Derivation of the Hybrid NK Wage Phillips Curve with Backward Wage Indexation

This derivation follows Holmberg (2006). Let $x$ represent the wage set by firms that are able to adjust their wages and $\theta$ represent the proportion of firms that are not able to reset their wages. Then,

$$\hat{W}_t = (1-\theta)x_t + \theta \hat{W}_{t-1}. \quad (A1)$$

Subtracting $\hat{W}_{t-1}$ from both sides of (A1) yields the following expression for wage inflation:

$$\pi^w_t = \hat{W}_t - \hat{W}_{t-1} = (1-\theta)(x_t - \hat{W}_{t-1}). \quad (A2)$$

Let $\omega$ represent the proportion of firms resetting wages that index their wages to lagged wage inflation, $x^b_t$ represent the wage set by backward-looking firms, and $x^f_t$ represent the wage set by forward-looking firms. Then

$$x_t = \omega x^b_t + (1-\omega)x^f_t. \quad (A3)$$

Solving (A1) for $x_t$ and substituting this expression into (A3) yields

$$\frac{1}{1-\theta} \hat{W}_t - \frac{\theta}{1-\theta} \hat{W}_{t-1} = \omega x^b_t + (1-\omega)x^f_t. \quad (A3)$$

Subtracting $\hat{W}_t$ from both sides of this equation yields

$$\frac{\theta}{1-\theta} \pi^w_t = \omega (x^b_t - \hat{W}_t) + (1-\omega)(x^f_t - \hat{W}_t). \quad (A4)$$

Let $\beta$ represent the discount factor and $\hat{W}^*_{t+j}$ represent firms’ expectation of the optimal wage in period $t+j$ when they set wages in period $t$. Then the wage set by forward-looking firms can be expressed as,

$$x^f_t = (1-\beta \theta) \sum_{j=0}^{\infty} \beta^j \theta^j E_t \hat{W}^*_{t+j} \quad (A5)$$

$$x^f_t = (1-\beta \theta) \hat{W}^*_t + \beta \theta E_t x^f_{t+1}. \quad (A6)$$
Backward-looking firms are assumed to set wages equal to the wage set last period by firms that adjusted wages, indexed to lagged wage inflation, where the $\lambda_i$’s represent the weight given to each lag of wage inflation in the indexation process. Thus, the wage set by backward-looking firms is

$$x_i^b = x_{i-1} + \sum_{i=1}^{T} \lambda_i \pi_{i-1}^w.$$  \hfill (A7)

Combining (A1) and (A7) yields

$$x_i^b - \hat{W}_i = -\pi_i^w + \frac{\theta}{1-\theta} \pi_{i-1}^w + \sum_{i=1}^{T} \lambda_i \pi_{i-1}^w.$$  \hfill (A8)

Substituting (A15) from Campbell (2019) into (A6) yields the relationship,

$$x_i^f = (1-\beta\theta)\left(\hat{W}_i - \frac{e_{wa} - e_u}{e_{ww}} du_i \right) + \beta\theta E_t x_{i+1}^{f,e}$$

$$x_i^f - \hat{W}_i = -(1-\beta\theta)\frac{e_{wa} - e_u}{e_{ww}} du_i - \beta\theta \hat{W}_i + \beta\theta E_t x_{i+1}^{f,e}. \hfill (A9)$$

From (A4)

$$\frac{\theta}{1-\theta} \pi_i^w = \omega(x_i^b - \hat{W}_i) + (1-\omega)(x_i^f - \hat{W}_i)$$

$$x_i^f = \frac{\theta}{(1-\theta)(1-\omega)} \pi_i^w - \frac{\omega}{1-\omega} (x_i^b - \hat{W}_i) + \hat{W}_i$$

$$E_t x_{i+1}^{f} = \frac{\theta}{(1-\theta)(1-\omega)} E_t \pi_{i+1}^w - \frac{\omega}{1-\omega} (E_t x_{i+1}^b - E_t \hat{W}_{i+1}) + E_t \hat{W}_{i+1}.$$  \hfill (A4)

Substituting (A8) yields

$$E_t x_{i+1}^{f} = \frac{\theta}{(1-\theta)(1-\omega)} E_t \pi_{i+1}^w - \frac{\omega}{1-\omega} \left[ -E_t \pi_{i+1}^w + \left( \frac{\theta}{1-\theta} + \lambda_1 \right) \pi_i^w + \sum_{i=1}^{T-1} \lambda_{i+1} \pi_{i-1}^w \right] + E_t \hat{W}_{i+1}$$

$$E_t x_{i+1}^{f} = \frac{\theta + (1-\theta)\omega}{(1-\theta)(1-\omega)} E_t \pi_{i+1}^w - \frac{\omega}{1-\omega} \left( \frac{\theta}{1-\theta} + \lambda_1 \right) \pi_i^w + \sum_{i=1}^{T-1} \lambda_{i+1} \pi_{i-1}^w \right) + E_t \hat{W}_{i+1}. \hfill (A10)$$
Substituting (A10) into (A9) yields

