PHD Qualifying Examination D
Differential Equations
AUGUST 2013

Instructions: In this three-hour examination, Part A and Part B carry equal weight in determining your overall performance. Please use separate blue books for Part A and Part B.

Part A: Answer 4 of the following 5 questions.

A1. Consider the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \tag{1}$$

where t_0, $x_0 \in \mathbb{R}$, $f \in C(\mathbb{R}^2, \mathbb{R})$ and satisfies a Lipschitz condition in x, i.e., there exists a $k > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq k|x_1 - x_2|$$

for all t, x_1, $x_2 \in \mathbb{R}$. Show that the solution of Eq. (1) is unique and exists on $\mathbb{R}$.

A2. Let $A \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ such that $A(t+\omega) = A(t)$ for some $\omega > 0$ and all $t \in \mathbb{R}$. Assume the equation $x' = A(t)x$ is stable. Show it is uniformly stable.

A3. (a) Use a Liapunov function to determine the stability of the zero solution of the system

$$\begin{cases} x' = -2x + 2y + 2xy^2 \\ y' = x - y - x^2 y. \end{cases}$$

(b) Can you use the linearization method to solve the problem in (a), and why?

A4. Consider the following BVP

$$u'' + u = h(t), \quad u'(0) = 0,\ u(\pi/2) + u'(\pi/2) = 0.$$

Determine if the BVP has a unique solution for each $h \in C[0, \pi/2]$; and if it does, find the Green's function.

A5. (a) Use a comparison theorem to show that every nontrivial solution of the equation $u'' + \dfrac{1}{1+t^2}\, u = 0$ has at most one zero on $[0, \pi]$.

(b) What can you say about the number of zeros of any solution of the equation $u'' + \left(1 + \dfrac{1}{1+t^2}\right) u = 0$ on $[0, \infty)$, and why?

Part B: Answer all 3 questions.

B1. Solve the Cauchy problem

$$u_x^2 - 3u_y^2 = u \qquad u(x,0) = x^2$$

B2. Let $f(x)$ be a given continuous function. Prove that there exists a positive number $\alpha_0 > 0$ such that $\forall \alpha \in (0, \alpha_0)$ the problem

$$-u''(x) + \alpha \cos(x) u'(x) + u(x) = f(x) \qquad \forall x \in (0,1)$$

$$u(0) = u(1) = 0$$

has a unique solution u which satisfies $\int_0^1 (u'(x))^2 dx < \infty$.

B3. Let Ω be he uniqt disk in R^2 and H is the set of C^2 functions u such that $u = 0$ and $\frac{\partial u}{\partial r} = 0$ on the boundary of Ω, where r is the radial variable.

prove that there exists a positive number $\lambda > 0$ such that

$$\frac{\int\int_\Omega (u_{xx} + u_{yy})^2 dxdy}{\int\int_\Omega u^2 dxdy} \geq \lambda \qquad \forall u \in H$$