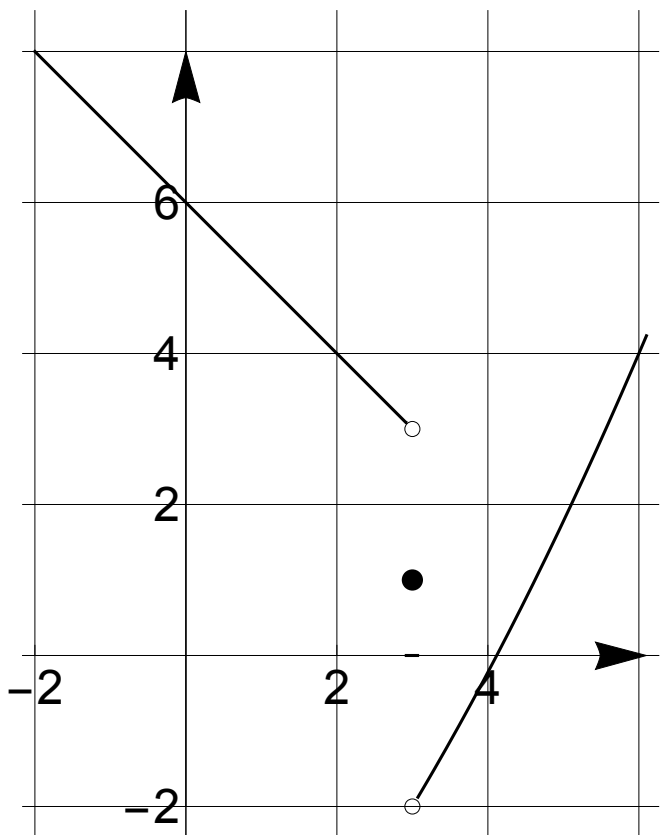


MATH 211 SAMPLE EXAM 1 SOLUTIONS



1. The plot shows the graph of the function  $f(x)$ . Determine the quantities.

(a)  $\lim_{x \rightarrow 3^+} f(x)$

**Solution:** Look at the graph. Let  $x$  approach 3 from the *right*. The corresponding  $y = f(x)$  approaches  $-2$ .

(b)  $\lim_{x \rightarrow 3^-} f(x)$

**Solution:** Look at the graph. Let  $x$  approach 3 from the *left*. The corresponding  $y = f(x)$  approaches 3, as far as one can tell.

(c)  $\lim_{x \rightarrow 3} f(x)$

**Solution:** The one-sided limits are different so the limit does not exist.

(d)  $f(2)$

**Solution:** It is clear from the graph that  $f(2) = 4$ .

(e)  $f(3)$

**Solution:** The black dot indicates that the function  $f$  is defined at  $x = 3$  and the value is  $f(3) = 1$ , as far as one can tell.

2. Let  $f(x) = \frac{1}{x+2} + 3$ . Compute the following

(a)  $\lim_{x \rightarrow \infty} f(x)$

**Solution:** As  $x$  becomes a larger and larger *positive* number, the denominator will also become

a larger and larger positive number. The first term will therefore be negligible and the limit is equal to 3.

(b)  $\lim_{x \rightarrow -\infty} f(x)$

**Solution:** As  $x$  becomes a larger and larger *negative* number, the denominator will also ultimately become

a larger and larger negative number. The first term will again be negligible and the limit is equal to 3.

(c)  $\lim_{x \rightarrow 0} f(x)$

**Solution:** The function is continuous away from  $x = -2$ , so plug in  $x = 0$  and get  $7/2$ .

(d)  $\lim_{x \rightarrow -2^+} f(x)$

**Solution:** As  $x$  approaches -2 from the *right*, the denominator stays *positive* and approaches 0. The first term becomes arbitrarily large *positive* so the limit is  $+\infty$ .

(e)  $\lim_{x \rightarrow -2^-} f(x)$

**Solution:** As  $x$  approaches -2 from the *left*, the denominator stays *negative* and approaches 0. The first term becomes arbitrarily large *negative* so the limit is  $-\infty$ .

(f)  $\lim_{x \rightarrow -2} f(x)$

**Solution:** The one-sided limits are different so the limit does not exist.

**3.** Is the function  $f(x)$  continuous at  $x = 2$ ? Why or why not?

$$f(x) = \begin{cases} 3x - 4 & x < 2 \\ 1 & x = 2 \\ 7 - 3x & x > 2 \end{cases}$$

**Solution:** Utilize continuity and the case  $x > 2$  to determine that

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 7 - 3x = 1.$$

Utilize continuity and the case  $x < 2$  to determine that

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 3x - 4 = 2.$$

The one-sided limits are different so the limit at  $x = 2$  does not exist.

The function is not continuous at  $x = 2$ .

4. Determine the largest possible domain for the following functions and indicate in each case, using clear and unambiguous stated reasons, if the function is continuous on that domain.

