

1. Find the tangent line to the curve  $r(t) = \langle \sin t, \cos t, t \rangle$  at  $(0, 1, 0)$ . Find tangent line at  $t = \pi/4$ . Find the length of the curve of  $r(t)$  over the interval  $0 \leq t \leq \pi/2$ .
2. Determine the value of the limit. If it exists, find the value. If it does not show why.

(a)  $\lim_{(x,y) \rightarrow (0,0)} \frac{y-x}{\sqrt{x^2+y^2}}$

(b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{\sqrt{x^2+y^2}}$

3. If  $z = e^x \tan y$ , where  $x = s^2 + t^2$  and  $y = st$ , find  $\frac{\partial z}{\partial t}$  when  $s = 1$  and  $t = 0$ .
4. Find an equation of the plane through the point  $(1, 5, 4)$  and perpendicular to the line  $x = 1 + 7t$ ,  $y = t$ ,  $z = 23t$ .
5. Let  $P(1, 2, 3)$ ,  $Q(1, -1, -2)$ , and  $R(0, 0, 0)$  be three points in  $R^3$ .
  - (a) Find an equation of the plane through  $P$ ,  $Q$ , and  $R$ .
  - (b) Find the area of the triangle formed by  $PQR$ .
  - (c) Find the equation of the line through  $P$  that is perpendicular to the plane from (a).
6. Given  $xy + e^{xyz} - z - e^y = 0$ , use implicit partial derivative to find  $\frac{\partial z}{\partial x}$  at the point  $P(1, 1, 1)$ .
7. Let  $f(x, y) = x^2 - 5xy$ 
  - (a) Find  $\nabla f(x, y)$ .
  - (b) Find the directional derivative at  $(2, 1)$  in the direction of  $\vec{v} = -i + 3j$ .
  - (c) Find the equation of the tangent line on  $f(x, y)$  at  $(2, 1)$ .
  - (d) Find the linearization  $L(x, y)$  of  $f$  at  $(2, 1)$ .
  - (e) Use the linearization to approximate  $f(1.9, 0.9)$ .
8. Find the local max, min, and saddle points for  $f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - xy + 4$  (if any exist).
9. Find the local max, min, and saddle points for  $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$ .
10. Use Lagrange Multipliers to find the maximum and minimum of  $f(x, y) = x^2y$  subject to the constraint  $x^2 + y^2 = 1$
11. Evaluate  $\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$  by changing the order of integration.
12. Setup the triple integral in the order of  $dz dx dy$  and again as  $dz dy dx$  to find the volume of the solid tetrahedron which is bounded by  $3x + y + z = 1$  and the coordinate planes (i.e., the first octant).

13. Set up, do not evaluate the triple integral in rectangular, cylindrical, and spherical coordinates to find the volume of the solid in the first octant bounded above by  $x^2 + y^2 + z^2$  and bounded below by  $z = \sqrt{x^2 + y^2}$ .

12

14. Find the Jacobian for the transformation  $x = u^2v + v^2$  and  $y = uv^2 - u^2$ .

15. Evaluate  $\iint_D (3x - y)^{3/2}(x + y)^5 dA$  where  $D$  is the region bounded by  $y = -x$ ,  $y = -x + 1$ ,  $y = 3x$ , and  $y = 3x - 1$ . Use the change of variables  $u = 3x - y$  and  $v = x + y$ .

16. Evaluate the line integral  $\int_C (xz + 2y) dS$ , where  $C$  is the line segment from  $(0, 1, 0)$  to  $(1, 0, 2)$ .

17. Let  $F(x, y) = (xy^2 + 2y)i + (x^2y + 2x + 2)j$  be a vector field.

(a) Show  $F$  is conservative.

(b) Find  $f$  such that  $\nabla f = F$ .

(c) Evaluate  $\int_C F dr$  where  $C$  is defined by  $r(t) = \langle e^t, 1 + t \rangle$ ,  $0 \leq t \leq 1$ .

(d) Evaluate  $\int_C F dr$  where  $C$  is a closed curve  $r(t) = \langle 2 \sin(t), \cos(t) \rangle$ ,  $0 \leq t \leq 2\pi$ .

(e) Use Green's Theorem to evaluate  $\int_C -y^3 dx + x^3 dy$  where  $C$  is a circle given by  $r(t) = \langle 2 \cos t, 2 \sin t \rangle$ ,  $0 \leq t \leq 2\pi$ .

18. Use Green's Theorem to evaluate  $\int_C -y^3 dx + x^3 dy$  where  $C$  is a circle given by  $r(t) = \langle 2 \cos t, 2 \sin t \rangle$ ,  $0 \leq t \leq 2\pi$ .

19. Use Green's Theorem to evaluate  $\int_C (e^x + y^2) dx + (e^y + x^2) dy$  where  $C$  is the positively oriented boundary of the region in the first quadrant bounded by  $y = x^2$  and  $y = 4$ .

1. Find the tangent line to the curve  $r(t) = \langle \sin t, \cos t, t \rangle$  at  $(0, 1, 0)$ . Find tangent line at  $t = \pi/4$ . Find the length of the curve of  $r(t)$  over the interval  $0 \leq t \leq \pi/2$ .

TANGENT LINE EQUATION:  $L(t) = \vec{v}t + P_0$

• DIRECTION VECTOR  $\vec{v}$  CAN BE FOUND BY  $r'(t)$  AT  $(0, 1, 0)$ .

•  $r'(t) = \langle \cos t, -\sin t, 1 \rangle$

• WHAT  $t$  VALUE GIVES THE POINT  $(0, 1, 0)$ ?

$\sin(t) = 0$ ,  $\cos(t) = 1$ ,  $t = 0$

ONLY  $t$  VALUE THAT WORKS FOR ALL 3

IS  $t = 0$

$\vec{v} = r'(0) = \langle \cos 0, -\sin 0, 1 \rangle = \langle 1, 0, 1 \rangle$

• FORMULA

$\vec{v}t + P_0 = \langle 1, 0, 1 \rangle t + \langle 0, 1, 0 \rangle$

$= \langle t, 1, t \rangle$

---

TANGENT AT  $t = \pi/4$ ?

$\vec{v} = r'(\pi/4) = \langle \cos \pi/4, -\sin \pi/4, 1 \rangle = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 \rangle$

•  $P_0 = r(\pi/4) = \langle \sin \pi/4, \cos \pi/4, \pi/4 \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} \rangle$

FORMULA:  $\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 \rangle t + \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} \rangle$

1.

$$= \left\langle \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}, t + \frac{\pi}{4} \right\rangle$$

---

LENGTH OF  $r(t)$  ON  $[0, \pi/2]$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

WHERE  $x = \sin(t)$ ,  $y = \cos(t)$ ,  $z = t$

•  $\frac{dx}{dt} = \cos t$ ,  $\frac{dy}{dt} = -\sin t$ ,  $\frac{dz}{dt} = 1$

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} dt \\ &= \int_0^{\pi/2} \sqrt{2} dt \\ &= \sqrt{2}t \Big|_0^{\pi/2} \\ &= \sqrt{2}\left(\frac{\pi}{2}\right) - \sqrt{2}(0) \\ &= \frac{\sqrt{2}}{2}\pi \end{aligned}$$



2. Determine the value of the limit. If it exists, find the value. If it does not show why.

$$(a) \lim_{(x,y) \rightarrow (0,0)} \frac{y-x}{\sqrt{x^2+y^2}}$$

$$(b) \lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{\sqrt{x^2+y^2}}$$

(a) TRY THE PATH METHOD:

$$x=0: \lim_{y \rightarrow 0^+} f(0,y) = \lim_{y \rightarrow 0^+} \frac{y}{\sqrt{y^2}} = \lim_{y \rightarrow 0^+} \frac{y}{y} = 1$$

$$y=0: \lim_{x \rightarrow 0^+} f(x,0) = \lim_{x \rightarrow 0^+} \frac{-x}{\sqrt{x^2}} = \lim_{x \rightarrow 0^+} \frac{-x}{x} = -1$$

SINCE THESE TWO PATHS LEAD TO DIFFERENT VALUES, THE LIMIT DNE

---

(b) CONVERT TO POLAR

$$\begin{aligned} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{2xy}{\sqrt{x^2+y^2}} &= \lim_{r \rightarrow 0} \frac{2r \cos \theta \cdot r \sin \theta}{\sqrt{r^2}} \\ &= \lim_{r \rightarrow 0} \frac{2r^2 \cos \theta \sin \theta}{r} \\ &= \lim_{r \rightarrow 0} \underbrace{2r}_{\downarrow 0} \underbrace{\cos \theta \sin \theta}_{\text{BOUNDED}} \end{aligned}$$

YOU CAN USE THE SQUEEZE TO SHOW

$$0 \cdot \text{BOUNDED} = 0$$

3. If  $z = e^x \tan y$ , where  $x = s^2 + t^2$  and  $y = st$ , find  $\frac{\partial z}{\partial t}$  when  $s = 1$  and  $t = 0$ .

