# Phillips curve model with effort depending on relative wage changes

$e[\bullet] = e[W_t / \Omega_t, \Delta W_t, u_t],$ where $\Delta W_t = \dfrac{W_t / \Omega_t}{W_{t-1} / \overline{W}_{t-1}}$

Demand: $Q = \theta^* P^{-\gamma}$

Thus, $P = \theta Q^{-\frac{1}{\gamma}}$ where $\theta = \theta^{*\frac{1}{\gamma}}$

Total revenue $= PQ = \theta Q^{\frac{\gamma-1}{\gamma}}$

Production function: $Q_t = A_t^{\phi} L_t^{\phi} K_0^{1-\phi} e[\bullet]^{\phi}$

Profit function:

$$\Pi = \theta_t \left[A_t^{\phi} L_t^{\phi} K_0^{1-\phi} e[\bullet]^{\phi}\right]^{\frac{\gamma-1}{\gamma}} - W_t L_t + \sum_{i=1}^{\infty} \frac{1}{(\overline{P}_{t+i}^e / \overline{P}_t)(1+r)^i} \left\{\theta_{t+i}^e \left[\left(A_{t+i}^e\right)^{\phi} L_{t+i}^{\phi} K_0^{1-\phi} e[\bullet]^{\phi}\right]^{\frac{\gamma-1}{\gamma}} - W_{t+i} L_{t+i}\right\}$$

$$\Pi = \theta_t A_t^{\frac{\phi(\gamma-1)}{\gamma}} L_t^{\frac{\phi(\gamma-1)}{\gamma}} K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[\bullet]^{\frac{\phi(\gamma-1)}{\gamma}} - W_t L_t$$
$$+ \sum_{i=1}^{\infty} \frac{1}{(\overline{P}_{t+i}^e / \overline{P}_t)(1+r)^i} \left\{\theta_{t+i}^e \left(A_{t+i}^e\right)^{\frac{\phi(\gamma-1)}{\gamma}} L_{t+i}^{\frac{\phi(\gamma-1)}{\gamma}} K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[\bullet]^{\frac{\phi(\gamma-1)}{\gamma}} - W_{t+i} L_{t+i}\right\}$$

$$P_t = \theta_t A_t^{-\frac{\phi}{\gamma}} L_t^{-\frac{\phi}{\gamma}} K_0^{-\frac{1-\phi}{\gamma}} e[\bullet]^{-\frac{\phi}{\gamma}}$$

$$\Omega_t = \left(\overline{W}_t\right)^{\omega} \left(\overline{W}_{t-1}\right)^{1-\omega}$$

$$\frac{d\Pi}{dL_t} = 0 = \frac{\phi(\gamma-1)}{\gamma} \theta_t A_t^{\frac{\phi(\gamma-1)}{\gamma}} L_t^{\frac{\phi(\gamma-1)}{\gamma}-1} K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[\bullet]^{\frac{\phi(\gamma-1)}{\gamma}} - W_t$$

$$L_t = W_t^{\frac{\gamma}{\phi(\gamma-1)-\gamma}} \left( \frac{\gamma}{\phi(\gamma-1)} \right)^{\frac{\gamma}{\phi(\gamma-1)-\gamma}} \theta_t^{-\frac{\gamma}{\phi(\gamma-1)-\gamma}} A_t^{-\frac{\phi(\gamma-1)}{\phi(\gamma-1)-\gamma}} K_0^{-\frac{(1-\phi)(\gamma-1)}{\phi(\gamma-1)-\gamma}} e[\bullet]^{-\frac{\phi(\gamma-1)}{\phi(\gamma-1)-\gamma}} \tag{1}$$

$$\begin{aligned}
\frac{d\Pi}{dW_t} = 0 = \phi \frac{\gamma-1}{\gamma} \theta_t A_t^{\frac{\phi(\gamma-1)}{\gamma}} L_t^{\frac{\phi(\gamma-1)}{\gamma}} K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[W_t / \Omega_t, \Delta W_t, u_t]^{\frac{\phi(\gamma-1)}{\gamma}-1} \\
\times \left( e_W [W_t / \Omega_t, \Delta W_t, u_t] \frac{1}{\Omega_t} + e_{\Delta W} [W_t / \Omega_t, \Delta W_t, u_t] \frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t} \right) - L_t \\
- \frac{1}{(1+r)(\overline{P}_{t+1} / \overline{P}_t)} \phi \frac{\gamma-1}{\gamma} \theta_{t+1} A_{t+1}^{\frac{\phi(\gamma-1)}{\gamma}} L_{t+1}^{\frac{\phi(\gamma-1)}{\gamma}} K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[W_{t+1} / \Omega_{t+1}, \Delta W_{t+1}, u_{t+1}]^{\frac{\phi(\gamma-1)}{\gamma}-1} \\
\times e_{\Delta W} [W_{t+1} / \Omega_{t+1}, \Delta W_{t+1}, u_{t+1}] \frac{W_{t+1}\overline{W}_t}{W_t^2 \Omega_{t+1}}
\end{aligned}$$

$$\begin{aligned}
0 = \phi \frac{\gamma-1}{\gamma} \theta_t A_t^{\frac{\phi(\gamma-1)}{\gamma}} L_t^{\frac{\phi(\gamma-1)}{\gamma}-1} K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[W_t / \Omega_t, \Delta W_t, u_t]^{\frac{\phi(\gamma-1)}{\gamma}-1} \\
\times \left( e_W [W_t / \Omega_t, \Delta W_t, u_t] \frac{1}{\Omega_t} + e_{\Delta W} [W_t / \Omega_t, \Delta W_t, u_t] \frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t} \right) - 1 \\
- \frac{1}{(1+r)} \phi \frac{\gamma-1}{\gamma} \theta_{t+1} A_{t+1}^{\frac{\phi(\gamma-1)}{\gamma}} L_{t+1}^{\frac{\phi(\gamma-1)}{\gamma}-1} K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[W_{t+1} / \Omega_{t+1}, \Delta W_{t+1}, u_{t+1}]^{\frac{\phi(\gamma-1)}{\gamma}-1} \\
\times e_{\Delta W} [W_{t+1} / \Omega_{t+1}, \Delta W_{t+1}, u_{t+1}] \frac{W_{t+1}\overline{W}_t}{W_t^2 \Omega_{t+1}} \overline{P}_t \overline{P}_{t+1}^{-1} \frac{L_{t+1}}{L_t}
\end{aligned} \tag{2}$$

