Derivation of the Hybrid New Keynesian Wage Phillips Curve with Multiple Lags of Backward Wage Indexation

This derivation follows Holmberg (2006). Let $\bar{w}_t$ represent the average wage, $w_t^{\dagger}$ represent the wage set by firms that are able to adjust their wages in the current period, and θ represent the proportion of firms that are not able to reset their wages.[1] Then,

$$\bar{w}_t = (1-\theta)w_t^{\dagger} + \theta\bar{w}_{t-1}. \tag{A1}$$

Subtracting $\bar{w}_{t-1}$ from both sides of (A1) yields the following expression for wage inflation:

$$\pi_t^w = \bar{w}_t - \bar{w}_{t-1} = (1-\theta)(w_t^{\dagger} - \bar{w}_{t-1}). \tag{A2}$$

Let ω represent the proportion of firms resetting wages that index their wages to lagged wage inflation, $w_t^{\dagger,b}$ represent the wage set by backward-looking firms, and $w_t^{\dagger,f}$ represent the wage set by forward-looking firms. Then

$$w_t^{\dagger} = \omega w_t^{\dagger,b} + (1-\omega)w_t^{\dagger,f}. \tag{A3}$$

Solving (A1) for $w_t^{\dagger}$ and substituting this expression into (A3) yields

$$\frac{1}{1-\theta}\bar{w}_t - \frac{\theta}{1-\theta}\bar{w}_{t-1} = \omega w_t^{\dagger,b} + (1-\omega)w_t^{\dagger,f}.$$

Subtracting $\bar{w}_t$ from both sides of the above equation results in the expression,

$$\frac{\theta}{1-\theta}\pi_t^w = \omega(w_t^{\dagger,b} - \bar{w}_t) + (1-\omega)(w_t^{\dagger,f} - \bar{w}_t). \tag{A4}$$

Let β represent the discount factor and $E_t w_{t+j}^*$ represent firms' expectation of the optimal wage in period $t+j$ when they set wages in period t. It is assumed that the wage set by forward-looking firms is a weighted average of the current and expected future values of the optimal wage, with the weights determined by the discount factor and by the probability that the wage set in this period will still be in effect in each future period, so that,

$$w_t^{\dagger,f} = (1-\beta\theta)\sum_{j=0}^{\infty}\beta^j\theta^j E_t w_{t+j}^* \tag{A5}$$

$$w_t^{\dagger,f} = (1-\beta\theta)w_t^* + \beta\theta E_t w_t^{\dagger,f}\ .^2 \tag{A6}$$

Backward-looking firms are assumed to set wages equal to the wage set last period by firms that adjusted wages, plus a weighted average of lagged wage inflation, where $\lambda_1, \lambda_2, ..., \lambda_T$ represent the weight given to each lag of wage inflation in the indexation process. Thus, the wage set by backward-looking firms is

$$w_t^{\dagger,b} = w_{t-1}^{\dagger} + \sum_{i=1}^{T} \lambda_i \pi_{t-i}^w \qquad \text{with} \qquad \sum_{i=1}^{T} \lambda_i = 1\,. \tag{A7}$$

By lagging (A1) by one period, $w_{t-1}^{\dagger}$ can be expressed as

$$w_{t-1}^{\dagger} = \frac{1}{1-\theta}\bar{w}_{t-1} - \frac{\theta}{1-\theta}\bar{w}_{t-2}\ .$$

Subtracting $\bar{w}_t$ from both sides of (A7) and substituting in the above expression for $w_{t-1}^{\dagger}$ yields

$$w_t^{\dagger,b} - \bar{w}_t = -\pi_t^w + \frac{\theta}{1-\theta}\pi_{t-1}^w + \sum_{i=1}^{T} \lambda_i \pi_{t-i}^w\,. \tag{A8}$$