FORMULA: 
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial x} = e^x \tan y \quad \frac{\partial x}{\partial t} = 2t \quad \frac{\partial z}{\partial y} = e^x \sec^2 y \quad \frac{\partial y}{\partial t} = s$$

• PUT THESE TOGETHER

$$\frac{\partial z}{\partial t} = (e^x \tan y)(2t) + (e^x \sec^2 y)(s)$$

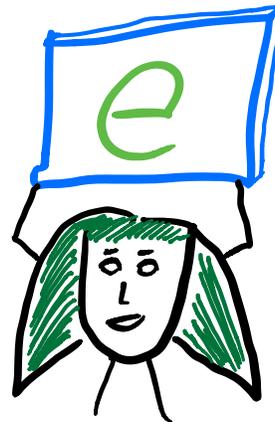
• TO EVALUATE AT  $s=1, t=0$  WE NEED TO FIND  $x$  AND  $y$

$$x = 1^2 + 0^2 = 1 \quad y = 1 \cdot 0 = 0$$

$$\frac{\partial z}{\partial t} \Big|_{\substack{s=1 \\ t=0}} = (e^1 \tan 0) \cdot 2 \cdot 0 + (e^1 \sec^2 0)(1)$$

$$= 0 + e$$

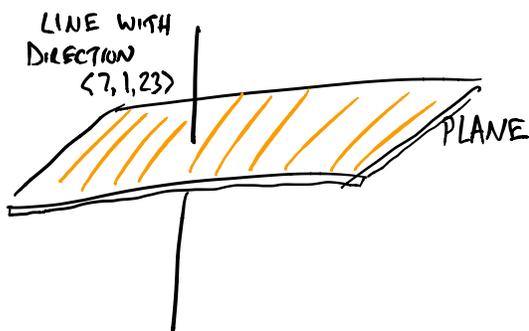
$$= e$$



4. Find an equation of the plane through the point  $(1, 5, 4)$  and perpendicular to the line  $x = 1 + 7t, y = t, z = 23t$ .

EQUATION OF PLANE:  $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

WHERE  $\vec{n} = \langle a, b, c \rangle$   
 "NORMAL VECTOR"



\* SINCE THE LINE IS  $\perp$  TO THE PLANE ITS DIRECTION VECTOR IS THE SAME AS THE NORMAL VECTOR TO THE PLANE

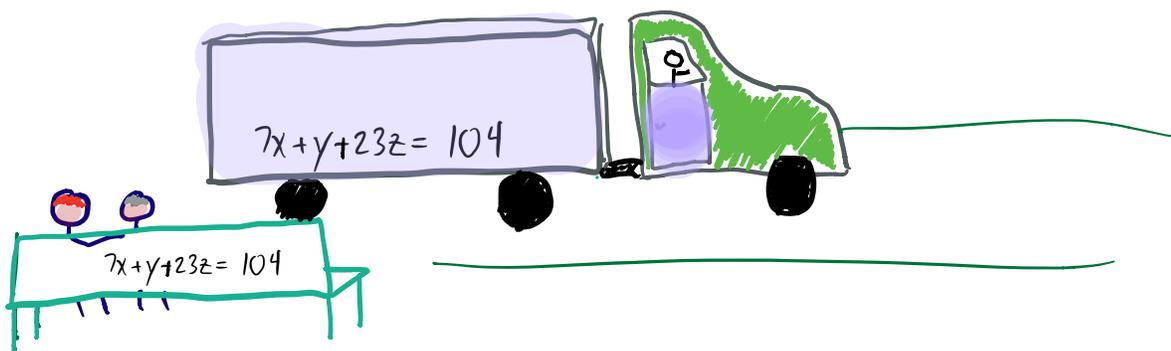
$\vec{n} = \langle 7, 1, 23 \rangle$  WITH  $P_0 = (1, 5, 4)$

EQUATION:

$$7(x-1) + 1(y-5) + 23(z-4) = 0$$

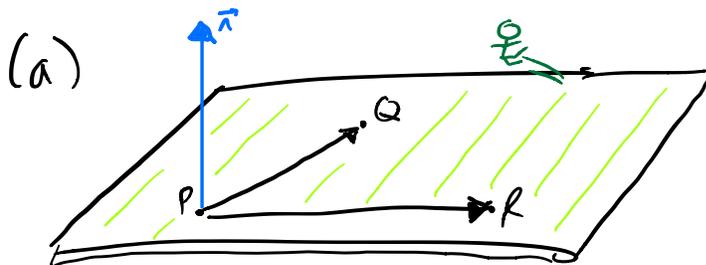
$$7x - 7 + y - 5 + 23z - 92 = 0$$

$$7x + y + 23z = 104$$



5. Let  $P(1, 2, 3)$ ,  $Q(1, -1, -2)$ , and  $R(0, 0, 0)$  be three points in  $R^3$ .

- Find an equation of the plane through  $P$ ,  $Q$ , and  $R$ .
- Find the area of the triangle formed by  $PQR$ .
- Find the equation of the line through  $P$  that is perpendicular to the plane from (a).



TO FWD  $\vec{n}$  WE NEED TO FWD  $\vec{PQ} \times \vec{PR}$ . SINCE  $\vec{PQ}$  AND  $\vec{PR}$  ARE ON THE PLANE THEIR CROSS PRODUCT WILL BE  $\perp$  TO IT

$$\vec{PQ} = \langle 1-1, -1-2, -2-3 \rangle = \langle 0, -3, -5 \rangle$$

$$\vec{PR} = \langle 0-1, 0-2, 0-3 \rangle = \langle -1, -2, -3 \rangle$$

$$\begin{aligned} \vec{PQ} \times \vec{PR} &= \begin{vmatrix} i & j & k \\ 0 & -3 & -5 \\ -1 & -2 & -3 \end{vmatrix} = i \begin{vmatrix} -3 & -5 \\ -2 & -3 \end{vmatrix} - j \begin{vmatrix} 0 & -5 \\ -1 & -3 \end{vmatrix} + k \begin{vmatrix} 0 & -3 \\ -1 & -2 \end{vmatrix} \\ &= i(9-10) - j(0-5) + k(0-3) \\ &= -1i + 5j - 3k \quad \text{so } \vec{n} = \langle -1, 5, -3 \rangle \end{aligned}$$

WE NEED TO PICK A POINT. LET'S USE  $R(0, 0, 0)$

$$\begin{aligned} a(x-x_0) + b(y-y_0) + c(z-z_0) &= 0 \\ -1(x-0) + 5(y-0) - 3(z-0) &= 0 \\ -x + 5y - 3z &= 0 \end{aligned}$$

5. Let  $P(1, 2, 3)$ ,  $Q(1, -1, -2)$ , and  $R(0, 0, 0)$  be three points in  $R^3$ .

- (a) Find an equation of the plane through  $P$ ,  $Q$ , and  $R$ .
- (b) Find the area of the triangle formed by  $PQR$ .
- (c) Find the equation of the line through  $P$  that is perpendicular to the plane from (a).