Substituting (1) into (2) yields

$$0=\phi\frac{\gamma-1}{\gamma}\theta_t A_t^{\frac{\phi(\gamma-1)}{\gamma}} W_t \frac{\gamma}{\phi(\gamma-1)}\theta_t^{-1} A_t^{-\frac{\phi(\gamma-1)}{\gamma}} K_0^{-\frac{(1-\phi)(\gamma-1)}{\gamma}} e[W_t/\Omega_t,\Delta W_t,u_t]^{-\frac{\phi(\gamma-1)}{\gamma}}$$
$$\times K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[W_t/\Omega_t,\Delta W_t,u_t]^{\frac{\phi(\gamma-1)}{\gamma}-1}$$
$$\times\left(e_W[W_t/\Omega_t,\Delta W_t,u_t]\frac{1}{\Omega_t}+e_{\Delta W}[W_t/\Omega_t,\Delta W_t,u_t]\frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t}\right)-1$$
$$-\frac{1}{(1+r)}\phi\frac{\gamma-1}{\gamma}\theta_{t+1}A_{t+1}^{\frac{\phi(\gamma-1)}{\gamma}} W_{t+1}\frac{\gamma}{\phi(\gamma-1)}\theta_{t+1}^{-1}A_{t+1}^{-\frac{\phi(\gamma-1)}{\gamma}} K_0^{-\frac{(1-\phi)(\gamma-1)}{\gamma}}$$
$$e[W_{t+1}/\Omega_{t+1},\Delta W_{t+1},u_{t+1}]^{-\frac{\phi(\gamma-1)}{\gamma}}\times K_0^{\frac{(1-\phi)(\gamma-1)}{\gamma}} e[W_{t+1}/\Omega_{t+1},\Delta W_{t+1},u_{t+1}]^{\frac{\phi(\gamma-1)}{\gamma}-1}$$
$$\times e_{\Delta W}[W_{t+1}/\Omega_{t+1},\Delta W_{t+1},u_{t+1}]\frac{W_{t+1}\overline{W}_t}{W_t^2\Omega_{t+1}}\theta_t A_t^{-\frac{\phi}{\gamma}} L_t^{-\frac{\phi}{\gamma}} K_0^{-\frac{1-\phi}{\gamma}} e\left[W_t/\Omega_t,\Delta W_t,u_t\right]^{-\frac{\phi}{\gamma}}$$
$$\theta_{t+1}^{-1}A_{t+1}^{\frac{\phi}{\gamma}} L_{t+1}^{\frac{\phi}{\gamma}} K_0^{\frac{1-\phi}{\gamma}} e\left[W_{t+1}/\Omega_{t+1},\Delta W_{t+1},u_{t+1}\right]^{\frac{\phi}{\gamma}} L_t^{-1}L_{t+1}$$

$$W_t e[W_t/\Omega_t,\Delta W_t,u_t]^{-1}\left(e_W[W_t/\Omega_t,\Delta W_t,u_t]\frac{1}{\Omega_t}+e_{\Delta W}[W_t/\Omega_t,\Delta W_t,u_t]\frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t}\right)$$
$$=1+\frac{1}{(1+r)}e[W_{t+1}/\Omega_{t+1},\Delta W_{t+1},u_{t+1}]^{\frac{\phi-\gamma}{\gamma}}\times e_{\Delta W}[W_{t+1}/\Omega_{t+1},\Delta W_{t+1},u_{t+1}]\frac{W_{t+1}^2\overline{W}_t}{W_t^2\Omega_{t+1}} \tag{3}$$
$$\theta_t\theta_{t+1}^{-1}A_t^{-\frac{\phi}{\gamma}}A_{t+1}^{\frac{\phi}{\gamma}} e\left[W_t/\Omega_t,\Delta W_t,u_t\right]^{-\frac{\phi}{\gamma}} L_t^{-\frac{\phi+\gamma}{\gamma}} L_{t+1}^{\frac{\phi+\gamma}{\gamma}}$$

Note that

$$u_t=\frac{N-L_t}{N}$$

$$du_t=\frac{-dL_t}{N}$$

Let $s_L=\frac{L}{N}$

Then, $du_t = \dfrac{-dL_t}{\dfrac{L_t}{s_L}} = -s_L \hat{L}_t$

Expressing *L* in terms of percentage deviations from its initial value yields

$$L_t = W_t^{\frac{\gamma}{\phi(\gamma-1)-\gamma}} \left( \frac{\gamma}{\phi(\gamma-1)} \right)^{\frac{\gamma}{\phi(\gamma-1)-\gamma}} \theta_t^{-\frac{\gamma}{\phi(\gamma-1)-\gamma}} A_t^{-\frac{\phi(\gamma-1)}{\phi(\gamma-1)-\gamma}} K_0^{-\frac{(1-\phi)(\gamma-1)}{\phi(\gamma-1)-\gamma}} e\left[W_t / \Omega_t, \Delta W_t, u_t\ \right]^{-\frac{\phi(\gamma-1)}{\phi(\gamma-1)-\gamma}}$$

