Ph.D. Qualifying Examination A
Algebra
January 2016

Instructions: For the two-hour examination, work **Part A** only. For the three-hour examination, work **Part A** and **Part B**.

Part A Solve seven of the following eight problems.

1. Let G be a finite group, let p be a prime, and let K be a normal subgroup of G of index p. Prove that, for all subgroups H of G, either
   (a) $H \leq K$, or
   (b) $G = HK$ and $|H : H \cap K| = p$.

2. Prove that there is no simple group of order 56.

3. Prove that if G is a (not necessarily finite) group in which the number of elements of order two is exactly three, then G is not simple.

4. Let F denote the field $\mathbb{Z}_2$, and let $R = F[x]/\langle x^2 + 1 \rangle$. Prove that R contains exactly four elements and that $R \not\cong \mathbb{Z}_4$ and $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, as rings.

5. Let F/K be a field extension, and suppose that $u \in F$ is algebraic over K.
   (a) Prove that there exists a unique monic irreducible polynomial $p(x) \in K[x]$ with $p(u) = 0$.
   (b) Prove that if $f(x)$ is any polynomial in $K[x]$ with $f(u) = 0$, then $p(x)$ divides $f(x)$ in $K[x]$.

6. Let $f(x) = x^4 - 4$.
   (a) Determine a splitting field F of the polynomial $f(x)$ over $\mathbb{Q}$.
   (b) Determine the isomorphism type of the Galois group $\mathrm{Aut}(F/\mathbb{Q})$.
   (c) Determine all subgroups of $\mathrm{Aut}(F/\mathbb{Q})$ and their fixed fields.

7. Explain how to construct a field K with 27 elements. What are the subfields of this field?

8. Let V be a finite-dimensional complex vector space, and let S, T be linear operators on V such that $ST = TS$. Recall that a subspace W of V is said to be invariant under T if $T(W) \subseteq W$.
   (a) Prove that if λ is an eigenvalue of S, then the eigenspace $V_\lambda = \{\vec{x} \in V \mid S(\vec{x}) = \lambda\vec{x}\}$ is invariant under T.
   (b) Prove that S and T have at least one common eigenvector (not necessarily associated to the same eigenvalue).

Part B Solve three of the following four problems.

1. Prove that, in a principal ideal domain, every nonzero prime ideal is maximal.

2. Let R be a ring, let M be an R-module, and let $\phi : M \to M$ be an R-module homomorphism.
   (a) Prove that if M is noetherian and ϕ is surjective, then ϕ is injective.
   (b) Prove that if M is artinian and ϕ is injective, then ϕ is surjective.

3. Let R be a commutative ring. Prove that R is semisimple if and only if R is isomorphic to a finite direct product of fields.

4. Let R be a commutative ring, and let I be an ideal of R. Prove that I is primary if and only if R/I is a nonzero ring with the property that every zero divisor is nilpotent.