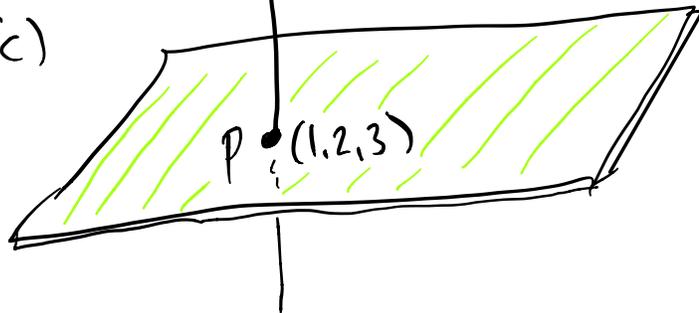
$$(b) \text{ AREA} = \frac{1}{2} |\vec{PQ} \times \vec{PR}|$$

$$|\vec{PQ} \times \vec{PR}| = | \langle -1, 5, -3 \rangle | = \sqrt{(-1)^2 + (5)^2 + (-3)^2} = \sqrt{35}$$

SO THE AREA IS

$$A = \frac{1}{2} \sqrt{35}$$

(c)



THE DIRECTION VECTOR OF THE LINE IS THE SAME AS THE  
NORMAL VECTOR  $\langle -1, 5, -3 \rangle$

$$\begin{aligned} L &= \vec{v}t + P = \langle -1, 5, -3 \rangle t + \langle 1, 2, 3 \rangle \\ &= \langle -t+1, 5t+2, -3t+3 \rangle \end{aligned}$$



6. Given  $xy + e^{xyz} - z - e^y = 0$ , use implicit partial derivative to find  $\frac{\partial z}{\partial x}$  at the point  $P(1, 1, 1)$ .

SINCE THIS IS IMPLICIT WE CAN USE

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{WHERE } F(x, y, z) = xy + e^{xyz} - z - e^y$$

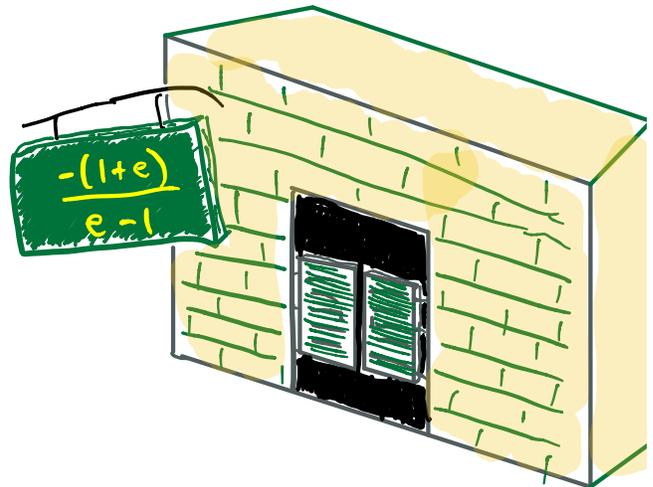
$$\bullet F_x = y + e^{xyz}(yz) - 0 - 0 = y + yze^{xyz}$$

$$\bullet F_z = 0 + e^{xyz}(xy) - 1 - 0 = xy e^{xyz} - 1$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(y + yze^{xyz})}{xy e^{xyz} - 1}$$

AT (1, 1, 1)?

$$= -\frac{(1 + e^1)}{e - 1}$$



7. Let  $f(x, y) = x^2 - 5xy$

- Find  $\nabla f(x, y)$ .
- Find the directional derivative at  $(2, 1)$  in the direction of  $\vec{v} = -i + 3j$ .
- Find the equation of the tangent line on  $f(x, y)$  at  $(2, 1)$ .
- Find the linearization  $L(x, y)$  of  $f$  at  $(2, 1)$ .
- Use the linearization to approximate  $f(1.9, 0.9)$ .

(a) FIND  $\nabla f(x, y)$ :

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x - 5y, -5x \rangle$$

$$\begin{aligned} (b) D_{\vec{v}} f(x, y) &= \frac{\nabla f(2, 1) \cdot \vec{v}}{|\vec{v}|} = \frac{\langle 4 - 5, -10 \rangle \cdot \langle -1, 3 \rangle}{\sqrt{(-1)^2 + (3)^2}} \\ &= \frac{\langle -1, -10 \rangle \cdot \langle -1, 3 \rangle}{\sqrt{10}} \\ &= \frac{-1(-1) - 10(3)}{\sqrt{10}} \\ &= \frac{-29}{\sqrt{10}} \end{aligned}$$

(c) FORMULA:  $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$$* f_x(x, y) = 2x - 5y, \quad f_x(2, 1) = 4 - 5 = -1$$

$$* f_y(x, y) = -5x, \quad f_y(2, 1) = -10$$

$$* z_0 = f(2, 1) = 2^2 - 5(2)(1) = 4 - 10 = -6$$

$$z+6 = -1(x-2) - 10(y-1)$$

$$z+6 = -x+2-10y+10$$

$$x+10y+z = 6$$

$$(d) \quad L(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

$$= f_x(2,1)(x-2) + f_y(2,1)(y-1) + (-6)$$

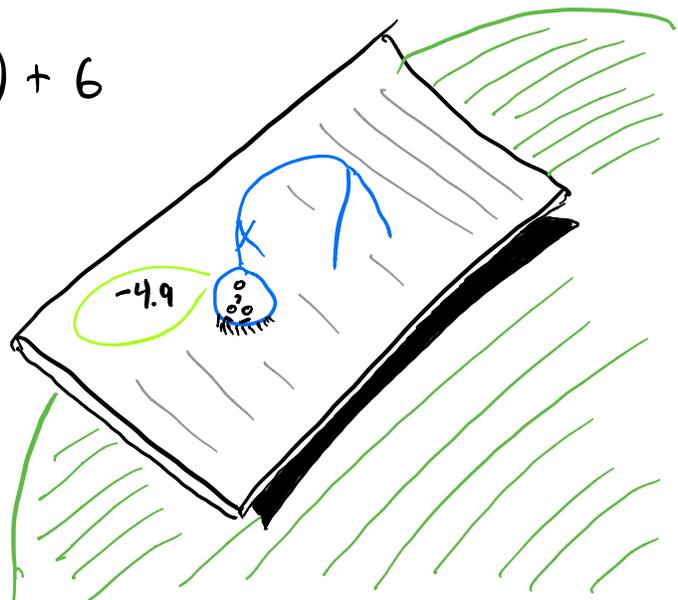
$$= -1(x-2) - 10(y-1) + (-6)$$

$$= -x+2-10y+10+(-6)$$

$$L(x,y) = -x-10y+6$$

$$(e) \quad L(1.9, .9) = -1.9 - 10(.9) + 6$$

$$= -4.9$$



8. Find the local max, min, and saddle points for  $f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - xy + 4$  (if any exist).

1. FIND  $f_x$  AND  $f_y$

$$f_x = x^2 - y \quad f_y = y^2 - x$$

2. SOLVE  $f_x = 0$  AND  $f_y = 0$

$$x^2 - y = 0 \quad y^2 - x = 0$$
$$\Rightarrow y = x^2$$

PLUG  $y = x^2$  INTO  $f_y = 0$  AND SOLVE

$$y^2 - x = 0$$
$$\Rightarrow (x^2)^2 - x = 0$$
$$\Rightarrow x^4 - x = 0$$
$$\Rightarrow x(x^3 - 1) = 0$$
$$\Rightarrow x(x-1)(x^2+x+1) = 0$$
$$\Rightarrow x = 0, x = 1$$