$$\hat{L}_t = \frac{\gamma}{\phi(\gamma-1)-\gamma}\hat{W}_t - \frac{\gamma}{\phi(\gamma-1)-\gamma}\hat{\theta}_t - \frac{\phi(\gamma-1)}{\phi(\gamma-1)-\gamma}\hat{A}_t - \frac{\phi(\gamma-1)}{\phi(\gamma-1)-\gamma}e^{-1}[e_W \frac{1}{\Omega_t} dW_t - e_W \frac{W_t}{\Omega_t^2} d\Omega_t$$
$$+ e_{\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t} dW_t + e_{\Delta W} \frac{W_t}{W_{t-1}\Omega_t} d\overline{W}_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}^2 \Omega_t} dW_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t^2} d\Omega_t + e_u du_t ]$$

$$\hat{L}_t = \frac{\gamma}{\phi(\gamma-1)-\gamma}\hat{W}_t - \frac{\gamma}{\phi(\gamma-1)-\gamma}\hat{\theta}_t - \frac{\phi(\gamma-1)}{\phi(\gamma-1)-\gamma}\hat{A}_t - \frac{\phi(\gamma-1)}{\phi(\gamma-1)-\gamma}e^{-1}[e_W \frac{W_t}{\Omega_t}\hat{W}_t - e_W \frac{W_t}{\Omega_t}\hat{\Omega}_t$$
$$+ e_{\Delta W} \frac{\overline{W}_{t-1} W_t}{W_{t-1}\Omega_t}\hat{W}_t + e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t}\hat{\overline{W}}_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t}\hat{W}_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t}\hat{\Omega}_t - s_L e_u \hat{L}_t ]$$

$$\hat{L}_t = \frac{1}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \Bigg[ \gamma\hat{W}_t - \gamma\hat{\theta}_t - \phi(\gamma-1)\hat{A}_t - \phi(\gamma-1)e^{-1}[e_W \frac{W_t}{\Omega_t}\hat{W}_t - e_W \frac{W_t}{\Omega_t}\hat{\Omega}_t$$
$$+ e_{\Delta W} \frac{\overline{W}_{t-1} W_t}{W_{t-1}\Omega_t}\hat{W}_t + e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t}\hat{\overline{W}}_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t}\hat{W}_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t}\hat{\Omega}_t ] \Bigg]$$

$$\hat{L}_t = \frac{1}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \Big[ \gamma\hat{W}_t - \gamma\hat{\theta}_t - \phi(\gamma-1)\hat{A}_t - \phi(\gamma-1)e^{-1}[e_W \hat{W}_t - e_W \hat{\Omega}_t$$
$$+ e_{\Delta W}\hat{W}_t + e_{\Delta W}\hat{\overline{W}}_{t-1} - e_{\Delta W}\hat{W}_{t-1} - e_{\Delta W}\hat{\Omega}_t ] \Big]$$

$$\hat{L}_t = \frac{[\gamma - \phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})]\hat{W}_t - \gamma\hat{\theta}_t - \phi(\gamma-1)\hat{A}_t + \phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})\hat{\Omega}_t}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \tag{4}$$

We now totally differentiate of left-hand side and right-hand side of (3) and divide by the original equations.

Let $LH = W_t \Omega_t^{-1} \frac{1}{\phi} e[W_t / \Omega_t, \Delta W_t, u_t]^{-1} \left( e_W [W_t / \Omega_t, \Delta W_t, u_t] + e_{\Delta W} [W_t / \Omega_t, \Delta W_t, u_t] \frac{\overline{W}_{t-1}}{W_{t-1}} \right)$

$$
\begin{aligned}
dLH = {} & dW_t \frac{LH}{W_t} - \frac{d\Omega_t}{\Omega_t^2} \frac{LH}{\Omega_t^{-1}} - \frac{LH}{e^{-1}} e^{-2} (e_W \frac{1}{\Omega_t} dW_t - e_W \frac{W_t}{\Omega_t^2} d\Omega_t + e_{\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t} dW_t + e_{\Delta W} \frac{W_t}{W_{t-1}\Omega_t} d\overline{W}_{t-1} \\
& - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}^2 \Omega_t} dW_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1} \Omega_t^2} d\Omega_t + e_u du_t) \\
& + \frac{LH}{e_W + e_{\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}}} (e_{WW} \frac{1}{\Omega_t} dW_t - e_{WW} \frac{W_t}{\Omega_t^2} d\Omega_t + e_{W\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t} dW_t \\
& + e_{W\Delta W} \frac{W_t}{W_{t-1}\Omega_t} d\overline{W}_{t-1} - e_{W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}^2 \Omega_t} dW_{t-1} - e_{W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t^2} d\Omega_t + e_{Wu} du_t \\
& + e_{W\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t} dW_t - e_{W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t^2} d\Omega_t + e_{\Delta W\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{\overline{W}_{t-1}}{W_{t-1}} dW_t + e_{\Delta W\Delta W} \frac{W_t}{W_{t-1}\Omega_t} \frac{\overline{W}_{t-1}}{W_{t-1}} d\overline{W}_{t-1} \\
& - e_{\Delta W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}^2 \Omega_t} \frac{\overline{W}_{t-1}}{W_{t-1}} dW_{t-1} - e_{\Delta W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t^2} \frac{\overline{W}_{t-1}}{W_{t-1}} d\Omega_t + e_{u\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}} du_t \\
& + e_{\Delta W} \frac{1}{W_{t-1}} d\overline{W}_{t-1} - e_{\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}^2} dW_{t-1})
\end{aligned}
$$

