An Efficiency Wage – Imperfect Information Model of the Phillips Curve

Appendix A

An expression for $\hat{L}_t$ can be obtained by totally differentiating equation (2) and dividing by the original equation. This yields

$$\hat{L}_t = \frac{\gamma\hat{W}_t - \hat{Y}_t - (\gamma-1)\hat{A}_t - \phi(\gamma-1)e^{-1}[e_W \frac{W_t}{\overline{W}_t^e}\hat{W}_t - e_W \frac{W_t}{\overline{W}_t^e}\hat{\overline{W}}_t^e + e_u du_t] - \gamma\hat{\overline{P}}_t}{\phi(\gamma-1)-\gamma}.$$

By making the substitutions $du_t = -s_L\hat{L}_t$, $W_t/\overline{W}_t^e = ee_w^{-1}$, and $\hat{\overline{P}}_t = \hat{M}_t - \hat{Y}_t$, the solution for $\hat{L}_t$ becomes

$$\text{(A1)} \qquad \hat{L}_t = \frac{[\phi(\gamma-1)-\gamma]\hat{W}_t - \phi(\gamma-1)\hat{\overline{W}}_t^e - (\gamma-1)\hat{Y}_t + \phi(\gamma-1)\hat{A}_t + \gamma\hat{M}_t}{\eta},$$

where

$$\eta = \phi + \gamma - \phi\gamma + \phi(\gamma-1)e^{-1}s_L e_u > 0.$$

An equation for $\hat{Y}_t$ can be obtained by setting Q_t=Y_t in the production function (from assumption 3) and totally differentiating. This yields

$$\text{(A2)} \qquad \hat{Y}_t = \phi\hat{A}_t + \phi\hat{L}_t + \phi e^{-1}[e_W\hat{W}_t - e_W\hat{\overline{W}}_t^e + e_u du_t]$$

$$= \phi\hat{A}_t + \phi\hat{L}_t + \phi e^{-1}[e_W\hat{W}_t - e_W\hat{\overline{W}}_t^e - e_u s_L\hat{L}_t]$$

Substituting (A2) into (A1) yields

(A3) $\hat{L}_t = \hat{M}_t - \hat{W}_t$.

Thus, the unemployment rate can be expressed as

(A4) $du_t = -s_L(\hat{M}_t - \hat{W}_t)$.

By substituting (A4) into (6), the following equation is obtained:

$$\left[1 - s_L \frac{e_u - e_{Wu}}{e_{WW}}\right]\hat{W}_t = \hat{\overline{W}}_t^e - s_L \frac{e_u - e_{Wu}}{e_{WW}} \hat{M}_t,$$

which can be rewritten as

(A5) $$\hat{W}_t = \frac{e_{WW}}{e_{WW} - s_L(e_u - e_{Wu})}\hat{\overline{W}}_t^e - \frac{s_L(e_u - e_{Wu})}{e_{WW} - s_L(e_u - e_{Wu})}\hat{M}_t.$$

The price level is given by the equation $\hat{P}_t = \hat{M}_t - \hat{Y}_t$. Substituting (A3) into (A2) yields the following expression for $\hat{Y}_t$:

$$\hat{Y}_t = \phi\hat{A}_t + \phi(1 - e^{-1}e_u s_L)\hat{M}_t + e^{-1}e_u s_L \hat{W}_t - \phi\hat{\overline{W}}_t^e.$$

Thus, the price level is given by the equation:

$$\hat{P}_t = [1 - \phi + \phi e^{-1}e_u s_L]\hat{M}_t - \phi\hat{A}_t - e^{-1}e_u s_L \hat{W}_t + \phi\hat{\overline{W}}_t^e.$$

Appendix B

From equation (12), wages are determined from the second-order difference equation:

(B1) $\hat{W}_t - 2a\hat{W}_{t-1} + a\hat{W}_{t-2} = (1-a)\hat{M}_t$,

where $0<a<1$. Equation (B1) has two imaginary roots:

$$\mu_1 = a + i\sqrt{a - a^2}$$

$$\mu_2 = a - i\sqrt{a - a^2} .$$

Then the solution to the difference equation is

(B2) $\hat{W}_t = (1-\alpha)\dfrac{1}{\mu_1 - \mu_2}\sum_{j=0}^{\infty}(\mu_1^{j+1} - \mu_2^{j+1})\hat{M}_{t-j}$.

We now solve for $\hat{W}_t$ when the growth rate of demand follows a stochastic process. It is assumed that

$$\hat{M}_t = 0 \qquad \text{for} \quad t \le 0$$

$$\hat{M}_t - \hat{M}_{t-1} = \hat{M}_{t-1} - \hat{M}_{t-2} + \varepsilon_t \qquad \text{for} \ \ t > 0 .$$

Thus, $\hat{M}_t$ can be expressed as

$$\hat{M}_t = 2\hat{M}_{t-1} - \hat{M}_{t-2} + \varepsilon_t ,$$

which means that

$$\hat{M}_t = \sum_{k=1}^{t} k\varepsilon_{t-k+1} .$$

This is equivalent to the equation,

$$\hat{M}_t = \sum_{k=1}^{t} (t-k+1)\varepsilon_k .$$

Let $b = \sqrt{a-a^2}$. Then $\hat{W}_t$ can be expressed as

$$\hat{W}_t = (1-a)\frac{1}{\mu_1-\mu_2}\sum_{j=0}^{t-1}\left[[(a+bi)^{j+1}-(a-bi)^{j+1}]\sum_{k=1}^{t-j}(t-k+1-j)\varepsilon_k \right]$$

$$= (1-a)\frac{1}{2bi}\sum_{k=1}^{t}\varepsilon_k\sum_{j=0}^{t-k}(t+1-k-j)\left[(a+bi)^{j+1}-(a-bi)^{j+1}\right]$$

$$= (1-a)\frac{1}{2bi}\sum_{k=1}^{t}\varepsilon_k\left[\sum_{j=0}^{t-k}(t+1-k)\left[(a+bi)^{j+1}-(a-bi)^{j+1}\right] - \sum_{j=0}^{t-k}\left[j(a+bi)^{j+1}-j(a-bi)^{j+1}\right]\right]$$