(a)  $f(x) = \frac{1}{x+2}$

**Solution:**  $x \neq -2$  (Avoid zero denominator.)

(b)  $g(x) = \frac{1}{x^2+4}$

**Solution:** All real numbers since the denominator is never 0.

(c)  $h(x) = \sqrt{x+1}$

**Solution:**  $x \geq -1$  (Avoid negatives inside square root.)

(d)  $f(x) = \sqrt{x^2 - 4}$

**Solution:**  $x \geq 2$  or  $x \leq -2$  (Avoid negatives inside square root.)

(e)  $f(x) = \frac{1}{\sqrt{x^2-9}}$

**Solution:**  $x > 3$  or  $x < -3$  (Avoid negative inside square root and zero in the denominator.)

5. Simplify as far as possible the difference quotient

$\frac{f(x+h)-f(x)}{h}$  in each case

(a)  $f(x) = x^2$

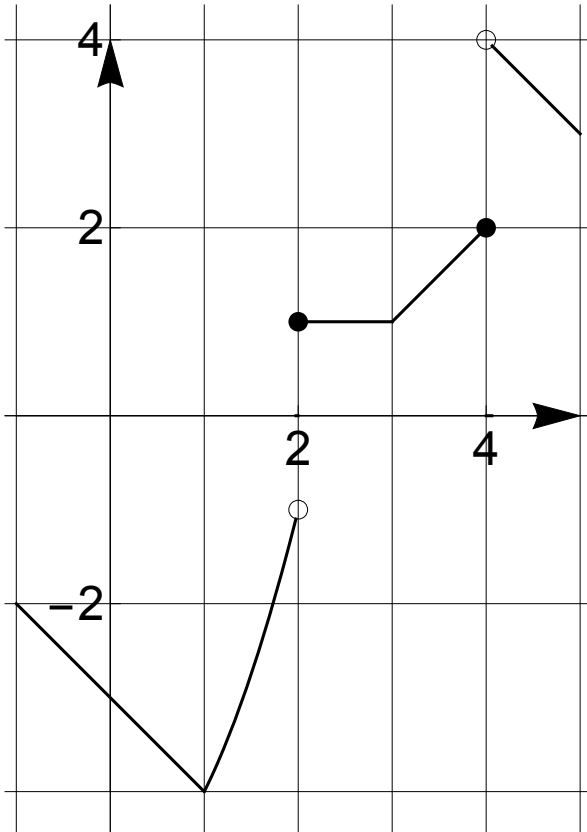
**Solution:**  $\frac{(x+h)^2-x^2}{h} = \frac{x^2+2xh+h^2-x^2}{h} = \frac{2xh+h^2}{h} = 2x+h$

(b)  $f(x) = \frac{1}{x}$

**Solution:**  $\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h} = \frac{x-(x+h)}{hx(x+h)} = \frac{-1}{x(x+h)}$

6. List the  $x$ -coordinates in the graph at which the function is not differentiable.

**Solution:**  $x = 1$  (corner),  $x = 2$  (discontinuous),  $x = 3$  (corner),  $x = 4$  (discontinuous)



7. A car travels from NIU in deKalb to O'Hare airport and its progress towards the destination is shown in the following diagram where the horizontal axis corresponds to time in minutes and the vertical axis corresponds to distance completed in miles at the given time. Answer the following:

(a) What is the average speed for the whole trip?

**Solution:**  $\frac{60 \text{ miles}}{80 \text{ minutes}} = \frac{60 \text{ miles}}{4/3 \text{ hour}} = 45 \text{ mph}$

(b) What is the average speed for the first half hour?

**Solution:**  $\frac{25 \text{ miles}}{30 \text{ minutes}} = \frac{25 \text{ miles}}{1/2 \text{ hour}} = 50 \text{ mph}$

(c) What is the average speed for the second half hour?

**Solution:**  $\frac{35-25 \text{ miles}}{30 \text{ minutes}} = \frac{10 \text{ miles}}{1/2 \text{ hour}} = 20 \text{ mph}$

(d) What is the average speed for the last half hour?

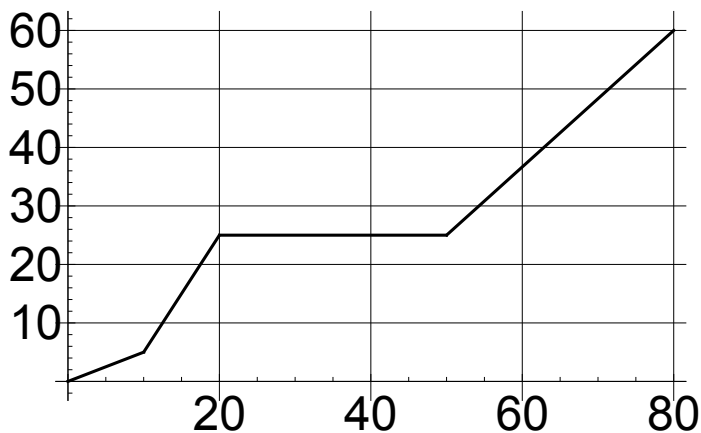
**Solution:**  $\frac{60-25 \text{ miles}}{30 \text{ minutes}} = \frac{35 \text{ miles}}{1/2 \text{ hour}} = 70 \text{ mph}$

(e) Using the diagram, describe, and support with carefully calculated quantities, what you believe happened during the trip?

**Solution:** Between the 10th and 20th minute the speed was

$\frac{25-5 \text{ miles}}{10 \text{ minutes}} = \frac{20 \text{ miles}}{1/6 \text{ hour}} = 120 \text{ mph}$ , so a speeding ticket was issued

between the 20th and 50th minute, which explains why the car was not moving during that time!



## MATH 211 SAMPLE EXAM 2 SOLUTIONS

1. Calculate the derivative of each of the following functions and simplify when appropriate.

(a)  $f(x) = 3x^5 - 7x^3 - 5x^2 - 11x + 13$

**Solution:** Split sums and differences. Pull out constants. Use the power rule.

$$(3x^5 - 7x^3 - 5x^2 - 11x + 13)' =$$

$$3(x^5)' - 7(x^3)' - 5(x^2)' - 11(x)' + (13)' = 15x^4 - 21x^2 - 10x - 11$$

(b)  $g(x) = 6\sqrt{x} - \frac{12}{\sqrt{x}}$

**Solution:** Split differences. Pull out constants. Use the power rule.

$$(6x^{1/2} - 12x^{-1/2})' = 3x^{-1/2} + 6x^{-3/2} = \frac{3}{\sqrt{x}} + \frac{6}{x\sqrt{x}}$$