$$
\begin{aligned}
\frac{dLH}{LH} = {} & \frac{dW_t}{W_t} - \frac{d\Omega_t}{\Omega_t} - e^{-1} (e_W \frac{W_t}{\Omega_t} \frac{dW_t}{W_t} - e_W \frac{W_t}{\Omega_t} \frac{d\Omega_t}{\Omega_t} + e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{dW_t}{W_t} + e_{\Delta W} \frac{\overline{W}_{t-1} W_t}{W_{t-1}\Omega_t} \frac{d\overline{W}_{t-1}}{\overline{W}_{t-1}} \\
& - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{dW_{t-1}}{W_{t-1}} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{d\Omega_t}{\Omega_t} + e_u du_t) \\
& + \frac{1}{e_W + e_{\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}}} (e_{WW} \frac{W_t}{\Omega_t} \frac{dW_t}{W_t} - e_{WW} \frac{W_t}{\Omega_t} \frac{d\Omega_t}{\Omega_t} + e_{W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{dW_t}{W_t} \\
& + e_{W\Delta W} \frac{\overline{W}_{t-1} W_t}{W_{t-1}\Omega_t} \frac{d\overline{W}_{t-1}}{\overline{W}_{t-1}} - e_{W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{dW_{t-1}}{W_{t-1}} - e_{W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{d\Omega_t}{\Omega_t} + e_{Wu} du_t \\
& + e_{W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{dW_t}{W_t} - e_{W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{d\Omega_t}{\Omega_t} + e_{\Delta W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{\overline{W}_{t-1}}{W_{t-1}} \frac{dW_t}{W_t} + e_{\Delta W\Delta W} \frac{\overline{W}_{t-1} W_t}{W_{t-1}\Omega_t} \frac{\overline{W}_{t-1}}{W_{t-1}} \frac{d\overline{W}_{t-1}}{\overline{W}_{t-1}} \\
& - e_{\Delta W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{\overline{W}_{t-1}}{W_{t-1}} \frac{dW_{t-1}}{W_{t-1}} - e_{\Delta W\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}\Omega_t} \frac{\overline{W}_{t-1}}{W_{t-1}} \frac{d\Omega_t}{\Omega_t} + e_{u\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}} du_t \\
& + e_{\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}} \frac{d\overline{W}_{t-1}}{\overline{W}_{t-1}} - e_{\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}} \frac{dW_{t-1}}{W_{t-1}})
\end{aligned}
$$

$$\frac{dLH}{LH}=\hat{W}_t-\hat{\Omega}_t-e^{-1}(e_W\hat{W}_t-e_W\hat{\Omega}_t+e_{\Delta W}\hat{W}_t+e_{\Delta W}\hat{\overline{W}}_{t-1}-e_{\Delta W}\hat{W}_{t-1}-e_{\Delta W}\hat{\Omega}_t+e_u du_t)$$
$$+\frac{1}{e_W+e_{\Delta W}}(e_{WW}\hat{W}_t-e_{WW}\hat{\Omega}_t+e_{W\Delta W}\hat{W}_t+e_{W\Delta W}\hat{\overline{W}}_{t-1}-e_{W\Delta W}\hat{W}_{t-1}-e_{W\Delta W}\hat{\Omega}_t+e_{Wu}du_t$$
$$+e_{W\Delta W}\hat{W}_t-e_{W\Delta W}\hat{\Omega}_t+e_{\Delta W\Delta W}\hat{W}_t+e_{\Delta W\Delta W}\hat{\overline{W}}_{t-1}-e_{\Delta W\Delta W}\hat{W}_{t-1}-e_{\Delta W\Delta W}\hat{\Omega}_t+e_{u\Delta W}du_t$$
$$+e_{\Delta W}\hat{\overline{W}}_{t-1}-e_{\Delta W}\hat{W}_{t-1})$$

Since $\hat{\overline{W}}_{t-1}=\hat{W}_{t-1}$, it follows that

$$\frac{dLH}{LH}=\{1-e^{-1}e_W-e^{-1}e_{\Delta W}+\frac{1}{e_W+e_{\Delta W}}(e_{WW}+e_{W\Delta W}+e_{W\Delta W}+e_{\Delta W\Delta W})\}\hat{W}_t$$
$$-\{1-e^{-1}e_W-e^{-1}e_{\Delta W}+\frac{1}{e_W+e_{\Delta W}}(e_{WW}+e_{W\Delta W}+e_{W\Delta W}+e_{\Delta W\Delta W})\}\hat{\Omega}_t$$
$$+\{-e^{-1}e_u+\frac{1}{e_W+e_{\Delta W}}(e_{Wu}+e_{u\Delta W})\}du_t$$

$$\frac{dLH}{LH}=\{1-e^{-1}e_W-e^{-1}e_{\Delta W}+\frac{1}{e_W+e_{\Delta W}}(e_{WW}+2e_{W\Delta W}+e_{\Delta W\Delta W})\}\hat{W}_t$$
$$+\{-1+e^{-1}e_W+e^{-1}e_{\Delta W}-\frac{1}{e_W+e_{\Delta W}}(e_{WW}+2e_{W\Delta W}+e_{\Delta W\Delta W})\}\hat{\Omega}_t$$
$$+[e^{-1}e_u-\frac{1}{e_W+e_{\Delta W}}(e_{Wu}+e_{u\Delta W})]$$
$$\times s_L\frac{[\gamma-\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})]\hat{W}_t-\gamma\hat{\theta}_t-\phi(\gamma-1)\hat{A}_t+\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})\hat{\Omega}_t}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_Le^{-1}e_u}$$