The relationships,

$$\sum_{q=0}^{n} x^q = \frac{1-x^{n+1}}{1-x} \qquad \text{and} \qquad \sum_{q=0}^{n} qx^{q-1} = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} ,$$

allow $\hat{W}_t$ to be rewritten as

$$\hat{W}_t = (1-a)\frac{1}{2bi}\sum_{k=1}^{t}\varepsilon_k\Bigg[(t-k+1)(a+bi)\frac{1-(a+bi)^{t-k+1}}{1-a-bi}$$
$$-(t-k+1)(a-bi)\frac{1-(a-bi)^{t-k+1}}{1-a+bi}$$
$$-(a+bi)^2\frac{1-(t-k+1)(a+bi)^{t-k}+(t-k)(a+bi)^{t-k+1}}{(1-a-bi)^2}$$
$$+(a-bi)^2\frac{1-(t-k+1)(a-bi)^{t-k}+(t-k)(a-bi)^{t-k+1}}{(1-a+bi)^2}\Bigg]$$

$$= (1-a)\frac{1}{2bi}\sum_{k=1}^{t}\varepsilon_k\Bigg[(t-k+1)(a+bi)\frac{[1-(a+bi)^{t-k+1}][1-a+bi]}{[1-a-bi][1-a+bi]}$$
$$-(t-k+1)(a-bi)\frac{[1-(a-bi)^{t-k+1}][1-a-bi]}{[1-a+bi][1-a-bi]}$$
$$-(a+bi)^2\frac{[1-(t-k+1)(a+bi)^{t-k}+(t-k)(a+bi)^{t-k+1}][1-a+bi]^2}{[1-a-bi]^2[1-a+bi]^2}$$
$$+(a-bi)^2\frac{[1-(t-k+1)(a-bi)^{t-k}+(t-k)(a-bi)^{t-k+1}][1-a-bi]^2}{[1-a+bi]^2[1-a-bi]^2}\Bigg]$$

$$= (1-a)\frac{1}{2bi}\sum_{k=1}^{t}\varepsilon_k\Bigg[(t-k+1)(a+bi)\frac{[1-(a+bi)^{t-k+1}][1-a+bi][(1-a)^2+b^2]}{[(1-a)^2+b^2]^2}$$
$$-(t-k+1)(a-bi)\frac{[1-(a-bi)^{t-k+1}][1-a-bi][(1-a)^2+b^2]}{[(1-a)^2+b^2]^2}$$
$$-(a+bi)^2\frac{[1-(t-k+1)(a+bi)^{t-k}+(t-k)(a+bi)^{t-k+1}][1-a+bi]^2}{[(1-a)^2+b^2]^2}$$
$$+(a-bi)^2\frac{[1-(t-k+1)(a-bi)^{t-k}+(t-k)(a-bi)^{t-k+1}][1-a-bi]^2}{[(1-a)^2+b^2]^2}\Bigg]$$

Letting $\rho = \sqrt{a}$, then $b = \rho\sqrt{1-\rho^2}$.

This allows $\hat{W}_t$ to be expressed as,

$$\hat{W}_t = \sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho(a+bi)^{t-k+1}-\rho(a-bi)^{t-k+1}}{2\sqrt{1-\rho^2}\,i}\right]$$

$$\hat{W}_t = \sum_{k=1}^{t} \varepsilon_k \left[t-k+1-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}} \right].$$

This last equation is obtained from the fact that

$$(a+bi)^{t-k+1}-(a-bi)^{t-k+1} = 2i\rho^{t-k+1}\sin[\psi(t-k+1)],$$

$$\text{where } \rho=\sqrt{a^2+b^2} \text{ and } \psi = \arccos(a/\rho).$$

Given the above expression for wages, the change in wages between periods *t*-1 and *t* is

$$\begin{aligned}\hat{W}_t - \hat{W}_{t-1} &= \sum_{k=1}^{t} \varepsilon_k \left[t-k+1-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}} \right] \\ &\quad -\sum_{k=1}^{t-1} \varepsilon_k \left[t-k-\frac{\rho^{t-k+1}\sin[\psi(t-k)]}{\sqrt{1-\rho^2}} \right]\end{aligned}$$

$$= \sum_{k=1}^{t} \varepsilon_k \left[1-\rho^{t-k+1}\cos[\psi(t-k+1)]\right]$$

The unemployment rate can also be expressed in terms of the underlying demand shocks. From (9),

$$du_t = -s_L(\hat{M}_t - \hat{W}_t)$$

$$= -\frac{s_L}{\sqrt{1-\rho^2}} \sum_{k=1}^{t} \varepsilon_k \rho^{t-k+2} \sin[\psi(t-k+1)].$$

An expression for the price level can be derived by substituting (11) into (10), yielding

$$\hat{P}_t = [1-\phi+\phi e^{-1}e_u s_L]\hat{M}_t - \phi\hat{A}_t + \phi(\omega - e^{-1}e_u s_L)\hat{W}_t + 2\phi(1-\omega)\hat{W}_{t-1} - \phi(1-\omega)\hat{W}_{t-2}$$

$$\begin{aligned}
&= [1-\phi+\phi e^{-1}e_u s_L]\sum_{k=1}^{t}\varepsilon_k(t-k+1) \\
&\quad + \phi(\omega - e^{-1}e_u s_L)\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right] \\
&\quad + 2\phi(1-\omega)\sum_{k=1}^{t-1}\varepsilon_k\left[t-k-\frac{\rho^{t-k+1}\sin[\psi(t-k)]}{\sqrt{1-\rho^2}}\right] \\
&\quad - \phi(1-\omega)\sum_{k=1}^{t-2}\varepsilon_k\left[t-k-1-\frac{\rho^{t-k}\sin[\psi(t-k-1)]}{\sqrt{1-\rho^2}}\right]
\end{aligned}$$