(c)  $h(x) = (2 - 3x + 4x^2 - 5x^3)(3 - 7x^2 + 11x^4)$

**Solution:** This one strongly suggests using the product rule.

$$(2 - 3x + 4x^2 - 5x^3)'(3 - 7x^2 + 11x^4) +$$

$$(2 - 3x + 4x^2 - 5x^3)(3 - 7x^2 + 11x^4)' =$$

$$(-3 + 8x - 15x^2)(3 - 7x^2 + 11x^4) + (2 - 3x + 4x^2 - 5x^3)(-14x + 44x^3)$$

(d)  $i(x) = \frac{x^3 - 7}{x^2 + 2x + 1}$

**Solution:** There is no choice but the quotient rule.

$$\left(\frac{x^3 - 7}{x^2 + 2x + 1}\right)' = \frac{(x^3 - 7)'(x^2 + 2x + 1) - (x^3 - 7)(x^2 + 2x + 1)'}{(x^2 + 2x + 1)^2} = \frac{3x^2(x^2 + 2x + 1) - (x^3 - 7)(2x + 2)}{(x^2 + 2x + 1)^2}$$

(e)  $j(x) = (3 - 5x + 7x^2)^3$

**Solution:** The chain rule is much better than expanding.

$$((3 - 5x + 7x^2)^3)' = 3(3 - 5x + 7x^2)^2 \cdot (-5 + 14x)$$



2. Calculate the second derivative of the following functions and simplify as appropriate

(a)  $f(x) = (x - 9)(3 - x^2)$

**Solution:** Use algebra before calculating the second derivative.

$$((x - 9)(3 - x^2))' =$$

$$(x - 9)'(3 - x^2) + (x - 9)(3 - x^2)' = (3 - x^2) + (x - 9) \cdot (-2x) = 3 + 18x - 3x^2$$

$$((x - 9)(3 - x^2))'' = (3 + 18x - 3x^2)' = 18 - 6x$$

(b)  $g(x) = \frac{1-x^3}{(x-3)^2}$

**Solution:** Use algebra before calculating the second derivative. Knowing the rules inside out

is rewarded here. Efficiency is one of the values in Mathematics.

$$\left(\frac{1-x^3}{(x-3)^2}\right)' = \frac{((1-x^3)'(x-3)^2 - (1-x^3)((x-3)^2)')}{(x-3)^4} =$$

$$\frac{(-3x^2(x-3)^2 - (1-x^3) \cdot 2(x-3))}{(x-3)^4} = \frac{-3x^2(x-3)-2(1-x^3)}{(x-3)^3} = \frac{-2+9x^2-x^3}{(x-3)^3}$$

$$\left(\frac{1-x^3}{(x-3)^2}\right)'' = \left(\frac{-2+9x^2-x^3}{(x-3)^3}\right)' = \frac{((-2+9x^2-x^3)'(x-3)^3 - (-2+9x^2-x^3)((x-3)^3)')}{(x-3)^6} =$$

$$\frac{(18x-3x^2)(x-3)^3 - (-2+9x^2-x^3) \cdot 3(x-3)^2}{(x-3)^6} =$$

$$\frac{3 \cdot (x(6-x)(x-3) - (-2+9x^2-x^3))}{(x-3)^4} = 6 \cdot \frac{1-9x}{(x-3)^4}$$

- 3.** Determine exactly where **(a)** the function  $f(x)$  is *decreasing* and **(b)** where the function  $g(x)$  is *concave up*.

$$\text{(a) } f(x) = 3x^5 - 20x^3$$

**Solution:** Read the question very carefully. Only answer what is asked.

$(3x^5 - 20x^3)' = (15x^4 - 60x^2) = 15x^2(x^2 - 4)$  so the derivative is zero only if  $x = -2, 0, 2$ .

For **decreasing** one needs  $x^2 - 4 < 0$  so  $-2 < x < 2$ .

$$\text{(b) } g(x) = 13 + 9x - 3x^2 - x^3$$

$$\text{Solution: } (13 + 9x - 3x^2 - x^3)'' = (9 - 6x - 3x^2)' = -6 - 6x.$$

The second derivative is zero only if  $x = -1$ .

For **concave up** one needs  $-6 - 6x > 0$  so  $x < -1$ .

4. Find the relative extrema of the function  $f(x) = x^4 - 2x^3$ .

You must in each case specify the type of extremum and the exact coordinates of the point.

**Solution:** There are no asymptotes to worry about here.

$(x^4 - 2x^3)' = 4x^3 - 6x^2 = 2x^2(2x - 3)$ . The derivative is zero only if  $x = 0, 3/2$ .

The slope is negative when  $x < 0$  and  $0 < x < 3/2$ . The slope is positive when  $3/2 < x$ .

The only relative extremum is  $x = 3/2$ , a relative minimum with  $y$ -coordinate

$$f(3/2) = 81/16 - 108/16 = -27/16.$$

5. Determine each vertical, horizontal, and slanted asymptote of the graph of  $f(x) = \frac{x^3-1}{x^2+x-6}$ .

**Solution:** First worry about zero in the denominator.

$x^2 + x - 6 = (x - 2)(x + 3)$  so horizontal asymptotes  
 $x = -3, x = 2$ .

Observe that the degree on the numerator exceeds the degree of the denominator by 1.

Long division produces  $\frac{x^3-1}{x^2+x-6} = x - 1 + \frac{7x-7}{x^2+x-6}$  so there is a slanted asymptote with equation  $y = x - 1$ .

There is no horizontal asymptote.

6. Consider the following graph and extract or indicate as precisely the following:

(a) The equation of each vertical asymptote.

**Solution:** Look for vertical lines.

$$x = -1, x = 3$$

(b) The equation of each horizontal or oblique asymptote.

**Solution:** Look for horizontal or slanted lines.

$$y = -10, y = 10$$

(c) An approximation of all intervals where the function is increasing.

**Solution:** The graph must be heading up as  $x$  increases.

A rough approximation is given by:  $-0.3 < x < 1.2$

(d) An approximation of all intervals where the function is concave up.

**Solution:** The must be bending upwards as  $x$  increases.

$$-1 < x < 0 \text{ and } 3 < x$$

(e) An approximation of the  $y$ -intercept expressed as a point.

