Ph.D. Qualifying Examination A, Algebra, June 2015

PART A Work 7 of the following 8 problems. Each problem will be given equal weight.

Part A

A1 Prove that the matrix

$$\begin{pmatrix} -9 & 4 \\ -25 & 11 \end{pmatrix}$$

is not diagonalizable.

A2 Let V be a finite dimensional vector space over a field k, and let T and U be linear transformations from V to V. Assume that U is invertible. Prove that T is diagonalizable if and only if $U^{-1}TU$ is diagonalizable.

A3 Prove that there is no simple group of order 80. You may use Sylow's theorem.

A4 Let G be a group whose order is a power of the prime p. Prove that the center of G has more than one element.

A5 Let R be an integral domain, and let I be a nonzero principal ideal in R. Prove that $R^\times$ (the unit group of R) acts transitively on the set

$$\{r \in R \mid (r) = I\}.$$

A6 Prove that the polynomial $x^4 + 4x^3 + 6x^2 + 2x + 1$ is irreducible in $\mathbb{Q}[x]$.

A7 Prove that the polynomial $x^{p^n} - x \in \mathbb{F}_p[x]$ is separable. (Here, $n \in \mathbb{N}$.)

A8 Let E/K be an extension of fields, and let $f(x) \in K[x]$ have degree d. Prove that $f(x)$ has at most d roots in E.

PART B   Work 3 of the following 4 problems. All problems will be given equal weight.

Part B

B1 Let E/K and L/E be algebraic extensions of fields. Prove that L/K is an algebraic extension of fields.

B2 Let E_1 and E_2 be Galois extensions of the field K, and assume that L is a field containing both E_1 and E_2. Prove that the compositum E_1E_2 is a Galois extension of K.

B3 Let $\alpha \in \mathbb{C}$ satisfy the following conditions:

1. The element α satisfies a polynomial in $\mathbb{Q}[x]$ of degree 5.
2. The extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ has degree 120.

Prove that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois and contains precisely one intermediate field K which is Galois over $\mathbb{Q}$. Find $|K : \mathbb{Q}|$.

B4 Let k be a field, and consider the ring

$$A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in k \right\}.$$

Find a composition series for the left-regular A-module ${}_AA$. How many isomorphism classes of simple left A-modules are there?