FOR EACH  $x$  VALUE FIND ITS CORRESPONDING  $y$ -VALUE

$$x = 0, y = 0^2 = 0 \quad (0, 0)$$

$$x = 1, y = 1^2 = 1 \quad (1, 1)$$

3. FIND D

$$D = f_{xx} \cdot f_{yy} - [f_{xy}]^2$$

$$f_{xx} = 2x, \quad f_{yy} = 2y, \quad f_{xy} = -1$$

$$D = (2x)(2y) - (-1)^2 = 4xy - 1$$

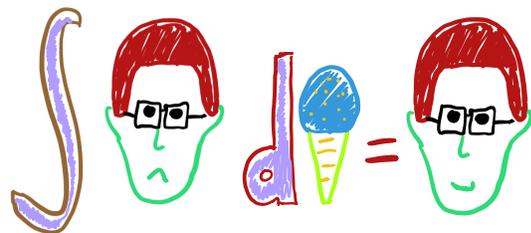
4. PLUG CRITICAL POINTS INTO D

$$D(0,0) = 4(0)(0) - 1 = -1$$

SINCE  $D < 0$ ,  $f(0,0)$  IS A SADDLE POINT

$$D(1,1) = 4(1)(1) - 1 = 3 > 0 \quad \text{AND} \quad f_{xx}(1,1) = 2(1) = 2 > 0$$

SINCE  $D > 0$  AND  $f_{xx} > 0$   $f(1,1)$  IS A LOCAL MIN



9. Find the local max, min, and saddle points for  $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$ .

1. FIND  $f_x$  AND  $f_y$

$$f_x = 6x^2 + y^2 + 10x \quad f_y = 2xy + 2y$$

2. SOLVE  $f_x = 0$  AND  $f_y = 0$

$$6x^2 + y^2 + 10x = 0 \quad 2xy + 2y = 0$$

$\Downarrow$

$$2y(x+1) = 0$$

$$y = 0 \text{ OR } x = -1$$

PLUG  $y=0$  INTO  $6x^2 + y^2 + 10x = 0$  TO FIND  $x$

$$6x^2 + 0 + 10x = 0$$

$$2x(3x+5) = 0$$

$$x=0, x=-5/3 \implies (0,0) \text{ AND } (-5/3, 0)$$

PLUG  $x=-1$  INTO  $6x^2 + y^2 + 10x = 0$  TO FIND  $y$

$$6(-1)^2 + y^2 + 10(-1) = 0$$

$$y^2 - 4 = 0$$

$$y = \pm 2 \implies (-1, -2) \text{ AND } (-1, 2)$$

3. FIND  $\Delta$

$$D = f_{xx} f_{yy} - [f_{xy}]^2$$

$$f_{xx} = 12x + 10, \quad f_{yy} = 2x + 2, \quad f_{xy} = 2y$$

$$D = (12x + 10)(2x + 2) - [2y]^2$$

4. CHECK POINTS

- $D(0,0) = (10)(2) - 0^2 = 20 > 0$ ,  $f_{xx}(0,0) = 10 > 0$

SO  $(0,0)$  IS A LOCAL MIN

- $D(-\frac{5}{3}, 0) = (12 \cdot -\frac{5}{3} + 10)(2 \cdot -\frac{5}{3} + 2) - 0^2 = \frac{40}{3} > 0$

AND  $f_{xx}(-\frac{5}{3}, 0) = -10 < 0$

SO  $(-\frac{5}{3}, 0)$  IS A LOCAL MAX

- $D(-1, -2) = (12(-1) + 10)(2(-1) + 2) - (-4)^2 = -16 < 0$

SADDLE POINT

- $D(-1, 2) = (12(-1) + 10)(2 \cdot (-1) + 2) - (4)^2 = -16 < 0$

SADDLE POINT

10. Use Lagrange Multipliers to find the maximum and minimum of  $f(x, y) = x^2y$  subject to the constraint  $\underbrace{x^2 + y^2 = 1}_g$

1. FWD of AND  $\nabla g$

$$\nabla f = \langle f_x, f_y \rangle = \langle 2xy, x^2 \rangle$$

$$\nabla g = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle$$

2. SET UP  $\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases}$

$$2xy = \lambda \cdot 2x, \quad x^2 = \lambda \cdot 2y$$

There are a couple of ways to start this problem. You can solve one equation for lambda and plug it into the other equation. Or you can set the first equation equal to 0, factor, and solve.

$$2xy = \lambda 2x \Rightarrow 2xy - 2x\lambda = 0$$

$$2x(y - \lambda) = 0$$

$$x = 0 \text{ OR } y = \lambda$$

These are two cases. They have to be taken separately.

CASE 1:  $x=0$  SINCE WE KNOW  $x=0$ , WE CAN PLUG

IT INTO  $x^2 + y^2 = 1$

$$0^2 + y^2 = 1 \rightarrow y = \pm 1$$

POINTS:  $(0, -1)$  AND  $(0, 1)$

CASE 2:  $y = \lambda$  LETS PLUG  $y = \lambda$  INTO THE OTHER EQUATION  $x^2 = \lambda \cdot 2y$

$$x^2 = y \cdot 2y = 2y^2$$

$$x^2 = 2y^2$$

REPLACE  $x^2$  WITH  $2y^2$  IN  $x^2 + y^2 = 1$  AND SOLVE

$$2y^2 + y^2 = 1$$

$$3y^2 = 1$$

$$y = \pm \sqrt{1/3}$$

USE  $x^2 = 2y^2$  TO FIND CORRESPONDING X-VALUES

$$\bullet y = -\sqrt{1/3} \rightarrow x^2 = 2\left(-\sqrt{1/3}\right)^2 = \frac{2}{3}$$

$$\rightarrow x = \pm \sqrt{2/3} \quad \left(\sqrt{2/3}, -\sqrt{1/3}\right)$$

$$\left(-\sqrt{2/3}, -\sqrt{1/3}\right)$$

$$\bullet y = \sqrt{1/3} \rightarrow x^2 = 2\left(\sqrt{1/3}\right)^2 = \frac{2}{3}$$

$$\rightarrow x = \pm \sqrt{2/3}$$

$$\left(\sqrt{2/3}, \sqrt{1/3}\right)$$

$$\left(-\sqrt{2/3}, \sqrt{1/3}\right)$$

ALL SIX OF OUR POINTS

$$\left(\pm\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) \quad \left(\pm\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right) \quad (0, 1) \quad (0, -1)$$

Now CHECK EACH OF THESE POINTS IN  $f(x, y) = x^2 y$

$$f\left(\pm\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = .385 \quad f\left(\pm\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right) = -.385$$

$$f(0, 1) = 0 \quad f(0, -1) = 0$$

---

MAX OCCURS AT  $\left(\pm\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right)$

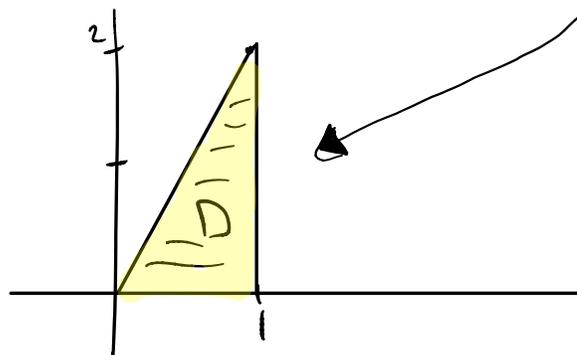
MIN OCCURS AT  $\left(\pm\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right)$

TOO EXHAUSTED  
FOR A PICTURE ...