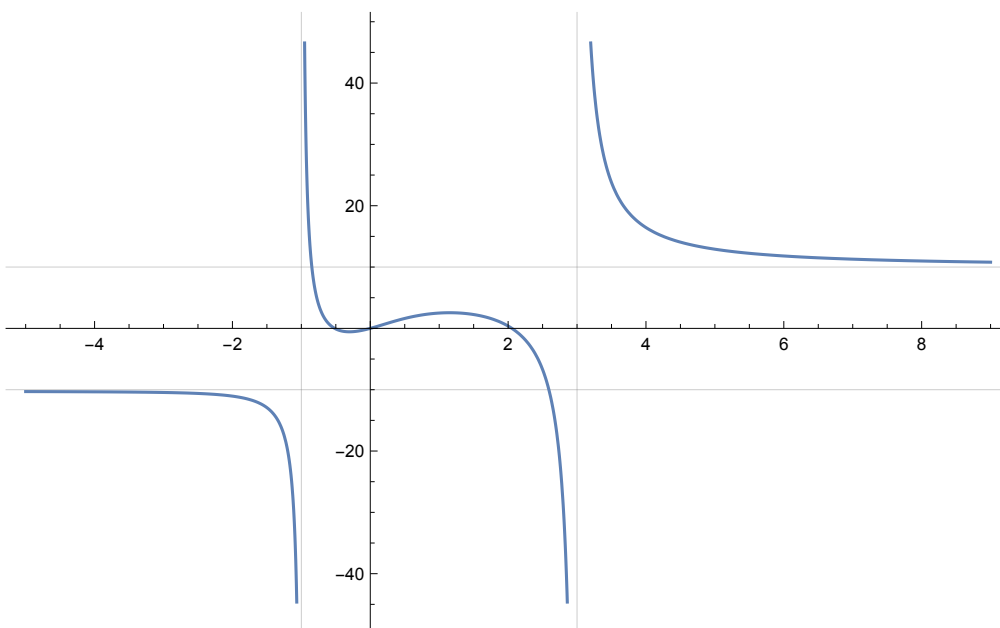
**Solution:** The graph seems to cross the  $y$ -axis at the point  $(0, 0)$ .

(f) An approximation of each  $x$ -intercept expressed as a point.

**Solution:** The graph seems to cross the  $x$ -axis at the points  $(0, 0)$  and  $(2, 0)$

(g) Indicate clearly in the graph each inflection point.

**Solution:** Where does the graph *change* concavity? How about the origin  $(0, 0)$ !



7. Consider the function  $f(x) = \frac{3(x-5)^2}{(x-2)^2} = 3 + \frac{63-18x}{(x-2)^2}$  with first derivative  $f'(x) = \frac{18(x-5)}{(x-2)^3}$  and second derivative  $f''(x) = \frac{18(13-2x)}{(x-2)^4}$ . Sketch the graph of  $f(x)$  in the coordinate system below and make sure all pertinent information is clearly indicated: asymptotes, extrema, intercepts, and inflection points. Give equations and coordinates of points as necessary and support each claim by the appropriate reasoning using algebra and calculus.

**Solution:** The formulas are given so there is no need to calculate derivatives.

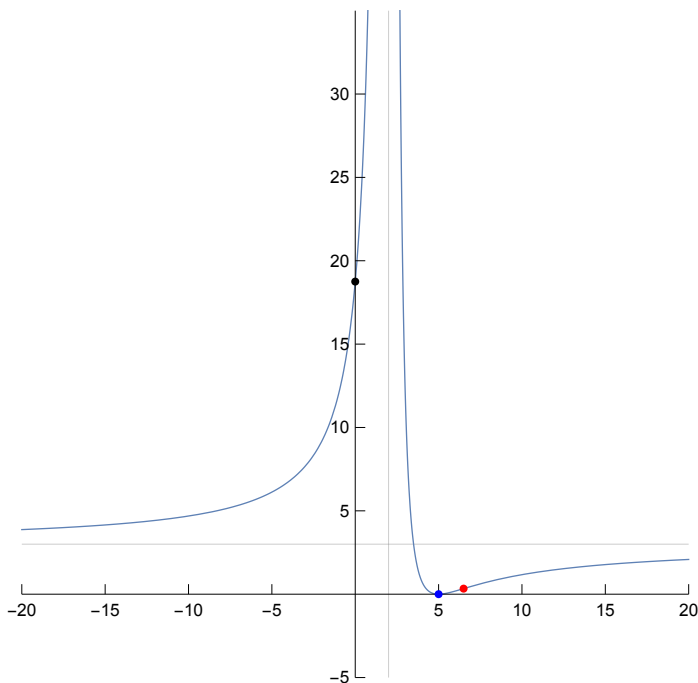
From the first formula for  $f$ : vertical asymptote  $x = 2$  (zero denominator),  $y$ -intercept:  $(0, 75/4)$  (plug in  $x = 0$ ),  $x$ -intercept:  $(5, 0)$  (zero numerator)

From the second formula for  $f$ : horizontal asymptote  $y = 3$  (limits as  $x \rightarrow \pm\infty$ )

From the derivatives:

relative minimum:  $(5, 0)$  (derivative 0, second derivative positive)

inflection point:  $(13/2, 1/3)$  (second derivative zero and changing sign)



## MATH 211 SAMPLE EXAM 3 SOLUTIONS

1. Find the absolute maximum and minimum value of the function over the indicated interval, and indicate the  $x$ -values at which it occurs:  $f(x) = 3 - 2x - 5x^2$ ;  $[-3, 3]$ .

**Solution:** Observe that the function is a polynomial of degree 2 so it is differentiable and therefore continuous. The domain is a closed interval. Together these two facts imply that there exists both an absolute maximum and an absolute minimum. The location of these must be where the derivative is zero or at an endpoint of the closed interval.

Now  $f(-3) = -36$  and  $f(3) = -48$ . The derivative is given by  $f'(x) = -2 - 10x$ , and it is zero only when  $x = -1/5$  so  $f(-1/5) = 16/5$ .