Substituting (17) from Campbell (2021) into (A6) results in the expression,

$$w_t^{\dagger,f} = (1-\beta\theta)\left(\bar{w}_t - \frac{e_{Wu} - e_u}{e_{WW}}\tilde{u}_t\right) + \beta\theta E_t w_{t+1}^{\dagger,f}$$

$$w_t^{\dagger,f} - \bar{w}_t = -(1-\beta\theta)\frac{e_{Wu} - e_u}{e_{WW}}\tilde{u}_t - \beta\theta\bar{w}_t + \beta\theta E_t w_{t+1}^{\dagger,f}\,. \tag{A9}$$

Solving (A4) for $w_t^{\dagger,f}$ yields

$$w_t^{\dagger,f} = \frac{\theta}{(1-\theta)(1-\omega)}\pi_t^w - \frac{\omega}{1-\omega}(w_t^{\dagger,b} - \bar{w}_t) + \bar{w}_t\ ,$$

so that

$$E_t w_{t+1}^{\dagger,f} = \frac{\theta}{(1-\theta)(1-\omega)}E_t\pi_{t+1}^w - \frac{\omega}{1-\omega}(E_t w_{t+1}^{\dagger,b} - E_t\bar{w}_{t+1}) + E_t\bar{w}_{t+1}\,. \tag{A10}$$

Substituting (A8), shifted ahead one period, into (A10) yields

$$E_t w_{t+1}^{\dagger,f} = \frac{\theta}{(1-\theta)(1-\omega)} E_t \pi_{t+1}^w - \frac{\omega}{1-\omega}\left(-E_t \pi_{t+1}^w + \left(\frac{\theta}{1-\theta} + \lambda_1\right)\pi_t^w + \sum_{i=1}^{T-1} \lambda_{i+1} \pi_{t-i}^w\right) + E_t \bar{w}_{t+1}$$

$$E_t w_{t+1}^{\dagger,f} = \frac{\theta + (1-\theta)\omega}{(1-\theta)(1-\omega)} E_t \pi_{t+1}^w - \frac{\omega}{1-\omega}\left(\left(\frac{\theta}{1-\theta} + \lambda_1\right)\pi_t^w + \sum_{i=1}^{T-1} \lambda_{i+1} \pi_{t-i}^w\right) + E_t \bar{w}_{t+1}\,. \tag{A11}$$

By substituting (A11) into (A9),

$$\begin{aligned} w_t^{\dagger,f} - \bar{w}_t &= -(1-\beta\theta)\frac{e_{Wu} - e_u}{e_{WW}} \tilde{u}_t - \beta\theta \bar{w}_t \\ &+ \beta\theta\left[\frac{\theta + (1-\theta)\omega}{(1-\theta)(1-\omega)} E_t \pi_{t+1}^w - \frac{\omega}{1-\omega}\left(\left(\frac{\theta}{1-\theta} + \lambda_1\right)\pi_t^w + \sum_{i=1}^{T-1} \lambda_{i+1} \pi_{t-i}^w\right) + E_t \bar{w}_{t+1}\right] \end{aligned}$$

$$\begin{aligned} w_t^{\dagger,f} - \bar{w}_t &= -(1-\beta\theta)\frac{e_{Wu} - e_u}{e_{WW}} \tilde{u}_t \\ &+ \beta\theta\left[\frac{\theta + (1-\theta)\omega}{(1-\theta)(1-\omega)} E_t \pi_{t+1}^w - \frac{\omega}{1-\omega}\left(\left(\frac{\theta}{1-\theta} + \lambda_1\right)\pi_t^w + \sum_{i=1}^{T-1} \lambda_{i+1} \pi_{t-i}^w\right) + E_t \pi_{t+1}^w\right] \end{aligned}$$

$$\begin{aligned} w_t^{\dagger,f} - \bar{w}_t &= -(1-\beta\theta)\frac{e_{Wu} - e_u}{e_{WW}} \tilde{u}_t \\ &+ \beta\theta\left[\frac{1}{(1-\theta)(1-\omega)} E_t \pi_{t+1}^w - \frac{\omega}{1-\omega}\left(\left(\frac{\theta}{1-\theta} + \lambda_1\right)\pi_t^w + \sum_{i=1}^{T-1} \lambda_{i+1} \pi_{t-i}^w\right)\right]. \end{aligned} \tag{A12}$$

