Derivation of the Hybrid NK Wage Phillips Curve with Backward Wage Indexation

This derivation follows Holmberg (2006). Let $x$ represent the wage set by firms that are able to adjust their wages and $\tau$ represent the proportion of firms that are not able to reset their wages. Then,

$$\hat{W}_t = (1 - \tau)x_t + \tau\hat{W}_{t-1}.$$  \hspace{1cm} (A1)

Subtracting $\hat{W}_{t-1}$ from both sides of (A1) yields the following expression for wage inflation:

$$\pi^w_t = \hat{W}_t - \hat{W}_{t-1} = (1 - \tau)(x_t - \hat{W}_{t-1}).$$  \hspace{1cm} (A2)

Let $\eta$ represent the proportion of firms resetting wages that index their wages to lagged wage inflation, $x^b_t$ represent the wage set by backward-looking firms, and $x^f_t$ represent the wage set by forward-looking firms. Then

$$x_t = \eta x^b_t + (1 - \eta)x^f_t.$$  \hspace{1cm} (A3)

Solving (A1) for $x_t$ and substituting this expression into (A3) yields

$$\frac{1}{1 - \tau}\hat{W}_t - \frac{\tau}{1 - \tau}\hat{W}_{t-1} = \eta x^b_t + (1 - \eta)x^f_t.$$  \hspace{1cm} (A4)

Let $\beta$ represent the discount factor and $\hat{W}_{t+j}^{*,e}$ represent firms’ expectation of the optimal wage in period $t+j$ when they set wages in period $t$. Then the wage set by forward-looking firms can be expressed as,

$$x^f_t = (1 - \beta \tau) \sum_{j=0}^{\infty} \beta^j \tau^j E_t \hat{W}_{t+j}^{*,e}$$  \hspace{1cm} (A5)

$$x^f_t = (1 - \beta \tau)\hat{W}_t^* + \beta \tau E_t x^f_{t+1}.$$  \hspace{1cm} (A6)
Backward-looking firms are assumed to set wages equal to the wage set last period by firms that adjusted wages, indexed to lagged wage inflation, where the $\lambda$’s represent the weight given to each lag of wage inflation in the indexation process. Thus, the wage set by backward-looking firms is

$$x_t^b = x_{t-1} + \sum_{i=1}^{T} \lambda_i \pi_t^{w_i}.$$  \hspace{1cm} (A7)

Combining (A1) and (A7) yields

$$x_t^w - \hat{W}_t = -\pi_t^w + \frac{\tau}{1 - \tau} \pi_{t-1}^w + \sum_{i=1}^{T} \lambda_i \pi_{t-1}^{w_i}.$$ \hspace{1cm} (A8)

Substituting (A15) from Campbell (2018) into (A6) yields the relationship,

$$x_t^f = (1 - \beta \tau) \left( \hat{W}_t - \frac{e_{wu} - e_u}{e_{ww}} du_t \right) + \beta \tau E_t x_{t+1}^{f,e}$$

$$x_t^f - \hat{W}_t = -(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{ww}} du_t - \beta \tau \hat{W}_t + \beta \tau E_t x_{t+1}^{f,e}.$$ \hspace{1cm} (A9)

From (A4)

$$\frac{\tau}{1 - \tau} \pi_t^w = \eta (x_t^b - \hat{W}_t) + (1 - \eta)(x_t^f - \hat{W}_t)$$

$$x_t^f = \frac{\tau}{(1 - \tau)(1 - \eta)} \pi_t^w - \frac{\eta}{1 - \eta} (x_t^b - \hat{W}_t) + \hat{W}_t$$

$$E_t x_{t+1}^f = \frac{\tau}{(1 - \tau)(1 - \eta)} E_t \pi_{t+1}^w - \frac{\eta}{1 - \eta} (E_t x_{t+1}^b - E_t \hat{W}_{t+1}) + E_t \hat{W}_{t+1}.$$ 

Substituting (A8) yields

$$E_t x_{t+1}^f = \frac{\tau}{(1 - \tau)(1 - \eta)} E_t \pi_{t+1}^w - \frac{\eta}{1 - \eta} \left[ -E_t \pi_{t+1}^w + \left( \frac{\tau}{1 - \tau} + \lambda_i \right) \pi_t^w + \sum_{i=1}^{T-1} \lambda_{t+i} \pi_{t-i}^w \right] + E_t \hat{W}_{t+1}$$

$$E_t x_{t+1}^f = \frac{\tau + (1 - \tau) \eta}{(1 - \tau)(1 - \eta)} E_t \pi_{t+1}^w - \frac{\eta}{1 - \eta} \left( \left( \frac{\tau}{1 - \tau} + \lambda_i \right) \pi_t^w + \sum_{i=1}^{T-1} \lambda_{t+i} \pi_{t-i}^w \right) + E_t \hat{W}_{t+1}. \hspace{1cm} (A10)$$
Substituting (A10) into (A9) yields