It follows that the absolute maximum is at  $x = -1/5$  and the absolute minimum is at  $x = 3$ .

2. Differentiate the following functions.

(a)  $f(x) = 2 e^x + 3 x + 4$

**Solution:** Use the fact that  $(e^x)' = e^x$ .

$$(2 e^x + 3 x + 4)' =$$

$$(2 e^x)' + (3 x)' + (4)' = 2 (e^x)' + 3 (x)' + 0 = 2 e^x + 3$$

(b)  $g(x) = x^3 + e^{2 x}$

**Solution:** Must use the chain rule here.

$$(x^3 + e^{2 x})' = (x^3)' + (e^{2 x})' = 3 x^2 + e^{2 x} \cdot 2 = 3 x^2 + 2 e^{2 x}$$

(c)  $h(x) = x^3 e^{2 x}$

**Solution:** Must use both the product rule and the chain rule.

$$(x^3 e^{2 x})' =$$

$$(x^3)' e^{2 x} + x^3 (e^{2 x})' = 3 x^2 e^{2 x} + x^3 e^{2 x} \cdot 2 = x^2 e^{2 x} (3 + 2 x)$$



**3.** Minimize  $R = x^2 + 2y^2$ , where  $x + y = 3$ .

**Solution:** Use the constraint to eliminate one of the variables. For instance,  $y = 3 - x$  so minimize

$R(x) = x^2 + 2(3 - x)^2$  where  $x$  may be any real number.

Observe that  $R'(x) = 2x - 4(3 - x) = 6x - 12$  and

$R''(x) = 6$ . It follows that  $R$  is minimized when  $x = 2$  and  $y = 1$  with minimal value 6.

4. Simplify the following expressions as far as possible:

(a)  $e^{\ln(5)}$

**Solution:** The two functions involved are inverses, so

$$e^{\ln(5)} = 5$$

(b)  $\ln \sqrt{e^4}$

**Solution:** Use known rules.

$$\ln \sqrt{e^4} = \ln\left((e^4)^{1/2}\right) = \ln(e^2) = 2$$

(c)  $\ln(1) + e^0 + \ln\left(\frac{1}{e^2}\right)$

**Solution:** Use known rules.

$$\ln(1) + e^0 + \ln\left(\frac{1}{e^2}\right) = 0 + 1 + \ln(1) - \ln(e^2) = 1 - 2 = -1$$

5. Complete the following:

(a) Give the exact value of  $\ln(e^{-999\,999\,999})$ .

**Solution:** A calculator is of little help here. The two functions involved are inverses.

$$\ln(e^{-999\,999\,999}) = -999\,999\,999$$

(b) Calculate the derivative of  $\ln(3)$ .

**Solution:** There is no variable here!

$\ln(3)$  is a constant so its derivative is 0.

(c) Let  $g(x) = x \ln(x) - x$ . Calculate  $g'(x)$ .

**Solution:** Use the fact that  $(\ln(x))' = \frac{1}{x}$ .

$$(x \ln(x) - x)' =$$

$$(x)' \ln(x) + x(\ln(x))' - (x)' = \ln(x) + x(1/x) - 1 = \ln(x)$$

(d) Let  $h(x) = \ln(x^7)$ . Calculate  $h'(x)$ .

**Solution:** Use algebraic rules before calculus!

$$(\ln(x^7))' = (7 \ln(x))' = 7 (\ln(x))' = 7/x$$

(e) Let  $i(x) = (\ln(x))^7$ . Calculate  $i'(x)$ .

**Solution:** No helpful algebraic rule here so the chain rule to the rescue!

$$((\ln(x))^7)' = 7 (\ln(x))^6 \cdot (\ln(x))' = 7 (\ln(x))^6 / x$$

6. The growth rate of the demand for coal in the world is 4% per year. Assume that this trend has been the same for a long time and that it will continue. Let  $M(t)$  be the demand for coal in billions of tons  $t$  years from now. Let  $M_0$  denote the current demand and answer the following questions.

(a) What is the value of  $k$  in the formula  $M(t) = M_0 e^{k t}$ ?

**Solution:**  $k = 0.04$ , i.e., the growth rate expressed as a decimal.

(b) If the demand has doubled  $T$  years from now, what is the value of  $M(T)$ ?

**Solution:** Since now corresponds to  $t = 0$  it follows that  $M_0$  is the current demand and double that is  $2 M_0$ .

(c) Determine the approximate value of  $T$  in years.

**Solution:** The formula simplifies to  $2 = e^{0.04 T}$ . Apply the natural logarithm to both sides and solve for  $T$  and get  $T = \ln(2)/0.04 \approx 17$  years.

(d) Is it necessary to know the exact value of  $M_0$  in order to determine  $T$ ?

**Solution:** No, it cancels.

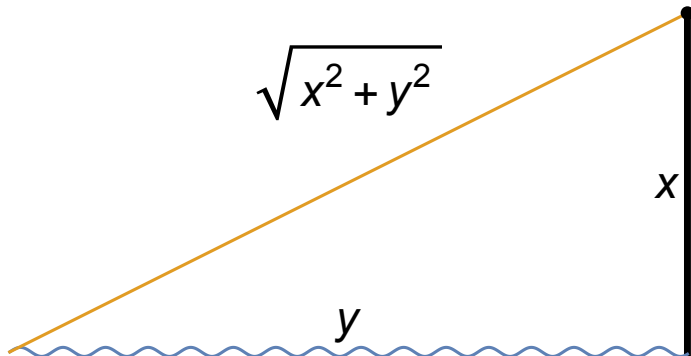
(e) Which year had a demand for coal half the level we have currently (2014)?