$$\begin{aligned}
&= [1-\phi+\phi e^{-1}e_u s_L]\sum_{k=1}^{t}\varepsilon_k(t-k+1) \\
&\quad + \phi(\omega - e^{-1}e_u s_L)\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right] \\
&\quad + 2\phi(1-\omega)\sum_{k=1}^{t-1}\varepsilon_k\left[t-k-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]-\rho^{t-k+1}\sqrt{1-\rho^2}\cos[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right] \\
&\quad - \phi(1-\omega)\sum_{k=1}^{t-2}\varepsilon_k\left[t-k-1-\frac{\rho^{t-k}(2\rho^2-1)\sin[\psi(t-k+1)]-2\sqrt{1-\rho^2}\,\rho^{t-k+1}\cos[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= [1-\phi+\phi e^{-1}e_u s_L]\sum_{k=1}^{t}\varepsilon_k(t-k+1) \\
&\qquad +\phi(\omega-e^{-1}e_u s_L)\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right] \\
&\qquad +2\phi(1-\omega)\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right]+2\phi(1-\omega)\rho^2\varepsilon_t \\
&\qquad -2\phi(1-\omega)\sum_{k=1}^{t}\varepsilon_k \\
&\qquad -\phi(1-\omega)\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho^{t-k}(2\rho^2-1)\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right]+2\phi(1-\omega)\sum_{k=1}^{t}\varepsilon_k \\
&\qquad -\phi(1-\omega)\varepsilon_t+2\phi(1-\omega)\rho^2(2\rho^2-1)\varepsilon_{t-1}-2\phi(1-\omega)\rho^2(2\rho^2-1)\varepsilon_{t-1} \\
&\qquad -\phi(1-\omega)(2\rho^2-1)\varepsilon_t
\end{aligned}
$$

$$
\begin{aligned}
&= [1-\phi+\phi e^{-1}e_u s_L]\sum_{k=1}^{t}\varepsilon_k(t-k+1) \\
&\qquad +\phi(\omega-e^{-1}e_u s_L)\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right] \\
&\qquad +2\phi(1-\omega)\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho^{t-k+2}\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right] \\
&\qquad -\phi(1-\omega)\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\rho^{t-k}(2\rho^2-1)\sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}}\right]
\end{aligned}
$$

$$
=\sum_{k=1}^{t}\varepsilon_k\left[t-k+1-\frac{\phi(\omega-e^{-1}e_u s_L)\rho^2+\phi(1-\omega)}{\sqrt{1-\rho^2}}\rho^{t-k}\sin[\psi(t-k+1)]\right].
$$

The rate of price inflation is

$$
\hat{P}_t-\hat{P}_{t-1}=\sum_{k=1}^{t}\varepsilon_k\left[1-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-1}\cos[\psi(t-k+1)]\right]
$$

Also,

$$\hat{P}_{t-1} - \hat{P}_{t-2} = \sum_{k=1}^{t-1} \varepsilon_k \left[1 - \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)\rho^{t-k-2} \cos[\psi(t-k)]\right]$$

$$\hat{P}_{t-1} - \hat{P}_{t-2} = \sum_{k=1}^{t-1} \varepsilon_k \Big[1 - \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)\rho^{t-k-1} \cos[\psi(t-k+1)]$$
$$- \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)\rho^{t-k-2} \sqrt{1-\rho^2} \sin[\psi(t-k+1)]\Big]$$

Subtracting $\hat{P}_t$ from $\hat{W}_t$ yields an expression for real wages. Accordingly,

$$\hat{W}_t - \hat{P}_t = \sum_{k=1}^{t} \varepsilon_k \left[t - k + 1 - \frac{\rho^{t-k+2} \sin[\psi(t-k+1)]}{\sqrt{1-\rho^2}} \right]$$
$$- \sum_{k=1}^{t} \varepsilon_k \left[t - k + 1 - \frac{\phi(\omega - e^{-1} e_u s_L)\rho^2 + \phi(1-\omega)}{\sqrt{1-\rho^2}} \rho^{t-k} \sin[\psi(t-k+1)] \right]$$

$$= \frac{\phi(\omega - e^{-1} e_u s_L)\rho^2 + \phi(1-\omega) - \rho^2}{\sqrt{1-\rho^2}} \sum_{k=1}^{t} \varepsilon_k \rho^{t-k} \sin[\psi(t-k+1)]$$

Appendix C

Demand shocks

Regression : $(\hat{W}_t - \hat{W}_{t-1}) = \beta_1 du_t + \beta_2 (\hat{P}_{t-1} - \hat{P}_{t-2}) + e_t$

$$\mathbf{X'X} = \begin{bmatrix} \sum_{t=1}^{T} du_t^2 & \sum_{t=1}^{T} du_t (\hat{P}_{t-1} - \hat{P}_{t-2}) \\ \sum_{t=1}^{T} (\hat{P}_{t-1} - \hat{P}_{t-2}) du_t & \sum_{t=1}^{T} (\hat{P}_{t-1} - \hat{P}_{t-2})^2 \end{bmatrix}$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{\begin{bmatrix} \sum_{t=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2 & -\sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2}) \\ -\sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2}) & \sum_{t=1}^{T} du_t^2 \end{bmatrix}}{\sum_{t=1}^{T} du_t^2 \sum_{t=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2 - \left(\sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2})\right)^2}$$

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} \sum_{t=1}^{T} du_t(\hat{W}_t - \hat{W}_{t-1}) \\ \sum_{t=1}^{T}(\hat{W}_t - \hat{W}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2}) \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}} = \frac{\begin{bmatrix} \sum_{t=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2 \sum_{t=1}^{T} du_t(\hat{W}_t - \hat{W}_{t-1}) - \sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2}) \sum_{t=1}^{T}(\hat{W}_t - \hat{W}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2}) \\ -\sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2}) \sum_{t=1}^{T} du_t(\hat{W}_t - \hat{W}_{t-1}) + \sum_{t=1}^{T} du_t^2 \sum_{t=1}^{T}(\hat{W}_t - \hat{W}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2}) \end{bmatrix}}{\sum_{t=1}^{T} du_t^2 \sum_{t=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2 - \left(\sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2})\right)^2}$$