$$\frac{dLH}{LH} = \left(1 - e^{-1}(e_W + e_{\Delta W}) + \frac{e_{WW} + 2e_{W\Delta W} + e_{\Delta W\Delta W}}{e_W + e_{\Delta W}}\right.$$
$$\left. + s_L\left[e^{-1}e_u - \frac{e_{Wu} + e_{u\Delta W}}{e_W + e_{\Delta W}}\right]\frac{\gamma - \phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u}\right)\hat{W}_t$$
$$+\left(-1 + e^{-1}(e_W + e_{\Delta W}) - \frac{e_{WW} + 2e_{W\Delta W} + e_{\Delta W\Delta W}}{e_W + e_{\Delta W}}\right.$$
$$\left. + s_L\left[e^{-1}e_u - \frac{e_{Wu} + e_{u\Delta W}}{e_W + e_{\Delta W}}\right]\frac{\phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u}\right)\hat{\Omega}_t$$
$$- s_L\left[e^{-1}e_u - \frac{e_{Wu} + e_{u\Delta W}}{e_W + e_{\Delta W}}\right]\frac{\gamma}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u}\hat{\theta}_t$$
$$- s_L\left[e^{-1}e_u - \frac{e_{Wu} + e_{u\Delta W}}{e_W + e_{\Delta W}}\right]\frac{\phi(\gamma-1)}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u}\hat{A}_t$$

Let

$$RH = 1 + \frac{1}{(1+r)}e[W_{t+1}/\Omega_{t+1}, \Delta W_{t+1}, u_{t+1}]^{\frac{\phi-\gamma}{\gamma}} \times e_{\Delta W}[W_{t+1}/\Omega_{t+1}, \Delta W_{t+1}, u_{t+1}]\frac{W_{t+1}^2\overline{W_t}}{W_t^2\Omega_{t+1}}$$
$$\theta_t\theta_{t+1}^{-1}A_t^{-\frac{\phi}{\gamma}}A_{t+1}^{\frac{\phi}{\gamma}}e[W_t/\Omega_t, \Delta W_t, u_t]^{-\frac{\phi}{\gamma}}L_t^{-\frac{\phi+\gamma}{\gamma}}L_{t+1}^{\frac{\phi+\gamma}{\gamma}}$$

Let $RH = 1 + Y$, where

$$Y = \frac{1}{(1+r)}e[W_{t+1}/\Omega_{t+1}, \Delta W_{t+1}, u_{t+1}]^{\frac{\phi-\gamma}{\gamma}} \times e_{\Delta W}[W_{t+1}/\Omega_{t+1}, \Delta W_{t+1}, u_{t+1}]\frac{W_{t+1}^2\overline{W_t}}{W_t^2\Omega_{t+1}}$$
$$\theta_t\theta_{t+1}^{-1}A_t^{-\frac{\phi}{\gamma}}A_{t+1}^{\frac{\phi}{\gamma}}e[W_t/\Omega_t, \Delta W_t, u_t]^{-\frac{\phi}{\gamma}}L_t^{-\frac{\phi+\gamma}{\gamma}}L_{t+1}^{\frac{\phi+\gamma}{\gamma}}$$

Let $s_R = \frac{Y}{1+Y}$

Then, $dRH = dY$

$$\frac{dRH}{RH} = \frac{dY}{1+Y} = \frac{dY}{\frac{Y}{s_R}} = s_R\frac{dY}{Y}$$

$$\frac{dRH}{RH} = s_R \left( \begin{array}{l} \frac{\phi-\gamma}{\gamma} e^{-1}[e_W \frac{1}{\Omega_{t+1}} dW_{t+1} - e_W \frac{W_{t+1}}{\Omega_{t+1}^2} d\Omega_{t+1} \\ + e_{\Delta W} \frac{\overline{W_t}}{W_t \Omega_{t+1}} dW_{t+1} + e_{\Delta W} \frac{W_{t+1}}{W_t \Omega_{t+1}} d\overline{W_t} - e_{\Delta W} \frac{W_{t+1}\overline{W_t}}{W_t^2 \Omega_{t+1}} dW_t - e_{\Delta W} \frac{W_{t+1}\overline{W_t}}{W_t \Omega_{t+1}^2} d\Omega_{t+1} + e_u du_{t+1}] \\ + e_{\Delta W}^{-1}[e_{W\Delta W} \frac{1}{\Omega_{t+1}} dW_{t+1} - e_{W\Delta W} \frac{W_{t+1}}{\Omega_{t+1}^2} d\Omega_{t+1} + e_{\Delta W\Delta W} \frac{\overline{W_t}}{W_t \Omega_{t+1}} dW_{t+1} \\ + e_{\Delta W\Delta W} \frac{W_{t+1}}{W_t \Omega_{t+1}} d\overline{W_t} - e_{\Delta W\Delta W} \frac{W_{t+1}\overline{W_t}}{W_t^2 \Omega_{t+1}} dW_t - e_{\Delta W\Delta W} \frac{W_{t+1}\overline{W_t}}{W_t \Omega_{t+1}^2} d\Omega_{t+1} + e_{u\Delta W} du_{t+1}] \\ + 2\hat{W}_{t+1} + \hat{\overline{W}}_t - 2\hat{W}_t - \hat{\Omega}_{t+1} + \hat{\theta}_t - \hat{\theta}_{t+1} - \frac{\phi}{\gamma}\hat{A}_t + \frac{\phi}{\gamma}\hat{A}_{t+1} \\ -\frac{\phi}{\gamma} e^{-1}[e_W \frac{1}{\Omega_t} dW_t - e_W \frac{W_t}{\Omega_t^2} d\Omega_t + e_{\Delta W} \frac{\overline{W}_{t-1}}{W_{t-1}\Omega_t} dW_t \\ + e_{\Delta W} \frac{W_t}{W_{t-1}\Omega_t} d\overline{W}_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1}^2 \Omega_t} dW_{t-1} - e_{\Delta W} \frac{W_t \overline{W}_{t-1}}{W_{t-1} \Omega_t^2} d\Omega_t + e_u du_t] - \frac{\phi+\gamma}{\gamma}\hat{L}_t + \frac{\phi+\gamma}{\gamma}\hat{L}_{t+1} \end{array} \right)$$