**Solution:** This time one gets

$$T = \ln(1/2)/0.04 = (\ln(1) - \ln(2))/0.04 = -\ln(2)/.04 \approx -17 \text{ so } 2014 - 17 = 1997.$$

7. A large plot of land is situated along a straight section of a river. There is enough money to put up a 3 mile long fence and a single anchor post. The fence will run from the anchor post in two directions. One direction will be perpendicular to the river, and the other will be at some other angle. There will be no fence along the river. It follows that the enclosed region forms a right-angle triangle. Assume the side perpendicular to the river has length  $x$  and the side along the river has length  $y$ . The remaining side will be the hypotenuse of the triangle. With these constraints determine the configuration that encloses the largest area.

**Hint:** It is more expedient to maximize the square of the area!



**Solution:** This is a right-angle triangle and the area is  $\frac{xy}{2}$ . The constraint is  $x + \sqrt{x^2 + y^2} = 3$ . Maximize  $x^2 y^2$  after dismissing the  $1/4$ . Observe that  $x^2 + y^2 = (3 - x)^2$  so maximize  $x^2((3 - x)^2 - x^2) = x^2(9 - 6x) = 9x^2 - 6x^3$ . The derivative is  $18x - 18x^2 = 18x(1 - x)$ , which is zero at  $x = 1$  ( $x = 0$  is of no interest.) It follows that the hypotenuse has length 2 and  $y = \sqrt{3}$ . The second derivative is  $18 - 36x$  so  $-18$  when  $x = 1$ . Also, the area is zero if  $x = 0$  or  $x = 3/2$  with  $y = 0$ . There must be a maximum in  $(0, 3/2)$  and there is only one candidate! The maximal area is  $\sqrt{3}/2$  square miles, which is approximately 0.87 square miles.

## MATH 211 SAMPLE FINAL EXAM SOLUTIONS

1. Consider  $f(x)$  as defined below and complete the subsequent tasks.

$$f(x) = \begin{cases} 3x - 1 & x < 2 \\ 5 & x = 2 \\ 9 - 2x & x > 2 \end{cases}$$

- (a) Calculate

$$\lim_{x \rightarrow 2^-} f(x)$$

**Solution:** Utilize continuity and the case  $x < 2$  to determine that

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 3x - 1 = 5.$$

- (b) Calculate

$$\lim_{x \rightarrow 2^+} f(x)$$

**Solution:** Utilize continuity and the case  $x > 2$  to determine that

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 9 - 2x = 5.$$

- (c) Calculate

$$\lim_{x \rightarrow 2} f(x)$$

The two one-sided limits are equal, so the limit exists and its value is 5.

- (d) Decide if  $f(x)$  is a continuous function or not. Make sure you support your claim with rigorous reasoning.

**Solution:** Observe that  $f(2) = 5$  and this is equal to the limit so the function is in fact continuous.

2. Calculate

$$\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}$$

**Hint:** Either use the fact that  $x = (\sqrt{x})^2$  and  $9 = 3^2$ , or multiply by the conjugate.

**Solution:** Use the hint to write

$$\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \rightarrow 9} \frac{(\sqrt{x})^2-3^2}{\sqrt{x}-3} = \lim_{x \rightarrow 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} =$$

$$\lim_{x \rightarrow 9} (\sqrt{x} + 3) = \sqrt{9} + 3 = 6$$

Alternatively, multiply by the conjugate

$$\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x})^2-9} =$$

$$\lim_{x \rightarrow 9} (\sqrt{x} + 3) = \sqrt{9} + 3 = 6$$

**3.** Calculate

$$\lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 + 5x - 14}$$

**Solution:** Observe that the rational function is not defined at  $x = 2$  as the denominator is 0.

This is mitigated by the fact that the numerator is also 0 when  $x = 2$ .

Now  $x^2 - x - 2 = (x - 2)(x + 1)$  and

$x^2 + 5x - 14 = (x - 2)(x + 7)$  so

$$\lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^2 + 5x - 14} = \lim_{x \rightarrow 2} \frac{(x-2)(x+1)}{(x-2)(x+7)} = \lim_{x \rightarrow 2} \frac{x+1}{x+7} = \frac{2+1}{2+7} = \frac{3}{9} = \frac{1}{3}.$$



4. Calculate the derivative of

$$f(x) = 3x^2 - \frac{6}{x^2} - \frac{1}{e^x} + 4\sqrt{x} + \ln(x) + e^{2x} + \ln(3x) + \frac{1}{x+5}$$

**Solution:**

$$f'(x) = 3(x^2)' - 6(x^{-2})' - (e^{-x})' + 4(x^{1/2})' +$$

$$(\ln(x))' + (e^{2x})' + (\ln(3) + \ln(x))' + ((x+5)^{-1})' =$$

$$3 \cdot 2x - 6(-2x^{-3}) - e^{-x} \cdot (-1) + 4\left(\frac{1}{2}x^{-1/2}\right) + \frac{1}{x} +$$

$$e^{2x} \cdot 2 + (\ln(3))' + \frac{1}{x} + (-(x+5)^{-2}) =$$

$$6x + \frac{12}{x^3} + \frac{1}{e^x} + \frac{2}{\sqrt{x}} + \frac{2}{x} + 2e^{2x} - \frac{1}{(x+5)^2}$$

5. Calculate the derivative of the following functions.

Simplify only if appropriate.

(a)  $f(x) = (e^x + x^2)(\ln(x) - \sqrt{x})$

**Solution:** Use the product rule.

$$f'(x) = (e^x + x^2)'(\ln(x) - \sqrt{x}) + (e^x + x^2)(\ln(x) - \sqrt{x})' = \\ (e^x + 2x)(\ln(x) - \sqrt{x}) + (e^x + x^2)\left(\frac{1}{x} - \frac{1}{2\sqrt{x}}\right)$$

(b)  $g(x) = \frac{e^x}{x^3+1}$

**Solution:** Use the quotient rule.

$$g'(x) = \frac{(e^x)'(x^3+1) - e^x(x^3+1)'}{(x^3+1)^2} = \frac{e^x(x^3+1) - e^x 3x^2}{(x^3+1)^2} = \frac{e^x(x^3 - 3x^2 + 1)}{(x^3+1)^2}$$

(c)  $h(x) = e^{x^4} + \ln(\sqrt{x})$

**Solution:** Use the chain rule, but only if necessary.

$$h'(x) =$$

$$(e^{x^4})' + (\ln(\sqrt{x}))' = e^{x^4} \cdot 4x^3 + \left(\frac{1}{2} \ln(x)\right)' = 4x^3 e^{x^4} + \frac{1}{2x}$$

6. Sketch the graph of  $f(x) = -x^3 + 3x - 2$ . List the coordinates of where the **extrema** or **points of inflection** occur. State where the function is **increasing** or **decreasing**, as well as where it is **concave up** or **concave down**. Indicate each coordinate of the  **$y$ -intercept** as well as each  **$x$ -intercept**.

