Qualifying Examination B
Analysis
June 2017

Instructions: For the two-hour examination, work **Part A** only. For the three-hour examination, work **Part A** and **Part B**.

Part A: Solve seven of the following eight problems.

1. Let $f : [0,1] \to \mathbf{R}$ be Lebesgue integrable function. Define $F(x) = \int_0^x f$. Prove that function $F(x)$ has bounded variation on $[0,1]$. What does this imply for differentiability of F and why? (For the last question, please, make sure to mention monotone functions and their differentiability)

2. Let m be a σ-additive measure generated by an outer measure m^*. Suppose E is a measurable set with $m(E) > 0$. Let $A_n \subset \mathbf{R}$ be such that

(a) they all have the same outer measure : $m^*(A_n) = m^*(A_1)$;

(b) they are mutually disjoint: $A_n \cap A_k = \emptyset$ for all $n \neq k$;

(c) the outer measure of their union satisfies: $m^*(\cup_{n=1}^{\infty} A_n) < 2017\pi^e$;

(d) $E \subset \cup_{n=1}^{\infty} A_n$.

Prove that at least one set A_n should be non-measurable.

3. Let f_n be a non-negative integrable function for every $n \in \mathbf{N}$ and $\int f_n = \frac{1}{n}$. Define the sequence of numbers $a_n = \lambda(\{x : f_n(x) \geq n\})$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges. Hint: use Chebyshev's inequality.

4. (a) Give the definition of a σ-algebra of sets;

(b) Give the definition of a σ-algebra of sets generated by collection of sets $\mathcal{C}$;

(c) Give the definition of the Borel σ-algebra in $\mathbf{R}$;

(d) Prove that if $B_n \subset \mathbf{R}$ are Borel sets, then there is a G_δ set $A \subset \mathbf{R}$ such that $\cup_{n=1}^{\infty} B_n \subset A$ and $\lambda(A \backslash \cup_{n=1}^{\infty} B_n) = 0$. Please, don't forget to mention the definition of a G_δ set.

5. Construct a bijective conformal map from the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to the sector $S = \{z : \pi/4 < \operatorname{Arg}(z) < 3\pi/4\}$. Make sure to fully justify your construction does what is required.

6. Find all complex solutions to the equation $\exp(z^2) = i$.

7. Determine whether or not there is an analytic function f defined on $\mathbb{C} \setminus \{0\}$ with real part $u(x + iy) = \dfrac{y}{x^2 + y^2}$.

8. Use complex analysis to compute the real integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^3}$.

Part B. Solve three of the following four problems.

1. (a) Let (X, Σ, μ) be a measure space. State the definition of $f_n \xrightarrow{\mu} f$.

(b) Suppose $f_n \xrightarrow{\mu} 0$ and $f_n(x) \geq 0$ for each $x \in X$. Prove that for $g(x) := \underline{\lim} f_n(x)$ we have $g(x) = 0$ almost everywhere.

(c) Give an example of a sequence of Lebesgue measurable non-negative functions $f_n : [0,1] \to [0, \infty)$ such that $f_n \xrightarrow{\lambda} 0$ but for $g(x) := \overline{\lim} f_n(x)$ we have $g(x) = 1$ for each $x \in [0,1]$.

2. Let $f_n : [0,1] \to \mathbf{R}$ be Lebesgue **measurable** functions. Suppose they converge pointwise to a monotone function $f : [0,1] \to \mathbf{R}$. Prove that f is **Riemann** integrable.

3. Prove that if $|f(z)| \leq 1 + |z|^2$ for all $z \in \mathbb{C}$, then f is a polynomial of degree at most two.

4. Find the number of solutions, counting multiplicity, in the domain $\{z \in \mathbb{C} : 1 < |z| < 2\}$ to the equation $z^9 + z^5 - 8z^3 + 2z + 1 = 0$.