Define

$$D = \sum_{t=1}^{T} du_t^2 \sum_{t=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2 - \left(\sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2})\right)^2$$

$$N_1 = \sum_{t=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2 \sum_{t=1}^{T} du_t(\hat{W}_t - \hat{W}_{t-1}) - \sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2}) \sum_{t=1}^{T}(\hat{W}_t - \hat{W}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2})$$

$$N_2 = -\sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2}) \sum_{t=1}^{T} du_t(\hat{W}_t - \hat{W}_{t-1}) + \sum_{t=1}^{T} du_t^2 \sum_{t=1}^{T}(\hat{W}_t - \hat{W}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2})$$

Then $\beta_1 = \dfrac{N_1}{D}$

$$\beta_2 = \frac{N_2}{D}.$$

$$\lim_{T\to\infty}\beta_1 = \frac{\lim_{T\to\infty}(N_1/T^3)}{\lim_{T\to\infty}(D/T^3)}$$

$$\lim_{T\to\infty}\beta_2 = \frac{\lim_{T\to\infty}(N_2/T^3)}{\lim_{T\to\infty}(D/T^3)}$$

Calculations of sums are derived in Appendix D.

$$\sum_{t=t_0}^{T} du_t^2 = \sum_{t=t_0}^{T}\left[-\frac{s_L\rho^2}{\sqrt{1-\rho^2}}\sum_{k=1}^{t}\varepsilon_k\rho^{t-k}\sin[\psi(t-k+1)]\right]^2$$

$$= \frac{\sigma^2 s_L^2\rho^4}{1-\rho^2}\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2(t-k)}\sin^2[\psi(t-k+1)]$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}du_t^2 = \frac{\sigma^2 s_L^2\rho^4}{1-\rho^2}\frac{1-\rho^4}{(1-\rho^2)(1+2\rho^2-3\rho^4)}$$

$$\sum_{t=t_0}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2$$

$$= \sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t-1}\varepsilon_k\left[1-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-2}\cos[\psi(t-k)]\right]\right\}^2$$

$$= \sigma^2\sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t-1}\left[1-2\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-2}\cos[\psi(t-k)]\right.\right.$$
$$\left.\left.+\phi^2\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)^2\rho^{2t-2k-4}\cos^2[\psi(t-k)]\right]\right\}$$

$$= \sigma^2 \frac{(T + t_0 - 2)(T - t_0 + 1)}{2}$$

$$+ \sigma^2 \sum_{t=t_0}^{T} \left\{ \sum_{k=1}^{t-1} \left[-2\phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \rho^{t-k-2} \cos[\psi(t-k)] \right.\right.$$

$$\left.\left. + \phi^2 \left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)^2 \rho^{2t-2k-4} \cos^2[\psi(t-k)] \right]\right\}$$

$$\lim_{T\to\infty} \frac{1}{T^2} \sum_{t=t_0}^{T} (\hat{P}_{t-1} - \hat{P}_{t-2})^2 = \frac{\sigma^2}{2}$$

$$\sum_{t=t_0}^{T} du_t (\hat{P}_{t-1} - \hat{P}_{t-2}) = \sum_{t=t_0}^{T} \left\{ -\frac{s_L \rho^2}{\sqrt{1-\rho^2}} \sum_{k=1}^{t} \varepsilon_k \rho^{t-k} \sin[\psi(t-k+1)] \right.$$

$$\left. \times \sum_{k=1}^{t-1} \varepsilon_k \left[1 - \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \rho^{t-k-1} \cos[\psi(t-k)]\right] \right\}$$

$$= \sum_{t=t_0}^{T} \left\{ -\frac{\sigma^2 s_L \rho^2}{\sqrt{1-\rho^2}} \sum_{k=1}^{t-1} \left[\rho^{t-k} \sin[\psi(t-k+1)] \right.\right.$$

$$\left.\left. \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \rho^{2t-2k-1} \sin[\psi(t-k+1)] \cos[\psi(t-k)] \right] \right\}$$

$$= \sum_{t=t_0}^{T} \left\{ -\frac{s_L \rho^2}{\sqrt{1-\rho^2}} \sum_{k=1}^{t} \varepsilon_k \rho^{t-k} \sin[\psi(t-k+1)] \right.$$

$$\times \sum_{k=1}^{t-1} \varepsilon_k \left[1 - \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \rho^{t-k-1} \cos[\psi(t-k+1)] \right.$$

$$\left.\left. - \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \rho^{t-k-2} \sqrt{1-\rho^2} \sin[\psi(t-k+1)] \right] \right\}$$

$$= -\frac{\sigma^2 s_L \rho^2}{\sqrt{1-\rho^2}} \sum_{t=t_0}^{T} \left\{ \sum_{k=1}^{t-1} \rho^{t-k} \sin[\psi(t-k+1)] \right.$$

$$- \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \left[\sum_{k=1}^{t-1} \rho^{2t-2k-1} \sin[\psi(t-k+1)] \cos[\psi(t-k+1) \right.$$

$$\left.\left. + \sqrt{1-\rho^2} \sum_{k=1}^{t-1} \rho^{2t-2k-2} \sin^2[\psi(t-k+1)] \right] \right\}$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}du_t(\hat{P}_{t-1}-\hat{P}_{t-2})$$

$$=-\frac{\sigma^2 s_L\rho^2}{\sqrt{1-\rho^2}}\left[\frac{\rho^2}{\sqrt{1-\rho^2}}-\phi\Big((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\Big)\frac{3(1-\rho^2)\rho^2\sqrt{1-\rho^2}}{(1-\rho^2)(1+2\rho^2-3\rho^4)}\right]$$

$$=-\sigma^2 s_L\rho^2\left[\frac{\rho^2(1+2\rho^2-3\rho^4)-3\rho^2(1-\rho^2)\phi\Big((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\Big)}{(1-\rho^2)(1+2\rho^2-3\rho^4)}\right]$$