**Solution:** The derivative is  $f'(x) = -3x^2 + 3 = 3(1 - x^2)$ , which is 0 only if  $x = -1$  or  $x = 1$ .

The second derivative is  $f''(x) = -6x$ , which is 0 only if  $x = 0$ .

Observe that  $f''(-1) > 0$  and  $f''(1) < 0$ . Moreover  $f(1) = 0$  and  $f(-1) = -4$ .

It follows that the point  $(1, 0)$  is a local (relative) maximum, and  $(-1, -4)$  is a local (relative) minimum.

The  $y$ -intercept is  $(0, f(0)) = (0, -2)$ . One  $x$ -intercept is discovered by accident as  $(1, f(1)) = (1, 0)$ .

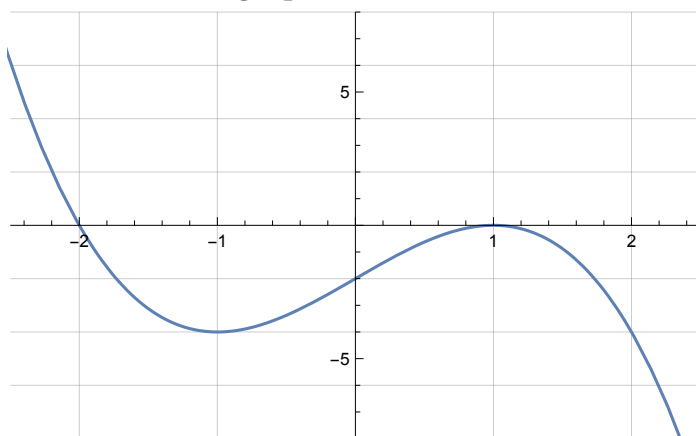
Armed with that information peel off one factor at a time

$$-x^3 + 3x - 2 = (x - 1)(-x^2 - x + 2) = (x - 1)(x + 2)(-x + 1) = -(x - 1)^2(x + 2)$$

The other  $x$ -intercept is  $(-2, 0)$ .

Finally, there is a point of inflection at the  $y$ -intercept.

Now draw the graph!



7. Calculate  $\int \left( 6x^2 + e^x + \frac{1}{x} + \frac{1}{\sqrt{x}} + \frac{1}{x^2} + \frac{4}{x^3} \right) dx$

**Solution:** Integrate term by term, remember the constant of integration. Check by taking the derivative.

$$\int \left( 6x^2 + e^x + \frac{1}{x} + \frac{1}{\sqrt{x}} + \frac{1}{x^2} + \frac{4}{x^3} \right) dx =$$
$$2x^3 + e^x + \ln(x) + 2\sqrt{x} - \frac{1}{x} - \frac{2}{x^2} + C$$

8. Complete the following tasks.

(a) Calculate  $\int_2^7 3x^2 + 4x \, dx$

**Solution:**  $\int_2^7 3x^2 + 4x \, dx = [x^3 + 2x^2]_2^7 =$

$$(7^3 + 2 \cdot 7^2) - (2^3 + 2 \cdot 2^2) = 425$$

(b) Calculate  $\int_0^3 e^{2x} \, dx$

**Solution:**  $\int_0^3 e^{2x} \, dx = \left[ \frac{e^{2x}}{2} \right]_0^3 = \frac{e^6}{2} - \frac{e^0}{2} = \frac{1}{2} (e^6 - 1)$

(c) The definite integral  $\int_1^e \frac{1}{3x-e} \, dx$  may be written as  $\frac{1}{a} (b + \ln(c) - \ln(d))$ .

Determine  $a$ ,  $b$ ,  $c$  and  $d$ .

**Solution:**

$$\int_1^e \frac{1}{3x-e} \, dx = \left[ \frac{\ln(3x-e)}{3} \right]_1^e =$$

$$\frac{1}{3} (\ln(3e - e) - \ln(3 - e)) = \frac{1}{3} (\ln(2e) - \ln(3 - e)) =$$

$$\frac{1}{3} (\ln(2) + \ln(e) - \ln(3 - e)) = \frac{1}{3} (1 + \ln(2) - \ln(3 - e))$$

so  $a = 3$ ,  $b = 1$ ,  $c = 2$  and  $d = 3 - e$ .

9. (a) Determine the area under the graph of  $f(x) = 4x^3 - 2x$  between  $x = 1$  and  $x = 3$ .

**Solution:** Use a definite integral!

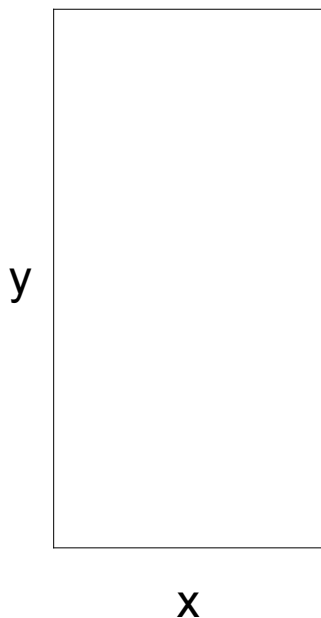
$$\int_1^3 (4x^3 - 2x) dx = [x^4 - x^2]_1^3 = (3^4 - 3^2) - (1^4 - 1^2) = 72$$

- (b) Calculate the **average** of  $f(x)$  on  $[1, 3]$ .

