A4. Let Π_n denote the set of polynomials of degree at most n, and let $p_n^*(x)$ be the best uniform approximation of a continuous function $f(x)$ on the interval $[a, b]$ from Π_n. Chebyshev's Equioscillation Theorem states that p_n^* exists and is unique, and also gives conditions that characterize p_n^*.

(a) State the conditions that characterize $p_n^*(x)$.

(b) Assume that there exists a polynomial $p_n^*(x) \in \Pi_n$ that satisfies the conditions from part (a). Prove that this polynomial is the best uniform approximation to $f(x)$ on $[a, b]$ from Π_n. (Do not rely on Chebyshev's theorem; this problem asks for a partial proof of Chebyshev's result.)

A5. Let $I = \int_a^b f(x)dx$, where $f^{(4)}(x)$ is continuous on the interval $[a, b]$, and let S_2 denote the (basic) Simpson's approximation to I.

(a) Derive the formula for S_2.

(b) Use the error formula for interpolating polynomials to show that

$$I - S_2 = \frac{-f^{(4)}(\xi)}{90} \left(\frac{b-a}{2} \right)^5$$

for some $\xi \in [a, b]$. You may use, without proof, the fact that $f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\eta)$ for some η in the smallest interval containing the nodes $x_0, \ldots, x_n$, and that this formula is valid for arbitrary (possibly multiple) nodes.

Part B. Answer four of the following five questions. Clearly indicate which problem is not to be graded.

B1. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n \geq 2$. Prove the following results directly. (Do <u>not</u> appeal to the singular value decomposition.)

(a) Prove that $\|U^*AV\|_2 = \|A\|_2$ for any unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.

(b) Let $\sigma_1 = \|A\|_2$. Prove that there exists unitary matrices U_1 and V_1 such that

$$U_1^*AV_1 = \begin{bmatrix} \sigma_1 & \mathbf{0}^* \\ \mathbf{0} & B \end{bmatrix},$$

where $B \in \mathbb{C}^{(m-1)\times(n-1)}$.

(c) Prove that $\|B\|_2 \leq \sigma_1$.

B2. Let $A \in \mathbb{C}^{m\times m}$, and consider the factorization $A = QL$, where $Q \in \mathbb{C}^{m\times m}$ is unitary and $L \in \mathbb{C}^{m\times m}$ is *lower* triangular.

(a) Give an explicit algorithm based on the efficient use of Householder transformations to transform A to a lower triangular matrix L. Your algorithm should implicitly determine the unitary matrix Q, but should not explicitly compute Q.

(b) Assuming the matrix A is real, determine the approximate number of flops your algorithm requires. (Neglect lower order terms in the flop count.)

B3. Let $A \in \mathbb{C}^{m\times n}$ be of rank n (full rank), and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization of A, where $Q \in \mathbb{C}^{m\times n}$ and $R \in \mathbb{C}^{n\times n}$. Also let $R(A)$ denote the range of A.

(a) Show that $y = \hat{Q}\hat{Q}^*b$ is the unique element of $R(A)$ such that $b - y$ is orthogonal to $R(A)$.

(b) Show that for all $z \in R(A)$, $\|b - y\|_2 \leq \|b - z\|_2$.

(c) Show how the reduced QR factorization of A can be used to find $x \in \mathbb{C}^n$ to minimize $\|b - Ax\|_2$. Outline the resulting algorithm.

B4. Let $A \in \mathbb{C}^{m\times m}$ be a hermitian matrix.

(a) Prove that if A has a Cholesky factorization, then A is positive definite.

(b) Prove that if A is positive definite, then A has a unique Cholesky factorization.

(c) Assume that A is positive definite and that R is its Cholesky factor. Prove that $\|A\|_2 = \|R\|_2^2$. (Hint: Use the singular value decomposition.)

B5. (a) Prove that every matrix $A \in \mathbb{C}^{m\times m}$ can be transformed by unitary similarity to Schur form.

(b) Prove that A is a normal matrix if and only if there exists a unitary matrix Q such that Q^*AQ is a diagonal matrix.