;)

11. Evaluate  $\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$  by changing the order of integration.

1. START BY SKETCHING THE DOMAIN  $D = \begin{cases} 0 \leq y \leq 2 \\ y/2 \leq x \leq 1 \end{cases}$



\*  $\frac{y}{2} = x \rightarrow y = 2x$

CHANGE DOMAIN TO HAVE  $D = \begin{cases} a \leq x \leq b \\ L(x) \leq y \leq U(x) \end{cases}$

$$D = \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 2x \end{cases}$$

•  $\int_0^2 \int_{y/2}^1 e^{x^2} dx dy = \int_0^1 \int_0^{2x} e^{x^2} dy dx$

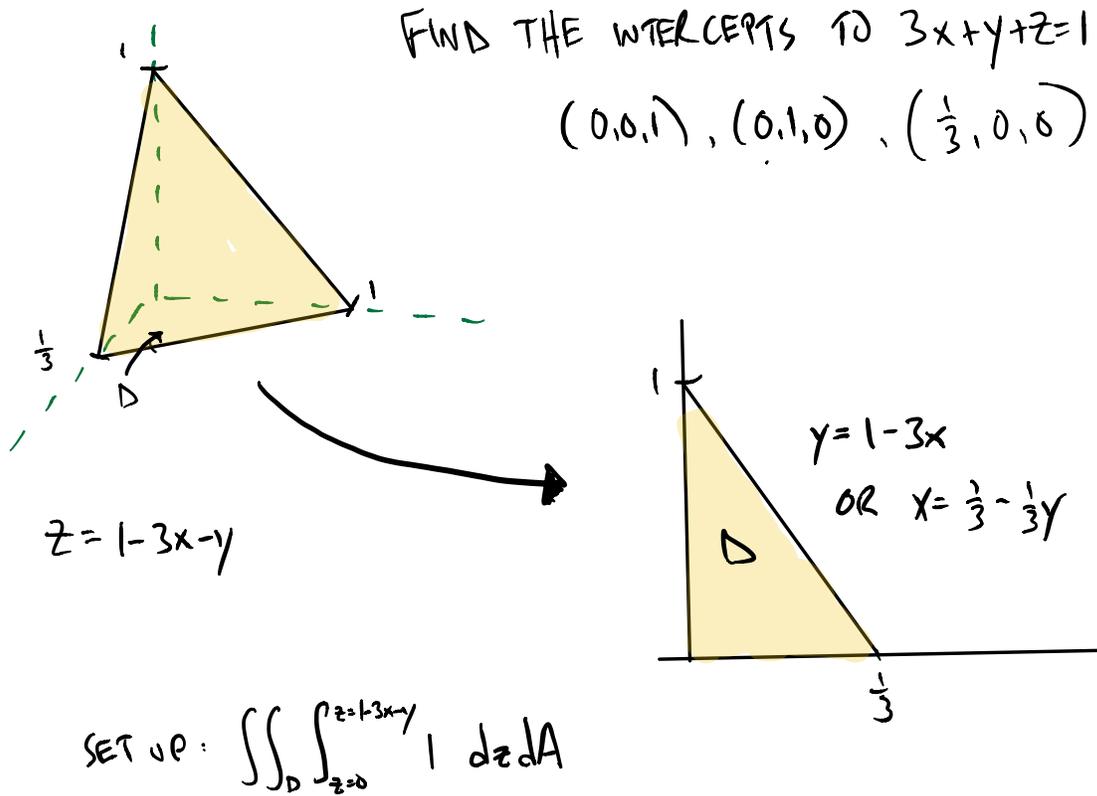
• INSIDE:  $\int_0^{2x} e^{x^2} dy = ye^{x^2} \Big|_{y=0}^{y=2x} = 2xe^{x^2} - 0e^{x^2} = 2xe^{x^2}$

• OUTSIDE  $\int_0^1 2xe^{x^2} dx$

LET  $u = x^2, du = 2x dx$

$$\int_{u=0}^{u=1} e^u du = e^u \Big|_0^1 = e^1 - e^0 = e - 1$$

12. Setup the triple integral in the order of  $dz dx dy$  and again as  $dz dy dx$  to find the volume of the solid tetrahedron which is bounded by  $3x+y+z=1$  and the coordinate planes (i.e., the first octant).

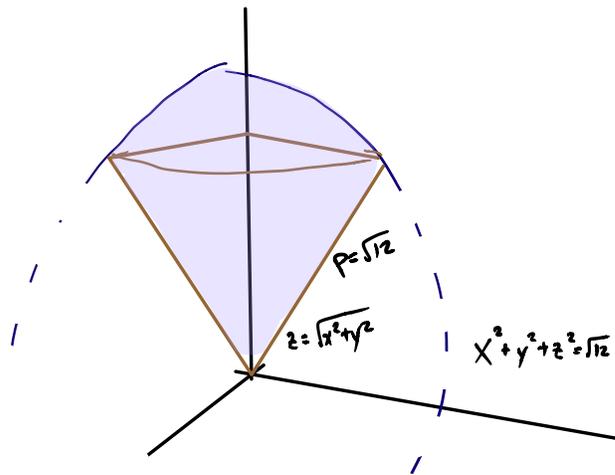


$$dz dy dx: D = \begin{cases} 0 \leq x \leq \frac{1}{3} \\ 0 \leq y \leq 1-3x \end{cases} \Rightarrow \int_0^{\frac{1}{3}} \int_0^{1-3x} \int_0^{1-3x-y} 1 dz dy dx$$

$$dz dx dy: D = \begin{cases} 0 \leq y \leq 1 \\ 0 \leq x \leq \frac{1}{3} - \frac{1}{3}y \end{cases} \Rightarrow \int_0^1 \int_0^{\frac{1}{3} - \frac{1}{3}y} \int_0^{1-3x-y} 1 dz dx dy$$

13. Set up, do not evaluate the triple integral in rectangular, cylindrical, and spherical coordinates to find the volume of the solid in the first octant bounded above by  $x^2 + y^2 + z^2 \stackrel{=12}{=}$  and bounded below by  $z = \sqrt{x^2 + y^2}$ .

Note:  $z = \sqrt{x^2 + y^2}$  is a cone at  $\phi = \pi/4$

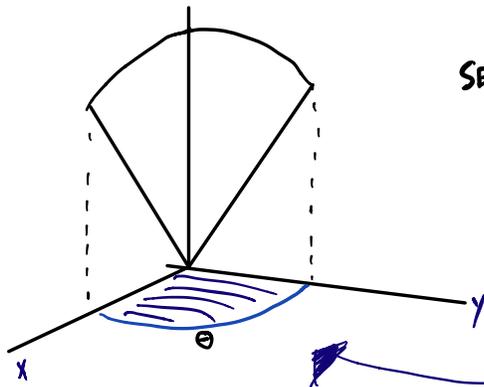


RECTANGULAR: MAKE SURE ALL FUNCTIONS ARE IN TERMS OF  $x$  AND  $y$

$$\text{TOP FUNCTION: } z = \sqrt{12 - x^2 - y^2}$$

$$\text{BOTTOM FUNCTION: } z = \sqrt{x^2 + y^2}$$

- FIND THE PROJECTION ONTO  $XY$  PLANE TO FIND  $D$



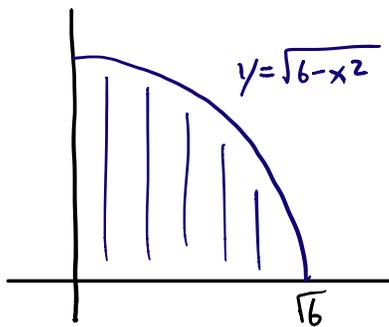
SET  $z = \sqrt{12 - x^2 - y^2}$  EQUAL TO  $z = \sqrt{x^2 + y^2}$

$$\sqrt{12 - x^2 - y^2} = \sqrt{x^2 + y^2}$$

$$12 - x^2 - y^2 = x^2 + y^2$$

$$12 = 2x^2 + 2y^2$$

$$6 = x^2 + y^2$$



$$D = \begin{cases} 0 \leq x \leq \sqrt{6} \\ 0 \leq y \leq \sqrt{6 - x^2} \end{cases}$$

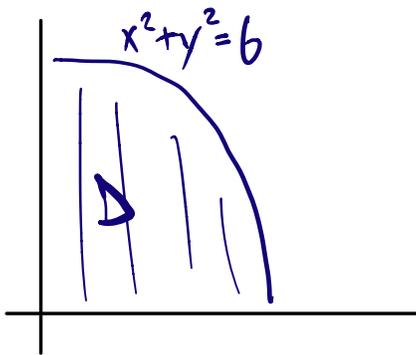
$$V = \int_0^{\sqrt{6}} \int_0^{\sqrt{6-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{12-x^2-y^2}} 1 \, dz \, dy \, dx$$



FASCINATING

# CYLINDRICAL:

The only difference between rectangular and cylindrical is the domain D must be written in polar form.



$$D = \begin{cases} 0 \leq r \leq \sqrt{6} \\ 0 \leq \theta \leq \pi/2 \end{cases}$$