\[
x'_i - \hat{W}_i = -(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{wu}} du_i - \beta \tau \hat{W}_i + \beta \tau E_i x'_{i+1}
\]

\[
x'_i - \hat{W}_i = -(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{wu}} du_i - \beta \tau \hat{W}_i 
+ \beta \tau \left[ \frac{\tau + (1 - \tau) \eta}{(1 - \tau)(1 - \eta)} E_i \pi^w_{i+1} - \frac{\eta}{1 - \eta} \left( \frac{\tau}{1 - \tau} + \lambda_i \right) \pi^w_i + \sum_{j=1}^{T-1} \lambda_{i+j} \pi^w_{i+j} \right] + E_i \hat{W}_{i+1}
\]

\[
x'_i - \hat{W}_i = -(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{wu}} du_i 
+ \beta \tau \left[ \frac{\tau + (1 - \tau) \eta}{(1 - \tau)(1 - \eta)} E_i \pi^w_{i+1} - \frac{\eta}{1 - \eta} \left( \frac{\tau}{1 - \tau} + \lambda_i \right) \pi^w_i + \sum_{j=1}^{T-1} \lambda_{i+j} \pi^w_{i+j} \right] + E_i \pi^w_{i+1}
\]

\[
x'_i - \hat{W}_i = -(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{wu}} du_i 
+ \beta \tau \left[ \frac{1}{(1 - \tau)(1 - \eta)} E_i \pi^w_{i+1} - \frac{\eta}{1 - \eta} \left( \frac{\tau}{1 - \tau} + \lambda_i \right) \pi^w_i + \sum_{j=1}^{T-1} \lambda_{i+j} \pi^w_{i+j} \right] 
\]

Substituting (A8) and (A11) into (A4) yields

\[
\frac{\tau}{1 - \tau} \pi^w_i = \eta (x_i^b - \hat{W}_i) + (1 - \eta) (x'_i - \hat{W}_i)
\]

\[
\frac{\tau}{1 - \tau} \pi^w_i = \eta \left( -\pi^w_i + \frac{\tau}{1 - \tau} \pi^w_{i+1} + \sum_{j=1}^{T-1} \lambda_j \pi^w_{i+j} \right) + (1 - \eta) \left\{ -(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{wu}} du_i 
+ \beta \tau \left[ \frac{1}{(1 - \tau)(1 - \eta)} E_i \pi^w_{i+1} - \frac{\eta}{1 - \eta} \left( \frac{\tau}{1 - \tau} + \lambda_i \right) \pi^w_i + \sum_{j=1}^{T-1} \lambda_{i+j} \pi^w_{i+j} \right] \right\}
\]
\[ \tau \pi^w_t = \eta \left( -(1 - \tau) \pi^w_t + \tau \pi^w_{t-1} + (1 - \tau) \sum_{i=1}^{T} \lambda_i \pi^w_{i-t} \right) - (1 - \eta)(1 - \tau)(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{ww}} du_t \]

\[ + \beta \tau E_t \pi^w_{t+1} - \beta \tau \eta \left( \tau + (1 - \tau) \lambda_i \right) \pi^w_t + (1 - \tau) \sum_{i=1}^{T-1} \lambda_i \pi^w_{i-t} \]

\[ \tau \pi^w_t = -(1 - \eta) \eta \pi^w_t + \tau \eta \pi^w_{t-1} + (1 - \eta) \eta \sum_{i=1}^{T} \lambda_i \pi^w_{i-t} - (1 - \eta)(1 - \tau)(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{ww}} du_t \]

\[ + \beta \tau E_t \pi^w_{t+1} - \beta \tau \eta \left( \tau + (1 - \tau) \lambda_i \right) \pi^w_t - \beta \tau \eta(1 - \tau) \sum_{i=1}^{T-1} \lambda_i \pi^w_{i-t} \]

\[ \left[ \tau + \eta(1 - \tau) + \beta \tau \eta \left( \tau + (1 - \tau) \lambda_i \right) \right] \pi^w_t = -(1 - \eta)(1 - \tau)(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{ww}} du_t \]

\[ + \beta \tau E_t \pi^w_{t+1} - \beta \tau \eta(1 - \tau) \sum_{i=1}^{T-1} \lambda_i \pi^w_{i-t} + \tau \eta \pi^w_{t-1} + (1 - \eta) \sum_{i=1}^{T} \lambda_i \pi^w_{i-t} \]

\[ \pi^w_t = \frac{-(1 - \eta)(1 - \tau)(1 - \beta \tau) \frac{e_{wu} - e_u}{e_{ww}} du_t + \beta \tau E_t \pi^w_{t+1} + \tau \eta \pi^w_{t-1} + \eta(1 - \tau) \sum_{i=1}^{T} (\lambda_i - \beta \tau \lambda_{i+1}) \pi^w_{i-t}}{\tau + \eta(1 - \tau) + \beta \tau \eta \left( \tau + (1 - \tau) \lambda_i \right)} . \]

References