$$\sum_{t=t_0}^{T}du_t(\hat{W}_t-\hat{W}_{t-1})=\sum_{t=t_0}^{T}\left\{-\frac{s_L\rho^2}{\sqrt{1-\rho^2}}\sum_{k=1}^{t}\varepsilon_k\rho^{t-k}\sin[\psi(t-k+1)]\right.$$
$$\left.\times\sum_{k=1}^{t}\varepsilon_k\Big[1-\rho^{t-k+1}\cos[\psi(t-k+1)]\Big]\right\}$$

$$=-\frac{\sigma^2 s_L\rho^2}{\sqrt{1-\rho^2}}\sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t}\rho^{t-k}\sin[\psi(t-k+1)]\right.$$
$$\left.-\sum_{k=1}^{t}\rho^{2t-2k+1}\sin[\psi(t-k+1)]\cos[\psi(t-k+1)]\right\}$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}du_t(\hat{W}_t-\hat{W}_{t-1})=-\frac{\sigma^2 s_L\rho^2}{\sqrt{1-\rho^2}}\left[\frac{1}{\sqrt{1-\rho^2}}-\frac{\rho^2\sqrt{1-\rho^2}}{1+2\rho^2-3\rho^4}\right]$$

$$=-\sigma^2 s_L\rho^2\left[\frac{1+\rho^2-2\rho^4}{(1-\rho^2)(1+2\rho^2-3\rho^4)}\right]$$

$$\sum_{t=t_0}^{T}(\hat{W}_t-\hat{W}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2})$$

$$=\sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t}\varepsilon_k\left[1-\rho^{t-k+1}\cos[\psi(t-k+1)]\right]\right.$$

$$\left.\times\sum_{k=1}^{t-1}\varepsilon_k\left[1-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-2}\cos[\psi(t-k)]\right]\right\}$$

$$=\sigma^2\sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t-1}\left[1-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-2}\cos[\psi(t-k)]\right.\right.$$

$$-\rho^{t-k+1}\cos[\psi(t-k+1)$$

$$\left.\left.+\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{2t-2k-1}\cos[\psi(t-k+1)]\cos[\psi(t-k)]\right]\right\}$$

$$=\sigma^2\frac{(T+t_0-2)(T-t_0+1)}{2}$$

$$+\sigma^2\sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t-1}\left[-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-2}\cos[\psi(t-k)]\right.\right.$$

$$-\rho^{t-k+1}\cos[\psi(t-k+1)$$

$$\left.\left.+\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{2t-2k-1}\cos[\psi(t-k+1)]\cos[\psi(t-k)]\right]\right\}$$

$$\lim_{T\to\infty}\frac{1}{T^2}\sum_{t=t_0}^{T}(\hat{W}_t-\hat{W}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2})=\frac{\sigma^2}{2}$$

Thus,

$$\lim_{T\to\infty}\frac{D}{T^3}=\lim_{T\to\infty}\left[\frac{\sum_{t=1}^{T}du_t^2}{T}\frac{\sum_{j=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2}{T^2}-\frac{\left(\sum_{t=1}^{T}du_t(\hat{P}_{t-1}-\hat{P}_{t-2})\right)^2}{T^3}\right]$$

$$=\frac{\sigma^4 s_L^2\rho^4(1-\rho^4)}{2(1-\rho^2)^2(1+2\rho^2-3\rho^4)}$$

$$\lim_{T\to\infty} \frac{N_1}{T^3}$$

$$= \lim_{T\to\infty} \left[\frac{\sum_{t=1}^{T} (\hat{P}_{t-1} - \hat{P}_{t-2})^2}{T^2} \frac{\sum_{t=1}^{T} du_t (\hat{W}_t - \hat{W}_{t-1})}{T} \right.$$

$$\left. - \frac{\sum_{t=1}^{T} du_t (\hat{P}_{t-1} - \hat{P}_{t-2})}{T} \frac{\sum_{t=1}^{T} (\hat{W}_t - \hat{W}_{t-1})(\hat{P}_{t-1} - \hat{P}_{t-2})}{T^2} \right]$$

$$= -\frac{\sigma^4 s_L \rho^2 (1+\rho^2 - 2\rho^4)}{2(1-\rho^2)(1+2\rho^2 - 3\rho^4)} + \sigma^4 s_L \rho^2 \left[\frac{\rho^2 (1+2\rho^2 - 3\rho^4)}{2(1-\rho^2)(1+2\rho^2 - 3\rho^4)} \right.$$
$$\left. - \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \frac{3\rho^2 (1-\rho^2)}{2(1-\rho^2)(1+2\rho^2 - 3\rho^4)} \right]$$

$$= \frac{\sigma^4 s_L \rho^2}{2(1-\rho^2)(1+2\rho^2 - 3\rho^4)} \left[-1 + 4\rho^4 - 3\rho^6 - 3\rho^2 (1-\rho^2) \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \right]$$

$$\lim_{T\to\infty} \frac{N_2}{T^3}$$
$$= \lim_{T\to\infty} \left[- \frac{\sum_{t=1}^{T} du_t (\hat{P}_{t-1} - \hat{P}_{t-2})}{T} \frac{\sum_{t=1}^{T} du_t (\hat{W}_t - \hat{W}_{t-1})}{T^2} + \frac{\sum_{t=1}^{T} du_t^2}{T} \frac{\sum_{t=1}^{T} (\hat{W}_t - \hat{W}_{t-1})(\hat{P}_{t-1} - \hat{P}_{t-2})}{T^2} \right]$$

$$= \frac{\sigma^2 s_L^2 \rho^4 (1-\rho^4)}{2(1-\rho^2)^2 (1+2\rho^2 - 3\rho^4)}$$

$$\lim_{T\to\infty} \hat{\beta}_1 = \lim_{T\to\infty} \frac{(N_1 / T^3)}{(D / T^3)}$$