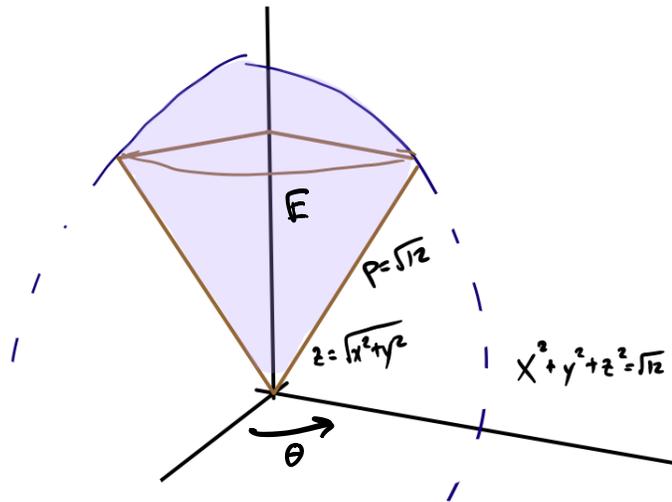
$$\text{WITH } \sqrt{x^2+y^2} \leq z \leq \sqrt{12-x^2-y^2}$$

NEED TO CONVERT TO POLAR

$$\sqrt{r^2} \leq z \leq \sqrt{12-r^2}$$

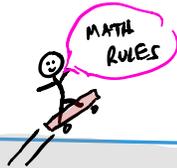
$$\int_0^{\pi/2} \int_0^{\sqrt{6}} \int_r^{\sqrt{12-r^2}} |r| dz dr d\theta$$

SPHERICAL: IT'S BEST TO GO BACK TO THE ORIGINAL PICTURE



$$E = \begin{cases} 0 \leq \theta \leq \pi/2 \\ 0 \leq \rho \leq \sqrt{2} \\ 0 \leq \phi \leq \pi/4 \end{cases}$$

\*  $z = \sqrt{x^2 + y^2}$  IS A CONE WITH  $\phi = \pi/4$



$$V = \int_0^{\pi/2} \int_0^{\sqrt{2}} \int_0^{\pi/4} 1 \cdot \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

14. Find the Jacobian for the transformation  $x = u^2v + v^2$  and  $y = uv^2 - u^2$ .

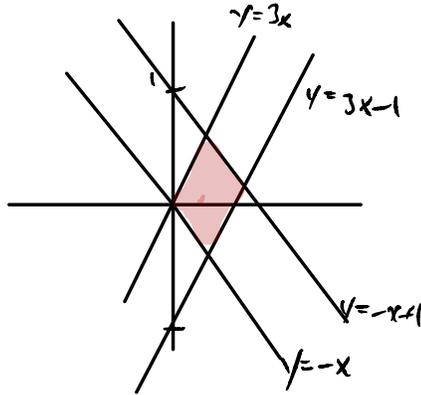
JACOBIAN J IS DEFINED BY

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\text{SO... } J = \begin{vmatrix} 2uv & u^2+2v \\ v^2-2u & 2uv \end{vmatrix} = (2uv)^2 - (u^2+2v)(v^2-2u)$$

15. Evaluate  $\int \int_D (3x - y)^{3/2} (x + y)^5 dA$  where  $D$  is the region bounded by  $y = -x$ ,  $y = -x + 1$ ,  $y = 3x$ , and  $y = 3x - 1$ . Use the change of variables  $u = 3x - y$  and  $v = x + y$ .

ORIGINAL DOMAIN :



There are usually two different ways to approach this problem. Either way the end goal is to get a domain in terms of  $u$  and  $v$ .

First way. Do you see how the equations for the bounds look like our substitutions?

IF  $y = -x$ , THEN  $x + y = 0$ . BUT SINCE  $v = x + y$  WE GET  $v = 0$

IF  $y = -x + 1$ , THEN  $x + y = 1$ . SINCE  $v = x + y$  WE GET  $v = 1$

IF  $y = 3x$ , THEN  $3x - y = 0$ . SINCE  $u = 3x - y$ , WE GET  $u = 0$

IF  $y = 3x - 1$ , THEN  $3x - y = 1$ . SINCE  $u = 3x - y$ , WE GET  $u = 1$

OUR NEW DOMAIN IS

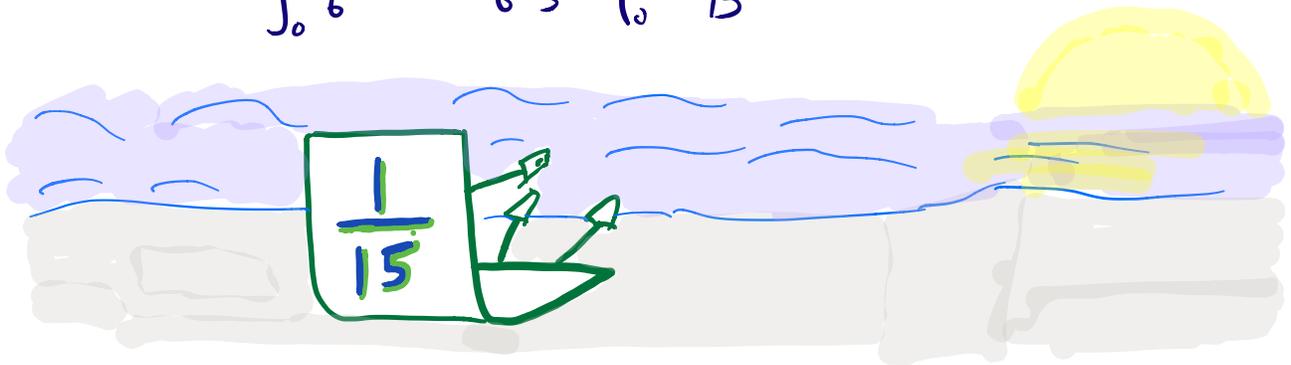
$$\begin{cases} 0 \leq u \leq 1 \\ 0 \leq v \leq 1 \end{cases}$$

WITH OUR NEW BOUNDS THE INTEGRAL BECOMES

$$\int_0^1 \int_0^1 u^{3/2} v^5 dr du$$

$$\text{INSIDE: } \int_0^1 u^{3/2} v^5 dr = u^{3/2} \cdot \frac{1}{6} v^6 \Big|_0^1 = \frac{1}{6} u^{3/2}$$

$$\text{OUTSIDE: } \int_0^1 \frac{1}{6} u^{3/2} du = \frac{1}{6} \cdot \frac{2}{5} u^{5/2} \Big|_0^1 = \frac{1}{15}$$



The second way to get the bounds for  $u$  and  $v$  would be

1. Solve  $u = 3x - y$  and  $v = x + y$  for  $x$  and  $y$ . Do this by solving the system of equations

$$\begin{cases} 3x - y = u \\ x + y = v \end{cases}$$

$$\text{ADD } 4x = u + v \rightarrow \boxed{x = \frac{1}{4}u + \frac{1}{4}v}$$

PLUG THIS INTO  $x + y = v$  TO FIND  $y$

$$\frac{1}{4}u + \frac{1}{4}v + y = v \rightarrow \boxed{y = -\frac{1}{4}u + \frac{3}{4}v}$$

2. Plug these two equations into all four bounds to get our new bounds. I'll do the first.

$$y = 3x - 1$$

$$-\frac{1}{4}u + \frac{3}{4}v = 3\left(\frac{1}{4}u + \frac{1}{4}v\right) - 1$$

$$-\frac{1}{4}u + \frac{3}{4}v = \frac{3}{4}u + \frac{3}{4}v - 1$$

$$-u = -1$$
$$u = 1$$

Do this for the remaining three bound equations and you will get

$$\begin{array}{l} u=0 \\ v=0 \\ v=1 \end{array}$$

16. Evaluate the line integral  $\int_C (xz + 2y) dS$ , where  $C$  is the line segment from  $(0, 1, 0)$  to  $(1, 0, 2)$ .