$$\frac{dRH}{RH} = s_R \left( \begin{array}{l} \frac{\phi-\gamma}{\gamma} e^{-1}[e_W \hat{W}_{t+1} - e_W \hat{\Omega}_{t+1} + e_{\Delta W}\hat{W}_{t+1} + e_{\Delta W}\hat{\overline{W}}_t - e_{\Delta W}\hat{W}_t - e_{\Delta W}\hat{\Omega}_{t+1} - e_u s_L \hat{L}_{t+1}] \\ + e_{\Delta W}^{-1}[e_{W\Delta W}\hat{W}_{t+1} - e_{W\Delta W}\hat{\Omega}_{t+1} + e_{\Delta W\Delta W}\hat{W}_{t+1} \\ + e_{\Delta W\Delta W}\hat{\overline{W}}_t - e_{\Delta W\Delta W}\hat{W}_t - e_{\Delta W\Delta W}\hat{\Omega}_{t+1} - e_{u\Delta W} s_L \hat{L}_{t+1}] \\ + 2\hat{W}_{t+1} + \hat{\overline{W}}_t - 2\hat{W}_t - \hat{\Omega}_{t+1} + \hat{\theta}_t - \hat{\theta}_{t+1} - \frac{\phi}{\gamma}\hat{A}_t + \frac{\phi}{\gamma}\hat{A}_{t+1} \\ -\frac{\phi}{\gamma} e^{-1}[e_W \hat{W}_t - e_W \hat{\Omega}_t + e_{\Delta W}\hat{W}_t \\ + e_{\Delta W}\hat{\overline{W}}_{t-1} - e_{\Delta W}\hat{W}_{t-1} - e_{\Delta W}\hat{\Omega}_t - e_u s_L \hat{L}_t] - \frac{\phi+\gamma}{\gamma}\hat{L}_t + \frac{\phi+\gamma}{\gamma}\hat{L}_{t+1} \end{array} \right)$$

$$\frac{dRH}{RH}=s_R\left(\begin{array}{l}
\frac{\phi-\gamma}{\gamma}e^{-1}[e_W\hat{W}_{t+1}-e_W\hat{\Omega}_{t+1} \\
+e_{\Delta W}\hat{W}_{t+1}+e_{\Delta W}\hat{\overline{W}}_t-e_{\Delta W}\hat{W}_t-e_{\Delta W}\hat{\Omega}_{t+1}] \\
+e_{\Delta W}^{-1}[e_{W\Delta W}\hat{W}_{t+1}-e_{W\Delta W}\hat{\Omega}_{t+1}+e_{\Delta W\Delta W}\hat{W}_{t+1} \\
+e_{\Delta W\Delta W}\hat{\overline{W}}_t-e_{\Delta W\Delta W}\hat{W}_t-e_{\Delta W\Delta W}\hat{\Omega}_{t+1}] \\
+2\hat{W}_{t+1}+\hat{\overline{W}}_t-2\hat{W}_t-\hat{\Omega}_{t+1}+\hat{\theta}_t-\hat{\theta}_{t+1}-\frac{\phi}{\gamma}\hat{A}_t+\frac{\phi}{\gamma}\hat{A}_{t+1} \\
-\frac{\phi}{\gamma}e^{-1}[e_W\hat{W}_t-e_W\hat{\Omega}_t+e_{\Delta W}\hat{W}_t \\
+e_{\Delta W}\hat{\overline{W}}_{t-1}-e_{\Delta W}\hat{W}_{t-1}-e_{\Delta W}\hat{\Omega}_t]+[\frac{\phi+\gamma}{\gamma}-\frac{\phi-\gamma}{\gamma}e^{-1}e_u s_L \\
-e_{\Delta W}^{-1}e_{u\Delta W}s_L]\hat{L}_{t+1}+[-\frac{\phi+\gamma}{\gamma}+\frac{\phi}{\gamma}e^{-1}e_u s_L]\hat{L}_t
\end{array}\right)$$

Since $\hat{\overline{W}}_t=\hat{W}_t$ and $\hat{\overline{W}}_{t-1}=\hat{W}_{t-1}$,

$$\frac{dRH}{RH}=s_R\left(\begin{array}{l}
\frac{\phi-\gamma}{\gamma}e^{-1}[e_W\hat{W}_{t+1}-e_W\hat{\Omega}_{t+1}+e_{\Delta W}\hat{W}_{t+1}-e_{\Delta W}\hat{\Omega}_{t+1}] \\
+e_{\Delta W}^{-1}[e_{W\Delta W}\hat{W}_{t+1}-e_{W\Delta W}\hat{\Omega}_{t+1}+e_{\Delta W\Delta W}\hat{W}_{t+1}-e_{\Delta W\Delta W}\hat{\Omega}_{t+1}] \\
+2\hat{W}_{t+1}-\hat{W}_t-\hat{\Omega}_{t+1}+\hat{\theta}_t-\hat{\theta}_{t+1}-\frac{\phi}{\gamma}\hat{A}_t+\frac{\phi}{\gamma}\hat{A}_{t+1} \\
-\frac{\phi}{\gamma}e^{-1}[e_W\hat{W}_t-e_W\hat{\Omega}_t+e_{\Delta W}\hat{W}_t-e_{\Delta W}\hat{\Omega}_t]+[\frac{\phi+\gamma}{\gamma}-\frac{\phi-\gamma}{\gamma}e^{-1}e_u s_L \\
-e_{\Delta W}^{-1}e_{u\Delta W}s_L]\frac{[\gamma-\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})]\hat{W}_{t+1}-\gamma\hat{\theta}_{t+1}-\phi(\gamma-1)\hat{A}_{t+1}+\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})\hat{\Omega}_{t+1}}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u} \\
+[-\frac{\phi+\gamma}{\gamma}+\frac{\phi}{\gamma}e^{-1}e_u s_L]\frac{[\gamma-\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})]\hat{W}_t-\gamma\hat{\theta}_t-\phi(\gamma-1)\hat{A}_t+\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})\hat{\Omega}_t}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}
\end{array}\right)$$