**Solution:** Since the average is  $\frac{1}{b-a} \int_a^b f(x) dx$  and  $a = 1$ ,  $b = 3$ , it follows that the answer is  $\frac{1}{3-1} \cdot 72 = 36$ .

10. A builder must install a fence that goes all the way around the perimeter of a rectangular dog playground.

The standard fence cost 10 [\$/ft]. The side of the playground that is seen from the street is to look nicer so it requires a fence that costs 20 [\$/ft]. If the area enclosed is  $5400 \text{ [ft}^2\text{]}$ , what dimensions are the most cost effective, and how much money is spent on the fence?




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**Solution:** The cost is given by  $20x + 10(x + 2y) = 30x + 20y$ . The area is given by  $xy = 5400$ . Use the constraint

to replace  $y$  to get a cost function in the variable  $x$ .

$f(x) = 30x + 20 \cdot \frac{5400}{x}$  with domain  $0 < x$ . It is clear that  $f(x)$  tends to infinity when  $x$  tends to 0 or  $\infty$ .

The derivative is  $f'(x) = 30 - 20 \cdot \frac{5400}{x^2}$ . The second derivative is  $f''(x) = 40 \cdot \frac{5400}{x^3}$  so concave up when  $x > 0$ .

Set the derivative equal to zero and get  $30 = 20 \cdot \frac{5400}{x^2}$ .

It must be that  $x^2 = 2 \cdot 1800 = 3600$ , so only  $x = 60$ .

It follows that  $y = 90$ , and the minimal cost is  $30 \cdot 60 + 20 \cdot 90 = 3600$  dollars.

**11.** The numbers of cells observed in a lab experiment is subject to exponential growth. Initially there are 3,000 cells. Four hours later there are 48,000 cells. Let  $t$  be the **number of hours elapsed since the initial observation**. Let  $N(t)$  be the **number of cells after  $t$  hours measured in thousands**. Since the growth is exponential it follows that  $N(t) = N_0 e^{k t}$ .

(a) Determine  $N_0$ .

**Solution:** Observe that  $N_0 = N(0) = N_0 e^{k \cdot 0} = 3$ .

(b) Determine  $k$ .

**Solution:**  $48 = 3 e^{k \cdot 4}$  so  $e^{4k} = 16$  and  $4k = \ln(16)$ . This simplifies as  $k = \frac{\ln(2^4)}{4} = \frac{4 \ln(2)}{4} = \ln(2)$ .

(c) How many cells are there after 8 hours?

**Solution:**  $N(8) = 3 e^{\ln(2) \cdot 8} = 3 \cdot 2^8 = 3 \cdot 256 = 768$ , so 768,000 (give or take ...)

(d) When are there 384,000 cells?

**Solution:**  $384 = 3 e^{\ln(2) \cdot t}$  implies  $e^{\ln(2) \cdot t} = 128$ . Apply the natural logarithm  $\ln(2) \cdot t = \ln(128) = \ln(2^7) = 7 \ln(2)$ , so  $t = 7$ , i.e., 7 hours.



12. Determine the area of the bounded region determined by the curves

$$f(x) = 4x - x^2 \text{ and } g(x) = x^2 - 6x + 8.$$

**Hint:** First use Algebra and Calculus to draw an accurate picture of  $f(x)$  and  $g(x)$  in the same coordinate system. Then use Calculus and your picture to finish the problem.

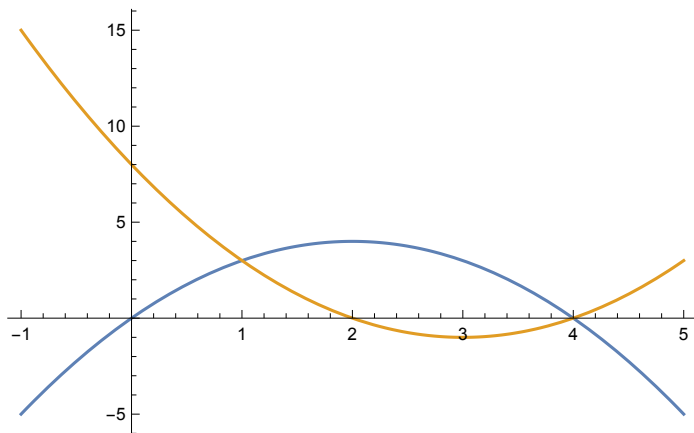
**Solution:** Both curves are parabolas. It is clear that  $f$  is concave down and  $f(x) = x(4 - x)$ , so  $x$ -intercepts at  $x = 0$  and  $x = 4$ .

Meanwhile,  $g$  is concave up and  $x = 4$  is an  $x$ -intercept. It follows that  $g(x) = (x - 4)(x - 2)$  so  $x = 2$  is also an  $x$ -intercept.

The two curves intersect when  $4x - x^2 = x^2 - 6x + 8$ , which reduces to  $x^2 - 5x + 4 = 0$  or  $(x - 4)(x - 1) = 0$ .

The point of intersection is  $(1, 3)$ .

The bounded region fits in a rectangle with base 3 and height 5, so the area must be less than 15.



It is clear that  $f$  is the upper function. The relevant definite integral is given by

$$\begin{aligned} \int_1^4 (4x - x^2) - (x^2 - 6x + 8) dx &= \int_1^4 (10x - 2x^2 - 8) dx = \left[ 5x^2 - 2 \cdot \frac{x^3}{3} - 8x \right]_1^4 \\ &= \left( 5 \cdot 4^2 - 2 \cdot \frac{4^3}{3} - 8 \cdot 4 \right) - \left( 5 \cdot 1^2 - 2 \cdot \frac{1^3}{3} - 8 \right) \\ &= 80 - 2 \cdot \frac{64}{3} - 32 - 5 + 2 \cdot \frac{1}{3} + 8 = 51 - 42 = 9 \end{aligned}$$