$$= \frac{(1-\rho^2)\left[-1+4\rho^4-3\rho^6-3\rho^2(1-\rho^2)\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\right]}{s_L\rho^2(1-\rho^4)}$$

$$= \frac{(1-\rho^2)\left[-1+\rho^4+3\rho^4-3\rho^6-3\rho^2(1-\rho^2)\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\right]}{s_L\rho^2(1-\rho^4)}$$

$$= -\frac{1-\rho^2}{s_L\rho^2}+3\frac{(1-\rho^2)\left[\rho^4(1-\rho^2)-\rho^2(1-\rho^2)\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\right]}{s_L\rho^2(1+\rho^2)(1-\rho^2)}$$

$$= -\frac{1-a}{s_L a}+3(1-\rho^2)\frac{\rho^2-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)}{s_L(1+\rho^2)}$$

$$= \frac{e_u-e_{Wu}}{(1-\omega)e_{WW}}+3(1-a)\frac{a-\phi\left((\omega-e^{-1}e_u s_L)a+1-\omega\right)}{s_L(1+a)}$$

$$\lim_{T\to\infty}\hat{\beta}_2 = \lim_{T\to\infty}\frac{(N_2/T^3)}{(D/T^3)} = 1$$

$$\underline{\text{Regression}: (\hat{P}_t-\hat{P}_{t-1}) = \beta_1 du_t + \beta_2(\hat{P}_{t-1}-\hat{P}_{t-2}) + e_t}$$

$$\hat{\boldsymbol{\beta}} = \frac{\begin{bmatrix} \sum_{t=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2\sum_{t=1}^{T}du_t(\hat{P}_t-\hat{P}_{t-1}) - \sum_{t=1}^{T}du_t(\hat{P}_{t-1}-\hat{P}_{t-2})\sum_{t=1}^{T}(\hat{P}_t-\hat{P}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2}) \\ -\sum_{t=1}^{T}du_t(\hat{P}_{t-1}-\hat{P}_{t-2})\sum_{t=1}^{T}du_t(\hat{P}_t-\hat{P}_{t-1}) + \sum_{t=1}^{T}du_t^2\sum_{t=1}^{T}(\hat{P}_t-\hat{P}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2}) \end{bmatrix}}{\sum_{t=1}^{T}du_t^2\sum_{t=1}^{T}(\hat{P}_{t-1}-\hat{P}_{t-2})^2 - \left(\sum_{t=1}^{T}du_t(\hat{P}_{t-1}-\hat{P}_{t-2})\right)^2}$$

$$\sum_{t=t_0}^{T} du_t(\hat{P}_t - \hat{P}_{t-1})$$

$$= \sum_{t=t_0}^{T}\left\{-\frac{s_L\rho^2}{\sqrt{1-\rho^2}}\sum_{k=1}^{t}\varepsilon_k\rho^{t-k}\sin[\psi(t-k+1)]\right.$$
$$\left.\times\sum_{k=1}^{t}\varepsilon_k\left[1-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-1}\cos[\psi(t-k+1)]\right]\right\}$$

$$= -\frac{\sigma^2 s_L\rho^2}{\sqrt{1-\rho^2}}\sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t}\rho^{t-k}\sin[\psi(t-k+1)]\right.$$
$$\left.-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{-2}\sum_{k=1}^{t}\rho^{2t-2k+1}\cos[\psi(t-k+1)]\sin[\psi(t-k+1)\right\}$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} du_t(\hat{P}_t - \hat{P}_{t-1})$$
$$= -\frac{\sigma^2 s_L\rho^2}{\sqrt{1-\rho^2}}\left[\frac{1}{\sqrt{1-\rho^2}}-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\frac{\sqrt{1-\rho^2}}{1+2\rho^2-3\rho^4}\right]$$

$$= -\sigma^2 s_L\rho^2\left[\frac{1+2\rho^2-3\rho^4}{(1-\rho^2)(1+2\rho^2-3\rho^4)}-\frac{\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)(1-\rho^2)}{(1-\rho^2)(1+2\rho^2-3\rho^4)}\right]$$

$$\sum_{t=t_0}^{T}(\hat{P}_t - \hat{P}_{t-1})(\hat{P}_{t-1} - \hat{P}_{t-2})$$
$$= \sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t}\varepsilon_k\left[1-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-1}\cos[\psi(t-k+1)]\right]\right.$$
$$\left.\times\sum_{k=1}^{t-1}\varepsilon_k\left[1-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-2}\cos[\psi(t-k)]\right]\right\}$$

$$= \sigma^2\sum_{t=t_0}^{T}\left\{\sum_{k=1}^{t-1}\left[1-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-2}\cos[\psi(t-k)]\right.\right.$$
$$-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\rho^{t-k-1}\cos[\psi(t-k+1)]$$
$$\left.\left.+\phi^2\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)^2\rho^{2t-2k-3}\cos[\psi(t-k)]\cos[\psi(t-k+1)]\right]\right\}$$

$$= \sigma^2 \frac{(T + t_0 - 2)(T - t_0 + 1)}{2}$$

$$+ \sigma^2 \sum_{t=t_0}^{T} \left\{ \sum_{k=1}^{t-1} \left[-\phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)\rho^{t-k-2} \cos[\psi(t-k)] \right.\right.$$

$$- \phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)\rho^{t-k-1} \cos[\psi(t-k+1)]$$

$$\left.\left. + \phi^2 \left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)^2 \rho^{2t-2k-3} \cos[\psi(t-k)]\cos[\psi(t-k+1)] \right] \right\}$$

$$\lim_{t \to \infty} \frac{1}{T^2} \sum_{t=t_0}^{T} (\hat{P}_t - \hat{P}_{t-1})(\hat{P}_{t-1} - \hat{P}_{t-2}) = \frac{\sigma^2}{2}$$

In this case, D is the same as before.