1. FIND THE PARAMETRIC EQUATIONS OF THE LINE

SEGMENT

$$P = (0, 1, 0) \quad Q = (1, 0, 2)$$

$$\text{DIRECTION: } \vec{PQ} = \langle 1-0, 0-1, 2-0 \rangle = \langle 1, -1, 2 \rangle$$

$$\text{INITIAL POINT: } P = \langle 0, 1, 0 \rangle$$

$$\text{VECTOR EQUATION: } \langle 1, -1, 2 \rangle t + \langle 0, 1, 0 \rangle$$

$$= \langle t, -t+1, 2t \rangle$$

$$x = t, \quad y = -t+1, \quad z = 2t \quad 0 \leq t \leq 1$$

$$\text{FORMULA: } \int f(x, y, z) dS = \int f(x(t), y(t), z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

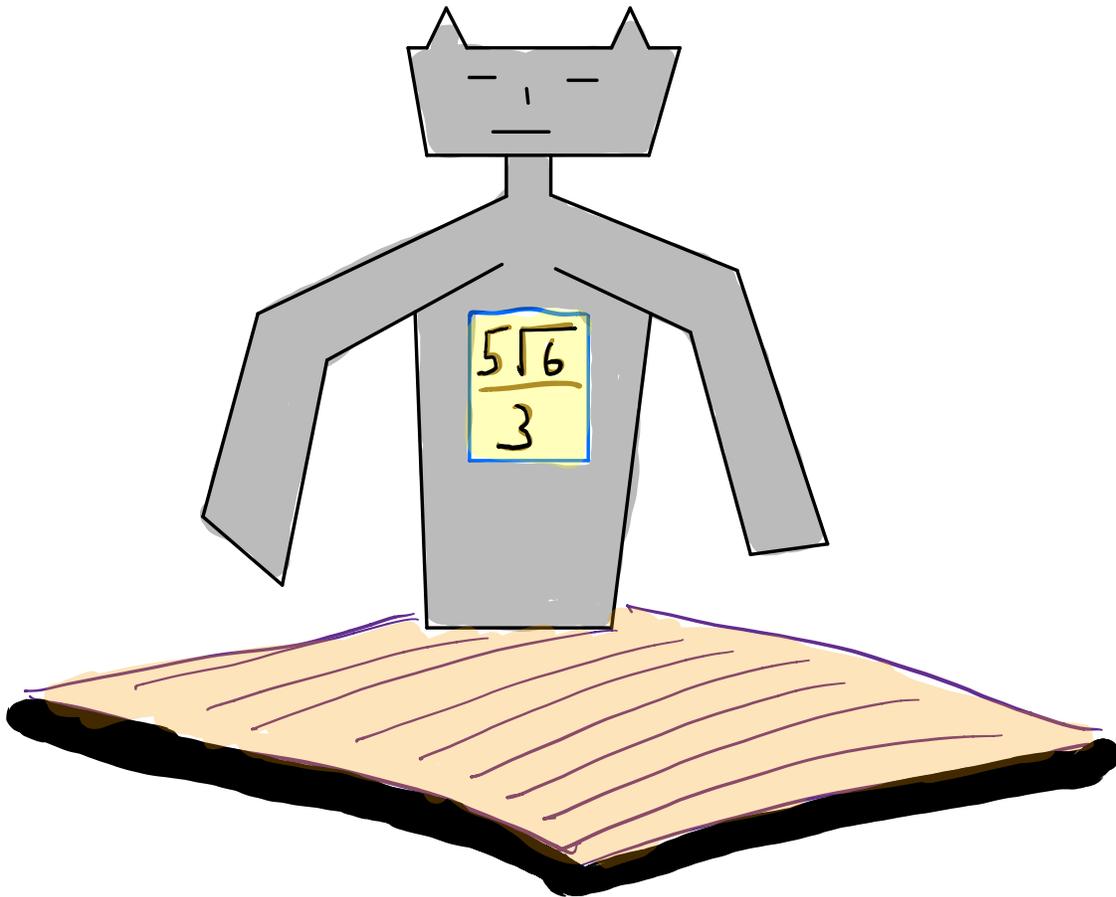
$$\int_0^1 [(t)(2t) + 2(-t+1)] \cdot \sqrt{(1)^2 + (-1)^2 + (2)^2} dt$$

$$= \int_0^1 [2t^2 - 2t + 2] \cdot \sqrt{6} \, dt$$

$$= \sqrt{6} \int_0^1 2t^2 - 2t + 2 \, dt$$

$$= \sqrt{6} \left[ \frac{2}{3}t^3 - t^2 + 2t \right]_0^1$$

$$= \frac{5\sqrt{6}}{3}$$



17. Let  $F(x, y) = (xy^2 + 2y)i + (x^2y + 2x + 2)j$  be a vector field.

(a) Show  $F$  is conservative.

(b) Find  $f$  such that  $\nabla f = F$ .

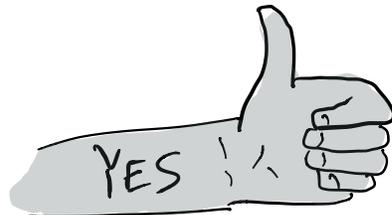
(c) Evaluate  $\int_C F dr$  where  $C$  is defined by  $r(t) = \langle e^t, 1 + t \rangle$ ,  $0 \leq t \leq 1$ .

(d) Evaluate  $\int_C F dr$  where  $C$  is a closed curve  $r(t) = \langle 2 \sin(t), \cos(t) \rangle$ ,  $0 \leq t \leq 2\pi$ .

$$a) P = xy^2 + 2y \quad \text{AND} \quad Q = x^2y + 2x + 2$$

IF  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$  THEN  $F$  IS CONSERVATIVE

$$\frac{\partial P}{\partial y} = 2xy + 2 \quad \frac{\partial Q}{\partial x} = 2xy + 2$$



---

b) FIND  $f$  SUCH THAT  $F = \nabla f$

IF THIS  $f$  EXISTS THEN  $P = f_x$  AND  $Q = f_y$

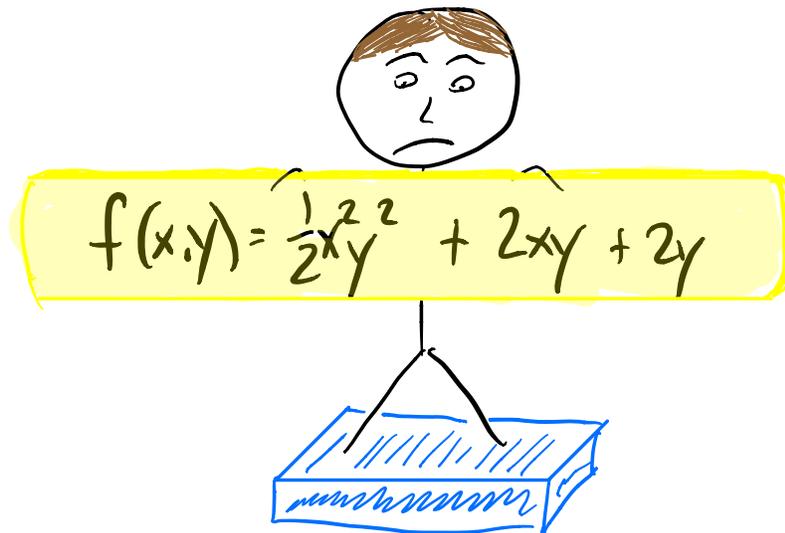
TO GET  $f$  WE MUST EVALUATE  $f = \int P dx$

$$\text{AND} \quad f = \int Q dy$$

$$f = \int P dx = \int xy^2 + 2y dx = \frac{1}{2}x^2y^2 + 2xy + g(y)$$

$$f = \int Q dy = \int x^2y + 2x + 2 dy = \frac{1}{2}x^2y^2 + 2xy + 2y + h(x)$$

SINCE THESE MUST MATCH WE USE  $g(y) = 2y$  AND  $h(x) = 0$



(c) Evaluate  $\int_C F dr$  where  $C$  is defined by  $r(t) = \langle e^t, 1+t \rangle$ ,  $0 \leq t \leq 1$ .

