Examination B (MS Students)
July 2015

Instructions: Write your answers to problems A1-A5, your answers to problems B1-B5, and your answers to problem B6-B10 in separate blue books. All candidates should attempt 4 of the 5 problems in Part A. Those taking the 2-hour examination should work 4 of the 10 problems in Part B. Those taking the 3-hour examination should work 8 of the 10 questions in part B. **Clearly indicate which problems you wish to have scored.**

PART A: Work 4 of the following 5 problems. Clearly indicate which problem is not to be graded.

A1. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x\cos\left(\dfrac{1}{x}\right) & x \in (0, 1] \\ 0 & x = 0 \end{cases}$

(a) Show that f is continuous at $x = 0$

(b) Show that f is uniformly continuous on $[0, 1]$

(c) Is f differentiable at $x = 0$? Explain.

A2. Define $f : \mathbb{R} \to \mathbb{R}$ as follows.

$$f(x) = \begin{cases} 8x \text{ if } x \text{ is a rational number} \\ 2x^2 \text{ if } x \text{ is not a rational number} \end{cases}$$

At what values of x does f have a limit? Is it continuous there? Is it differentiable there?

A3. State the Mean Value Theorem. Hence or otherwise prove that $|\sin(x)| < |x|$ for $x \in [-\pi/2, \pi/2]$

A4. Compute the following limits if they exist

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

A5. State the formal definition, in terms of ϵ and N, for a series to converge.

Which of the following series converge? Explain your answers

(a) $\displaystyle\sum \frac{n^2}{n^2+1}$

(b) $\displaystyle\sum \frac{1}{n\ln(n)}$

(c) $\displaystyle\sum \frac{7^{n+3}}{11^n}$

(d) $\displaystyle\sum \frac{1}{n^2+4n-1}$

PART B

B1.

1. (a) Determine the limit for the sequence $\left\{\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}dx\right\}$, and prove your result.

   (b) If $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ such that $0 \le f_n \le 1$ and such that $\lim_{n\to\infty} f_n(x) = 0$ for every x in $[0,1]$, prove

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

B2. (a) Define the Cantor ternary set C on the interval $[0,1]$.

(b) Prove C is closed.

(c) If $\sim C$ denotes the complement of C in $[0,1]$, prove $m(\sim C) = 1$, where m denotes Lebesgue measure.

B3. Let f be an absolutely continuous, nondecreasing function on a finite, closed interval $[a, b]$. Let E be a set of Lebesgue measure zero in $[a, b]$. Prove that the measurable set $f(E)$ has Lebesgue measure zero.

B4. Let M be the collection of Lebesgue measurable sets in $\mathbb{R}$. A set $A \in M$ is called an atom if $m(A) > 0$, and for all measurable sets $B \subset A, m(B) = 0$ or $m(A \backslash B) = 0$. Prove no atoms exist if m denotes Lebesgue measure.

B5. Let $\{f_n\}$ be a sequence of finite a.e. Lebesgue measurable functions on a set $X \subseteq \mathbb{R}$. We say $\{f_n\}$ converges **almost uniformly** to f on X, and write $f_n \to f[a.u.]$, if for every $\epsilon > 0$ there exists a measurable set $E \subseteq X$ such that $m(X \backslash E) < \epsilon$ and $\{f_n\}$ converges uniformly to f on E.

(a) State carefully and completely Egoroff's theorem.

(b) Prove that if $f_n \to f[a.u.]$ on X, then $f_n \to f(a.e.)$ on X and $f_n \to f[meas]$ on X.

B6. Can $\Omega = \{z;\ 1/2 < |z| < 1,\ \arg z \neq 0\}$ be mapped conformally onto a rectangle? If so, determine the mapping.

B7. Let f and g be analytic in a region G. If $\text{Re} f = \text{Re} g$ in G, then prove

$$f(z) = g(z) + ic, \qquad \text{for all} \quad z \in G,$$

where c is a real constant.

B8. Suppose that G is a bounded region, and let f_n be continuous in $\overline{G}$ and analytic in G for each $n \in \mathbb{Z}^+$. Prove that if $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on ∂G, then $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on G, where ∂G is the boundary of G..

B9. Find the number of zeros of $z^8 - 4z^5 + z^2 - 1 = 0$ in the unit disk $B(0,1) = \{z : \ |z| < 1\}$.

B10. Let f be an entire function and n an integer. If

$$|f(z)| \leq 4|z|^n \qquad \text{for} \quad |z| \geq 100,$$

show that f is a polynomial of degree at most n.