$$\lim_{T \to \infty} \frac{N_1}{T^3}$$

$$= \lim_{T \to \infty} \left[\frac{\sum_{t=1}^{T} (\hat{P}_{t-1} - \hat{P}_{t-2})^2}{T^2} \frac{\sum_{t=1}^{T} du_t (\hat{P}_t - \hat{P}_{t-1})}{T} - \frac{\sum_{t=1}^{T} du_t (\hat{P}_{t-1} - \hat{P}_{t-2})}{T} \frac{\sum_{t=1}^{T} (\hat{P}_t - \hat{P}_{t-1})(\hat{P}_{t-1} - \hat{P}_{t-2})}{T^2} \right]$$

$$= -\sigma^4 s_L \rho^2 \left[\frac{1 + 2\rho^2 - 3\rho^4}{2(1-\rho^2)(1+2\rho^2-3\rho^4)} - \frac{\phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)(1-\rho^2)}{2(1-\rho^2)(1+2\rho^2-3\rho^4)} \right]$$

$$+ \sigma^4 s_L \rho^2 \left[\frac{\rho^2(1+2\rho^2-3\rho^4) - 3\rho^2(1-\rho^2)\phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right)}{2(1-\rho^2)(1+2\rho^2-3\rho^4)} \right]$$

$$= -\frac{\sigma^4 s_L \rho^2}{2(1-\rho^2)(1+2\rho^2-3\rho^4)} \left[1 + \rho^2 - 5\rho^4 + 3\rho^6 \right.$$

$$\left. - (1-3\rho^2)(1-\rho^2)\phi\left((\omega - e^{-1} e_u s_L)\rho^2 + 1 - \omega\right) \right]$$

$$\lim_{T \to \infty} \frac{N_2}{T^3}$$

$$= \lim_{T\to\infty}\left[-\frac{\sum_{t=1}^{T} du_t(\hat{P}_{t-1}-\hat{P}_{t-2})}{T}\frac{\sum_{t=1}^{T} du_t(\hat{P}_t-\hat{P}_{t-1})}{T^2} + \frac{\sum_{t=1}^{T} du_t^2}{T}\frac{\sum_{t=1}^{T}(\hat{P}_t-\hat{P}_{t-1})(\hat{P}_{t-1}-\hat{P}_{t-2})}{T^2}\right]$$

$$= \frac{\sigma^2 s_L^2 \rho^4 (1-\rho^4)}{2(1-\rho^2)^2(1+2\rho^2-3\rho^4)}$$

$$\lim_{T\to\infty}\hat{\beta}_1 = \lim_{T\to\infty}\frac{(N_1/T^3)}{(D/T^3)}$$

$$= \frac{-(1-\rho^2)\left[1+\rho^2-5\rho^4+3\rho^6 - (1-3\rho^2)(1-\rho^2)\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\right]}{s_L(1-\rho^4)}$$

$$= -\frac{(1-\rho^2)(1-\rho^4)}{s_L\rho^2(1-\rho^4)} - \frac{(1-\rho^2)\left[\rho^2-4\rho^4+3\rho^6 - (1-3\rho^2)(1-\rho^2)\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\right]}{s_L\rho^2(1-\rho^4)}$$

$$= -\frac{(1-\rho^2)}{s_L\rho^2} - \frac{(1-\rho^2)\left[\rho^2(1-3\rho^2)(1-\rho^2) - (1-3\rho^2)(1-\rho^2)\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\right]}{s_L\rho^2(1-\rho^4)}$$

$$= -\frac{1-a}{s_L a} - \frac{(1-\rho^2)(1-3\rho^2)(1-\rho^2)\left[\rho^2-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\right]}{s_L\rho^2(1-\rho^2)(1+\rho^2)}$$

$$= \frac{e_u-e_{Wu}}{(1-\omega)e_{WW}} - \frac{(1-3\rho^2)(1-\rho^2)\left[\rho^2-\phi\left((\omega-e^{-1}e_u s_L)\rho^2+1-\omega\right)\right]}{s_L\rho^2(1+\rho^2)}$$

$$= \frac{e_u-e_{Wu}}{(1-\omega)e_{WW}} - \frac{(1-3a)(1-a)\left[a-\phi\left((\omega-e^{-1}e_u s_L)a+1-\omega\right)\right]}{s_L a(1+a)}$$

$$\lim_{T\to\infty}\hat{\beta}_2 = \lim_{T\to\infty}\frac{(N_2/T^3)}{(D/T^3)} = 1$$

Appendix D
Calculations of Sums

$$\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{t-k}\sin[\psi(t-k+1)]$$

$$=\sum_{t=t_0}^{T}\sum_{x=0}^{t-1}\rho^{x}\sin[\psi(x+1)]$$

$$=\sum_{t=t_0}^{T}\left\{\sum_{x=0}^{\infty}\rho^{x}\sin[\psi(x+1)]-\sum_{x=t}^{\infty}\rho^{x}\sin[\psi(x+1)]\right\}$$

$$=\sum_{t=t_0}^{T}\left\{\frac{1}{\sqrt{1-\rho^2}}-\rho^{t}\sum_{x=0}^{\infty}\rho^{x}\big[\sin[\psi t]\cos[\psi(x+1)]+\cos[\psi t]\sin[\psi(x+1)]\big]\right\}$$

$$=\sum_{t=t_0}^{T}\left\{\frac{1}{\sqrt{1-\rho^2}}-\frac{\rho^{t}\cos[\psi t]}{\sqrt{1-\rho^2}}\right\}$$

$$\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{t-k}\sin[\psi(t-k+1)]}{T}=\frac{1}{\sqrt{1-\rho^2}}$$

$$\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t-1}\rho^{t-k}\sin[\psi(t-k+1)]}{T}$$

$$=\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{t-k}\sin[\psi(t-k+1)]}{T}-\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sin[\psi]}{T}$$

$$=\frac{1}{\sqrt{1-\rho^2}}-\sqrt{1-\rho^2}=\frac{\rho^2}{\sqrt{1-\rho^2}}$$

$$\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2(t-k)}\sin^2[\psi(t-k+1)]$$