WE WILL USE THE FUNDAMENTAL THEOREM OF LINE INTEGRALS

$$\begin{aligned}\int_C F dr &= f(r(b)) - f(r(a)) \\ &= f(r(1)) - f(r(0))\end{aligned}$$

•  $r(1) = \langle e, 2 \rangle \rightarrow (e, 2)$

•  $r(0) = \langle 1, 1 \rangle \rightarrow (1, 1)$

$$= f(e, 2) - f(1, 1)$$

\* RECALL:  $f(x) = \frac{1}{2}x^2y^2 + 2xy + 2y$ .

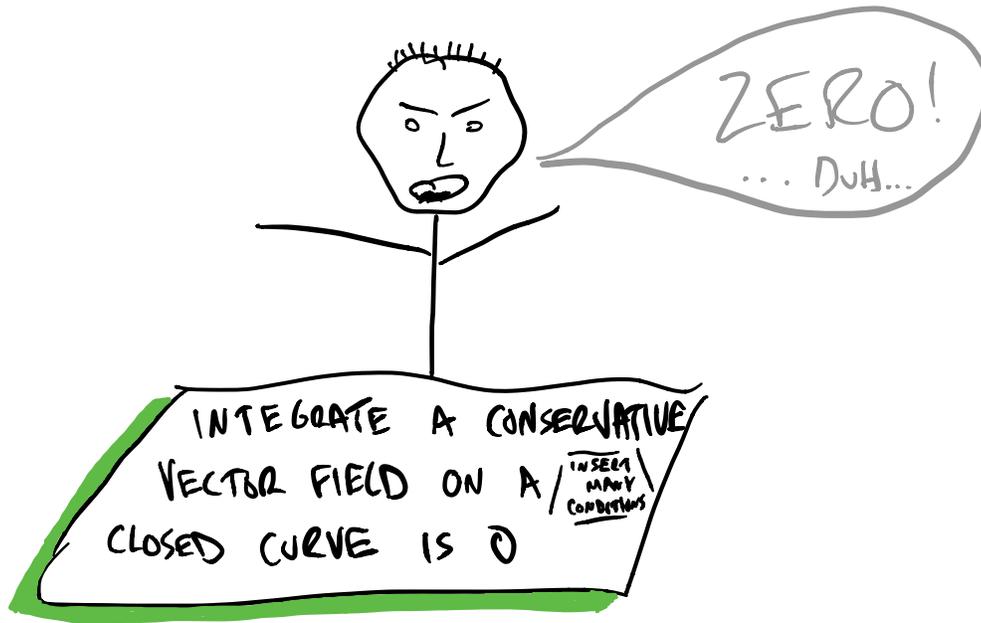
$$f(e, 2) = 2e^2 + 4e + 4$$

$$f(1, 1) = \frac{1}{2} + 2 + 2 = \frac{9}{2}$$

$$\text{FINAL: } (2e^2 + 4e + 4) - \left(\frac{9}{2}\right)$$

$$= 2e^2 + 4e - \frac{1}{2}$$

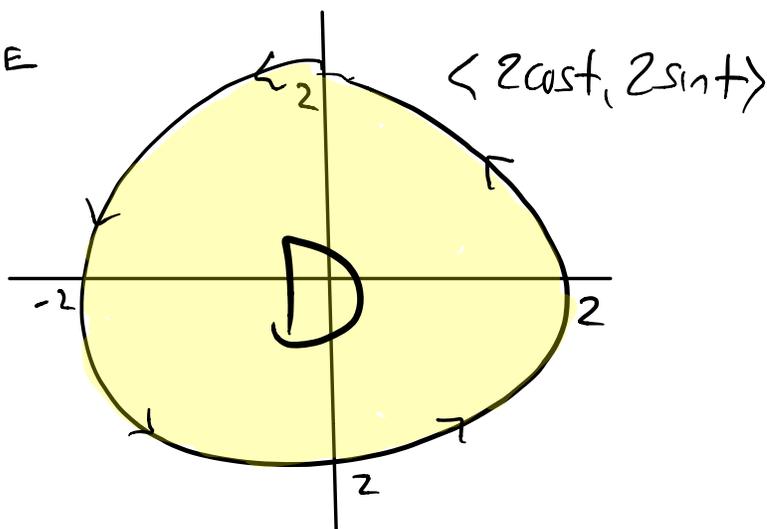
(d) Evaluate  $\int_C F dr$  where  $C$  is a closed curve  $r(t) = \langle 2\sin(t), \cos(t) \rangle$ ,  $0 \leq t \leq 2\pi$ .



18. Use Green's Theorem to evaluate  $\int_C -y^3 dx + x^3 dy$  where  $C$  is a circle given by  $r(t) = \langle 2 \cos t, 2 \sin t \rangle$ ,  $0 \leq t \leq 2\pi$ .

\*  $P(x,y) = -y^3$  AND  $Q(x,y) = x^3$

SKETCH THE CURVE



GREEN'S THEOREM:

$$\int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\frac{\partial P}{\partial y} = -3y^2 \quad \frac{\partial Q}{\partial x} = 3x^2 \quad \rightarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 - (-3y^2) = 3x^2 + 3y^2$$

$$\boxed{\iint_D 3x^2 + 3y^2 dA}$$

NEXT WE NEED TO FIND D. SINCE ITS A CIRCLE

LETS CONVERT TO POLAR

$$0 \leq r \leq 2$$

$$0 \leq \theta \leq 2\pi$$

\* NOTE  
 $x^2 + y^2 = r^2$

$$\int_0^{2\pi} \int_0^2 3r^2 \cdot r dr d\theta$$

INSIDE:  $\int_0^2 3r^3 dr = \frac{3}{4} r^4 \Big|_0^2 = 12$

OUTSIDE:  $\int_0^{2\pi} 12 d\theta = 12\theta \Big|_0^{2\pi} = 24\pi$

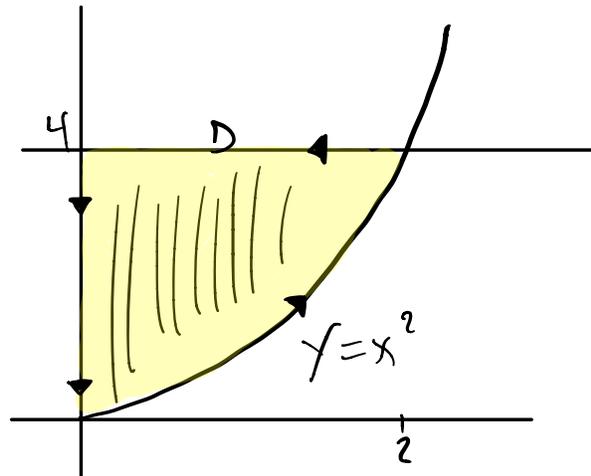
$$24\pi$$

19. Use Green's Theorem to evaluate  $\int_C (e^x + y^2) dx + (e^y + x^2) dy$  where  $C$  is the positively oriented boundary of the region in the first quadrant bounded by  $y = x^2$  and  $y = 4$ .

$$* P = e^x + y^2 \quad Q = e^y + x^2$$

$$\frac{\partial P}{\partial y} = 2y \quad \frac{\partial Q}{\partial x} = 2x$$

\* SKETCH CURVE



$$\text{GREEN'S THEOREM: } \int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\text{FIND } D: \begin{cases} 0 \leq x \leq 2 \\ x^2 \leq y \leq 4 \end{cases}$$



$$\int_0^2 \int_{x^2}^4 (2x - 2y) dy dx$$

# EVALUATE

INSIDE:

$$\int_{x^2}^4 2x - 2y \, dy = 2xy - y^2 \Big|_{y=x^2}^{y=4}$$

$$= [2x(4) - (4)^2] - [2x(x^2) - (x^2)^2]$$

$$= 8x - 16 - 2x^3 + x^4$$

OUTSIDE:  $\int_0^2 8x - 16 - 2x^3 + x^4 \, dx = 4x^2 - 16x - \frac{1}{2}x^4 + \frac{1}{5}x^5 \Big|_0^2$

$$= -88/5$$