\[ x^f_t - \hat{W}_t = -(1 - \beta \theta) \frac{e_{wu} - e_u}{e_{ww}} du_t - \beta \theta \hat{W}_t + \beta \theta E_t x^f_{t+1} \]

\[ x^f_t - \hat{W}_t = -(1 - \beta \theta) \frac{e_{wu} - e_u}{e_{ww}} du_t - \beta \theta \hat{W}_t \\
+ \beta \theta \left[ \frac{\theta + (1 - \theta) \omega}{(1 - \theta)(1 - \omega)} E_t \pi^w_{t+1} - \frac{\omega}{1 - \omega} \left( \frac{\theta}{1 - \theta} + \lambda_i \right) \pi^w_t + \sum_{i=1}^{T-1} \lambda_i \pi^w_{t-i} + E_t \pi^w_{t+1} \right] \]

\[ x^f_t - \hat{W}_t = -(1 - \beta \theta) \frac{e_{wu} - e_u}{e_{ww}} du_t \\
+ \beta \theta \left[ \frac{1}{(1 - \theta)(1 - \omega)} E_t \pi^w_{t+1} - \frac{\omega}{1 - \omega} \left( \frac{\theta}{1 - \theta} + \lambda_i \right) \pi^w_t + \sum_{i=1}^{T-1} \lambda_i \pi^w_{t-i} \right] \]  \hspace{1cm} (A11)

Substituting (A8) and (A11) into (A4) yields

\[ \frac{\theta}{1 - \theta} \pi^w_t = \omega(x^b_t - \hat{W}_t) + (1 - \omega)(x^f_t - \hat{W}_t) \]

\[ \frac{\theta}{1 - \theta} \pi^w_t = \omega \left( -\pi^w_t + \frac{\theta}{1 - \theta} \pi^w_{t-1} + \sum_{i=1}^{T} \lambda_i \pi^w_{t-i} \right) + (1 - \omega) \left\{ -(1 - \beta \theta) \frac{e_{wu} - e_u}{e_{ww}} du_t \\
+ \beta \theta \left[ \frac{1}{(1 - \theta)(1 - \omega)} E_t \pi^w_{t+1} - \frac{\omega}{1 - \omega} \left( \frac{\theta}{1 - \theta} + \lambda_i \right) \pi^w_t + \sum_{i=1}^{T-1} \lambda_i \pi^w_{t-i} \right] \right\} \]
\[
\theta \pi_t^w = \omega \left( - (1 - \theta) \pi_t^w + \theta \pi_{t-1}^w + (1 - \theta \sum_{i=1}^{T} \lambda_t \pi_{t-i}^w \right) - (1 - \omega)(1 - \theta)(1 - \beta \theta) \frac{e_{uw} - e_u}{e_{ww}} du_t + \beta \theta E_t \pi_{t+1}^w - \beta \theta \omega \left( \theta + (1 - \theta) \lambda_t \right) \pi_t^w + (1 - \theta) \sum_{i=1}^{T-1} \lambda_{t-i+1} \pi_{t-i}^w
\]

\[
\theta \pi_t^w = -(1 - \theta) \omega \pi_t^w + \omega \pi_{t+1}^w + (1 - \theta) \omega \sum_{i=1}^{T} \lambda_t \pi_{t-i}^w - (1 - \omega)(1 - \theta)(1 - \beta \theta) \frac{e_{uw} - e_u}{e_{ww}} du_t + \beta \theta E_t \pi_{t+1}^w - \beta \theta \omega \left( \theta + (1 - \theta) \lambda_t \right) \pi_t^w - \beta \theta \omega(1 - \theta) \sum_{i=1}^{T-1} \lambda_{t-i+1} \pi_{t-i}^w
\]

\[
\left[ \theta + \omega(1 - \theta) + \beta \theta \omega(\theta + (1 - \theta) \lambda_t) \right] \pi_t^w = -(1 - \omega)(1 - \theta)(1 - \beta \theta) \frac{e_{uw} - e_u}{e_{ww}} du_t + \beta \theta E_t \pi_{t+1}^w - \beta \theta \omega(1 - \theta) \sum_{i=1}^{T-1} \lambda_{t-i+1} \pi_{t-i}^w + \theta \omega \pi_{t+1}^w + (1 - \theta) \omega \sum_{i=1}^{T} \lambda_t \pi_{t-i}^w
\]

\[
\pi_t^w = \frac{\beta \theta E_t \pi_{t+1}^w + \theta \omega \pi_{t+1}^w + \omega(1 - \theta) \sum_{i=1}^{T} (\lambda_t - \beta \theta \lambda_{t-i}) \pi_{t-i}^w - (1 - \omega)(1 - \theta)(1 - \beta \theta) \frac{e_{uw} - e_u}{e_{ww}} du_t}{\theta + \omega(1 - \theta) + \beta \theta \omega(\theta + (1 - \theta) \lambda_t)}.
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References