$$=\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2(t-k)}\frac{1}{2}\big[1-\cos[2\psi(t-k+1)]\big]$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\sum_{x=0}^{t-1}\rho^{2x}\left[1-\cos[2\psi(x+1)]\right]$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\left\{\frac{1-\rho^{2t}}{1-\rho^2} - \sum_{x=0}^{\infty}\rho^{2x}\cos[2\psi(x+1)] + \sum_{x=t}^{\infty}\rho^{2x}\cos[2\psi(x+1)]\right\}$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\left\{\frac{1-\rho^{2t}}{1-\rho^2} - \frac{\rho^2-1}{1+2\rho^2-3\rho^4} + \rho^{2t}\cos(2\psi t)\sum_{x=0}^{\infty}\rho^{2x}\cos[2\psi(x+1)] \right.$$
$$\left. - \rho^{2t}\sin(2\psi t)\sum_{x=0}^{\infty}\rho^{2x}\sin[2\psi(x+1)]\right\}$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\left\{\frac{1-\rho^{2t}}{1-\rho^2} - \frac{\rho^2-1}{1+2\rho^2-3\rho^4} + \rho^{2t}\cos(2\psi t)\frac{\rho^2-1}{1+2\rho^2-3\rho^4} \right.$$
$$\left. - 2\rho^{2t}\sin(2\psi t)\frac{\rho\sqrt{1-\rho^2}}{1+2\rho^2-3\rho^4}\right\}$$

$$\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2(t-k)}\sin^2[\psi(t-k+1)]}{T} = \frac{1}{2}\left[\frac{1}{1-\rho^2} - \frac{\rho^2-1}{1+2\rho^2-3\rho^4}\right]$$

$$= \frac{1-\rho^4}{(1-\rho^2)(1+2\rho^2-3\rho^4)}$$

$$\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t-1}\rho^{2t-2k-2}\sin^2[\psi(t-k+1)]}{T}$$

$$= \lim_{T\to\infty}\rho^{-2}\left[\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2(t-k)}\sin^2[\psi(t-k+1)]}{T} - \frac{\sum_{t=t_0}^{T}\sin^2[\psi]}{T}\right]$$

$$= \rho^{-2}\left[\frac{1-\rho^4}{(1-\rho^2)(1+2\rho^2-3\rho^4)} - (1-\rho^2)\right]$$

$$= \rho^2 \frac{5-8\rho^2+3\rho^4}{(1-\rho^2)(1+2\rho^2-3\rho^4)}$$

$$\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2t-2k+1}\sin[\psi(t-k+1)]\cos[\psi(t-k+1)]$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2t-2k+1}\sin[2\psi(t-k+1)]$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\sum_{x=0}^{t-1}\rho^{2x+1}\sin[2\psi(x+1)]$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\left\{\sum_{x=0}^{\infty}\rho^{2x+1}\sin[2\psi(x+1)] - \sum_{x=t}^{\infty}\rho^{2x+1}\sin[2\psi(x+1)]\right\}$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\left\{\frac{2\rho^2\sqrt{1-\rho^2}}{1+2\rho^2-3\rho^4} - \rho^{2t+1}\sum_{x=0}^{\infty}\rho^{2x}\left[\sin[2\psi t]\cos[2\psi(x+1)] + \cos[2\psi t]\sin[2\psi(x+1)]\right]\right\}$$

$$= \frac{1}{2}\sum_{t=t_0}^{T}\left\{\frac{2\rho^2\sqrt{1-\rho^2}}{1+2\rho^2-3\rho^4} - \rho^{2t+1}\left[\sin[2\psi t]\sum_{x=0}^{\infty}\rho^{2x}\cos[2\psi(x+1)] + \cos[2\psi t]\sum_{x=0}^{\infty}\rho^{2x}\sin[2\psi(x+1)]\right]\right\}$$

$$\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2t-2k+1}\sin[\psi(t-k+1)]\cos[\psi(t-k+1)]}{T} = \frac{\rho^2\sqrt{1-\rho^2}}{1+2\rho^2-3\rho^4}$$

$$\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t-1}\rho^{2t-2k-1}\sin[\psi(t-k+1)]\cos[\psi(t-k+1)]}{T}$$

$$= \lim_{T\to\infty} \rho^{-2}\left[\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t}\rho^{2t-2k+1}\sin[\psi(t-k+1)]\cos[\psi(t-k+1)]}{T} - \frac{\sum_{t=t_0}^{T}\rho\sin\psi\cos\psi}{T}\right]$$

$$= \frac{\sqrt{1-\rho^2}}{1+2\rho^2-3\rho^4} - \sqrt{1-\rho^2}$$

$$= \rho^2\sqrt{1-\rho^2}\,\frac{3\rho^2-2}{1+2\rho^2-3\rho^4}$$

$$\lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t-1}\rho^{2t-2k-2}\sin[\psi(t-k+1)]\cos[\psi(t-k)]}{T}$$

$$= \lim_{T\to\infty}\frac{\sum_{t=t_0}^{T}\sum_{k=1}^{t-1}\rho^{2t-2k-2}\sin[\psi(t-k+1)]\left(\rho\cos[\psi(t-k+1)] + \sqrt{1-\rho^2}\sin[\psi(t-k+1)]\right)}{T}$$

$$= \lim_{T\to\infty}\frac{\left[\sum_{t=t_0}^{T}\sum_{k=1}^{t-1}\rho^{2t-2k-1}\sin[\psi(t-k+1)]\cos[\psi(t-k+1)] + \sqrt{1-\rho^2}\,\rho^{2t-2k-2}\left(\sin^2[\psi(t-k+1)]\right)\right]}{T}$$

$$= \rho^2\sqrt{1-\rho^2}\,\frac{3\rho^2-2}{1+2\rho^2-3\rho^4} + \sqrt{1-\rho^2}\,\rho^2\,\frac{5-8\rho^2+3\rho^4}{(1-\rho^2)(1+2\rho^2-3\rho^4)}$$

$$= \rho^2\sqrt{1-\rho^2}\,\frac{3(1-\rho^2)}{(1-\rho^2)(1+2\rho^2-3\rho^4)}$$