$$\frac{dRH}{RH} = s_R \left( \begin{array}{l} \left( \frac{\phi-\gamma}{\gamma} e^{-1}(e_W + e_{\Delta W}) + e_{\Delta W}^{-1}(e_{W\Delta W} + e_{\Delta W \Delta W}) \right. \\ \left. + 2 + [\frac{\phi+\gamma}{\gamma} - \frac{\phi-\gamma}{\gamma} e^{-1} e_u s_L - e_{\Delta W}^{-1} e_{u\Delta W} s_L] \frac{\gamma - \phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \right) \hat{W}_{t+1} \\ + \left( -1 - \frac{\phi}{\gamma} e^{-1}(e_W + e_{\Delta W}) + [\frac{\phi}{\gamma} e^{-1} e_u s_L - \frac{\phi+\gamma}{\gamma}] \frac{\gamma - \phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \right) \hat{W}_t \\ + \left( -\frac{\phi-\gamma}{\gamma} e^{-1}(e_W + e_{\Delta W}) - e_{\Delta W}^{-1}(e_{W\Delta W} + e_{\Delta W \Delta W}) - 1 \right. \\ \left. + [\frac{\phi+\gamma}{\gamma} - \frac{\phi-\gamma}{\gamma} e^{-1} e_u s_L - e_{\Delta W}^{-1} e_{u\Delta W} s_L] \frac{\phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \right) \hat{\Omega}_{t+1} \\ + \left( \frac{\phi}{\gamma} e^{-1}(e_W + e_{\Delta W}) + [\frac{\phi}{\gamma} e^{-1} e_u s_L - \frac{\phi+\gamma}{\gamma}] \frac{\phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \right) \hat{\Omega}_t \\ - \left( 1 + [\frac{\phi+\gamma}{\gamma} - \frac{\phi-\gamma}{\gamma} e^{-1} e_u s_L - e_{\Delta W}^{-1} e_{u\Delta W} s_L] \frac{\gamma}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \right) \hat{\theta}_{t+1} \\ + \left( 1 - [\frac{\phi}{\gamma} e^{-1} e_u s_L - \frac{\phi+\gamma}{\gamma}] \frac{\gamma}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \right) \hat{\theta}_t \\ + \left( \frac{\phi}{\gamma} - [\frac{\phi+\gamma}{\gamma} - \frac{\phi-\gamma}{\gamma} e^{-1} e_u s_L - e_{\Delta W}^{-1} e_{u\Delta W} s_L] \frac{\phi(\gamma-1)}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \right) \hat{A}_{t+1} \\ - \left( \frac{\phi}{\gamma} + [\frac{\phi}{\gamma} e^{-1} e_u s_L - \frac{\phi+\gamma}{\gamma}] \frac{\phi(\gamma-1)}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u} \right) \hat{A}_t \end{array} \right)$$

Setting the percentage deviations of the left-hand side and the right-hand side equal to one another yields:

$$
\begin{aligned}
&- s_R\Bigg(\frac{\phi-\gamma}{\gamma}e^{-1}(e_W+e_{\Delta W})+e_{\Delta W}^{-1}(e_{W\Delta W}+e_{\Delta W\Delta W})\\
&\qquad +2+\left[\frac{\phi+\gamma}{\gamma}-\frac{\phi-\gamma}{\gamma}e^{-1}e_u s_L-e_{\Delta W}^{-1}e_{u\Delta W}s_L\right]\frac{\gamma-\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\Bigg)\hat{W}_{t+1}\\
&+\Bigg(1-e^{-1}(e_W+e_{\Delta W})+\frac{e_{WW}+2e_{W\Delta W}+e_{\Delta W\Delta W}}{e_W+e_{\Delta W}}\\
&\qquad +s_L\left[e^{-1}e_u-\frac{e_{Wu}+e_{u\Delta W}}{e_W+e_{\Delta W}}\right]\frac{\gamma-\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\\
&\qquad +s_R+s_R\frac{\phi}{\gamma}e^{-1}(e_W+e_{\Delta W})-s_R\left[\frac{\phi}{\gamma}e^{-1}e_u s_L-\frac{\phi+\gamma}{\gamma}\right]\frac{\gamma-\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\Bigg)\hat{W}_t\\
&= s_R\Bigg(-\frac{\phi-\gamma}{\gamma}e^{-1}(e_W+e_{\Delta W})-e_{\Delta W}^{-1}(e_{W\Delta W}+e_{\Delta W\Delta W})-1\\
&\qquad +\left[\frac{\phi+\gamma}{\gamma}-\frac{\phi-\gamma}{\gamma}e^{-1}e_u s_L-e_{\Delta W}^{-1}e_{u\Delta W}s_L\right]\frac{\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\Bigg)\hat{\Omega}_{t+1}\\
&+\Bigg(1-e^{-1}(e_W+e_{\Delta W})+\frac{e_{WW}+2e_{W\Delta W}+e_{\Delta W\Delta W}}{e_W+e_{\Delta W}}\\
&\qquad -s_L\left[e^{-1}e_u-\frac{e_{Wu}+e_{u\Delta W}}{e_W+e_{\Delta W}}\right]\frac{\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\\
&\qquad +s_R\frac{\phi}{\gamma}e^{-1}(e_W+e_{\Delta W})+s_R\left[\frac{\phi}{\gamma}e^{-1}e_u s_L-\frac{\phi+\gamma}{\gamma}\right]\frac{\phi(\gamma-1)e^{-1}(e_W+e_{\Delta W})}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\Bigg)\hat{\Omega}_t\\
&- s_R\left(1+\left[\frac{\phi+\gamma}{\gamma}-\frac{\phi-\gamma}{\gamma}e^{-1}e_u s_L-e_{\Delta W}^{-1}e_{u\Delta W}s_L\right]\frac{\gamma}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\right)\hat{\theta}_{t+1}\\
&+\Bigg(s_L\left[e^{-1}e_u-\frac{e_{Wu}+e_{u\Delta W}}{e_W+e_{\Delta W}}\right]\frac{\gamma}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\\
&\qquad +s_R-s_R\left[\frac{\phi}{\gamma}e^{-1}e_u s_L-\frac{\phi+\gamma}{\gamma}\right]\frac{\gamma}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\Bigg)\hat{\theta}_t\\
&+ s_R\left(\frac{\phi}{\gamma}-[\frac{\phi+\gamma}{\gamma}-\frac{\phi-\gamma}{\gamma}e^{-1}e_u s_L-e_{\Delta W}^{-1}e_{u\Delta W}s_L]\frac{\phi(\gamma-1)}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\right)\hat{A}_{t+1}\\
&+\Bigg(s_L\left[e^{-1}e_u-\frac{e_{Wu}+e_{u\Delta W}}{e_W+e_{\Delta W}}\right]\frac{\phi(\gamma-1)}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\\
&\qquad -s_R\frac{\phi}{\gamma}-s_R\left[\frac{\phi}{\gamma}e^{-1}e_u s_L-\frac{\phi+\gamma}{\gamma}\right]\frac{\phi(\gamma-1)}{\phi\gamma-\phi-\gamma-\phi(\gamma-1)s_L e^{-1}e_u}\Bigg)\hat{A}_t
\end{aligned}
$$