Substituting (A8) and (A12) into (A4) yields

$$\begin{aligned} \frac{\theta}{1-\theta}\pi_t^w &= \omega\left(-\pi_t^w + \frac{\theta}{1-\theta}\pi_{t-1}^w + \sum_{i=1}^{T} \lambda_i \pi_{t-i}^w\right) + (1-\omega)\left\{-(1-\beta\theta)\frac{e_{Wu} - e_u}{e_{WW}} \tilde{u}_t\right. \\ &\left. + \beta\theta\left[\frac{1}{(1-\theta)(1-\omega)} E_t \pi_{t+1}^w - \frac{\omega}{1-\omega}\left(\left(\frac{\theta}{1-\theta} + \lambda_1\right)\pi_t^w + \sum_{i=1}^{T-1} \lambda_{i+1} \pi_{t-i}^w\right)\right]\right\} \end{aligned}$$

$$\theta\pi_t^w = -(1-\theta)\omega\pi_t^w + \theta\omega\pi_{t-1}^w + (1-\theta)\omega\sum_{i=1}^{T}\lambda_i\pi_{t-i}^w - (1-\omega)(1-\theta)(1-\beta\theta)\frac{e_{Wu}-e_u}{e_{WW}}\tilde{u}_t$$
$$+\beta\theta E_t\pi_{t+1}^w - \beta\theta\omega\left(\theta+(1-\theta)\lambda_1\right)\pi_t^w - \beta\theta\omega(1-\theta)\sum_{i=1}^{T-1}\lambda_{i+1}\pi_{t-i}^w.$$

Collecting terms results in the equation,

$$\left[\theta+\omega(1-\theta)+\beta\theta\omega\left(\theta+(1-\theta)\lambda_1\right)\right]\pi_t^w = \beta\theta E_t\pi_{t+1}^w + \theta\omega\pi_{t-1}^w + (1-\theta)\omega\sum_{i=1}^{T}\lambda_i\pi_{t-i}^w$$
$$-\beta\theta\omega(1-\theta)\sum_{i=1}^{T-1}\lambda_{i+1}\pi_{t-i}^w - (1-\omega)(1-\theta)(1-\beta\theta)\frac{e_{Wu}-e_u}{e_{WW}}\tilde{u}_t.$$

Dividing by the coefficient on π_t^w yields

$$\pi_t^w = \frac{\beta\theta E_t\pi_{t+1}^w + \theta\omega\pi_{t-1}^w + \omega(1-\theta)\sum_{i=1}^{T}(\lambda_i - \beta\theta\lambda_{i+1})\pi_{t-i}^w - (1-\omega)(1-\theta)(1-\beta\theta)\frac{e_{Wu}-e_u}{e_{WW}}\tilde{u}_t}{\theta+\omega[1-\theta+\beta\theta\left(\theta+(1-\theta)\lambda_1\right)]}.$$ [3]

References

Campbell, Carl (2021), ‘The wage Phillips curve and the dynamic labor demand curve as joint determinants of wage inflation and unemployment,’ Working Paper, Northern Illinois University.

Holmberg, Karolina (2006), ‘Derivation and estimation of a new Keynesian Phillips curve in a small open economy,’ Sveriges Riksbank Working Paper Series #197.

Romer, David (2019), *Advanced Macroeconomics*, Fifth edition, McGraw-Hill Education: New York.

---

[1] $\bar{w}_t$ and $w_t^{\dagger}$ represent percentage deviations from steady-state values.

[2] See Romer (2019), equations 7.56 and 7.57 for statements of (A5) and (A6), where x_t, 1-α, and p_t^* in Romer correspond to $w_t^{\dagger,f}$, θ, and w_t^* in the present derivation.

[3] In the term with the summation sign, $\lambda_{T+1}=0$, so the last term of the summation is $\omega(1-\theta)\lambda_T\pi_{t-T}^w$.