MS Qualifying Examination B
Analysis
August 2015

Instructions: Write your answers to problems A1-A5, your answers to Problems B1-B5, and your answers to problems B6-B10 in separate blue books. All candidates should attempt 4 of the 5 problems in part A. Those taking the two hour examination should work 4 of the 10 problems in Part B. Those taking the three hour examination should work 8 of the 10 problems in part B. **Clearly indicate which problems you wish to have scored.**

Part A: Work 4 of the following 5 problems. Clearly indicate which problem is not to be graded.

A1. It is always true that a sequence of real numbers that is decreasing and bounded below is convergent? Either state and prove a theorem relating to this situation, or give a counterexample.

A2. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. State the Mean Value Theorem. By considering the interval $[0, x]$. Prove that $1 + x < e^x$ for all $x > 0$.

A3. Let $f_n : [0, 1] \to \mathbb{R}$ be given for each n by

$$f_n(x) = x^n \text{ and define } f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

Prove that $\{f_n\}$ is <u>NOT</u> uniformly convergent to f. State clearly any theorems that you use.

A4. (a) Show that $\lim_{(x,y)\to(a,b)} x^2 + y^2$ exists at any point $(a, b) \in \mathbb{R}^2$.

(b) Prove that $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + y^2}$ does not exist.

A5. Prove that the power series $E(x) = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$ is uniformly convergent on $[-b, b]$ for any $b > 0$. State a theorem which guarantees that $E'(x) = \sum_{k=0}^{\infty} \left(\dfrac{x^k}{k!}\right)'$. Hence, or otherwise, show that $E(x)$ is the solution of the differential equation $y' - y = 0$ with $y(0) = 1$.

Part B.

B1. Let f be a measurable function in an interval $I = [a, b]$, and $|f(x)| < \infty$ a.e. in I, prove that for any $\epsilon > 0$, there are a positive number M and a measurable set A such that $|f(x)| \leq M$ for all $x \in A$ and $m(I \backslash A) \leq \epsilon$.

B2. Let f be integrable on a measurable set E. Prove

(a) $\lim_{t\to\infty} m(E|f| \geq t)) = 0$.

(b) $\lim_{t\to\infty} \int_{E(|f|\geq t)} |f| = 0$.

B3. Let $\mathbb{Q}$ be a set of rational numbers in $\mathbb{R}$. Let

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{Q} \cap [0, 1], \\ xe^x, & \text{if} \quad x \in [0, 1] \backslash \mathbb{Q}. \end{cases}$$

Prove that f is measurable and find $(R) \int_0^1 f(x)dx$ and $\int_{[0,1]} f(x)dx$ if they exist.

B4. Let E be a measurable set and $\{f_n\}$ a sequence of integrable functions such that $f_n \to f$ a.e. in E, with f integrable. Prove

$$\int_E |f - f_n| \to 0 \text{ if } \int_E |f_n| \to \int_E |f|.$$

B5. Let f be of bounded variation on $[a, b]$. Show that

$$\int_a^b |f'| \leq T_a^b(f).$$

B6. (a) Show $f(z) = \bar{z}e^z$ is not analytic for any $z \in \mathbb{C}$.

(b) For which z is $\text{Log}(z^2 - z) = g(z)$ analytic where Log is the principal logarithm?

B7. (a) Find a conformal map h which maps the disk $S = \{z \big| |z| < 3\}$ onto the right half plane.

(b) Let f be the function defined on $D = \{z \big| |z| < 1\}$ by $f(z) = 4z^3 + 15iz^2 + 18z + 1$. Is f a conformal map on D? Why or why not?

B8. (a) State Cauchy's Residue Theorem

(b) Determine the value of $\displaystyle\int_{|z|=1} z^3 \sin(\frac{2}{z})dz$.

(c) Suppose f is an analytic function on $\Omega = \{z \big| 0 < |z| < 1\}$ and g is an analytic function on $f(\Omega)$. If f has a removable singularity at zero, does $g \circ f$ also have a removable singularity at zero? Why or why not?

B9. Prove that there is <u>no</u> nonconstant analytic function f on $\mathbb{C}$ such that $f(z) = 0$ for all $z \in \mathbb{R}$.

B10. Let p be an nth degree polynomial with zeros $a_1, a_2, ..., a_n$. If R is a real number such that $|a_i| < R$ for $i = 1, 2, 3, ..., n$, evaluate

$$\frac{1}{2\pi i} \int_{|z|=R} z \frac{p'(z)}{P(z)} dz.$$