This can be expressed as

$$(a_1 + a_2)\hat{W}_{t+1} + (b_1 + b_2)\hat{W}_t = a_1\hat{\theta}_{t+1} + b_1\hat{\theta}_t + a_2\hat{\Omega}_{t+1} + b_2\hat{\Omega}_t \,, \tag{5}$$

where

$$a_1 = -s_R\left(1 + \left[\frac{\phi+\gamma}{\gamma} - \frac{\phi-\gamma}{\gamma}e^{-1}e_u s_L - e_{\Delta W}^{-1} e_{u\Delta W} s_L\right]\frac{\gamma}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u}\right)$$

$$\begin{aligned} a_2 = s_R \Bigg( & -\frac{\phi-\gamma}{\gamma}e^{-1}(e_W + e_{\Delta W}) - e_{\Delta W}^{-1}(e_{W\Delta W} + e_{\Delta W\Delta W}) - 1 \\ & + \left[\frac{\phi+\gamma}{\gamma} - \frac{\phi-\gamma}{\gamma}e^{-1}e_u s_L - e_{\Delta W}^{-1}e_{u\Delta W} s_L\right]\frac{\phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u}\Bigg) \end{aligned}$$

$$\begin{aligned} b_1 = \Bigg( & s_L\left[e^{-1}e_u - \frac{e_{Wu} + e_{u\Delta W}}{e_W + e_{\Delta W}}\right]\frac{\gamma}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u} \\ & + s_R - s_R\left[\frac{\phi}{\gamma}e^{-1}e_u s_L - \frac{\phi+\gamma}{\gamma}\right]\frac{\gamma}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u}\Bigg) \end{aligned}$$

$$\begin{aligned} b_2 = \Bigg( & 1 - e^{-1}(e_W + e_{\Delta W}) + \frac{e_{WW} + 2e_{W\Delta W} + e_{\Delta W\Delta W}}{e_W + e_{\Delta W}} \\ & - s_L\left[e^{-1}e_u - \frac{e_{Wu} + e_{u\Delta W}}{e_W + e_{\Delta W}}\right]\frac{\phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u} \\ & + s_R\frac{\phi}{\gamma}e^{-1}(e_W + e_{\Delta W}) + s_R\left[\frac{\phi}{\gamma}e^{-1}e_u s_L - \frac{\phi+\gamma}{\gamma}\right]\frac{\phi(\gamma-1)e^{-1}(e_W + e_{\Delta W})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1}e_u}\Bigg) \end{aligned}$$

Let $\hat{\Omega}_t = \omega\hat{W}_t + (1-\omega)\hat{W}_{t-1}$

Then, (5) can be expressed as

$$[a_1 + a_2(1-\omega)]\hat{W}_{t+1} + [b_1 - a_2(1-\omega) + b_2(1-\omega)]\hat{W}_t - [b_2(1-\omega)]\hat{W}_{t-1} = a_1\hat{\theta}_{t+1} + b_1\hat{\theta}_t$$

Lagging variables by one period yields

$$[a_1 + a_2(1-\omega)]\hat{W}_t + [b_1 - a_2(1-\omega) + b_2(1-\omega)]\hat{W}_{t-1} - [b_2(1-\omega)]\hat{W}_{t-2} = a_1\hat{\theta}_t + b_1\hat{\theta}_{t-1} \tag{6}$$

Dividing by the coefficient on $\hat{W}_t$ yields

$$\hat{W}_t - b\hat{W}_{t-1} + c\hat{W}_{t-2} = d_1\hat{\theta}_t + d_2\hat{\theta}_{t-1},$$

where

$$b = -\frac{b_1 - a_2(1-\omega) + b_2(1-\omega)}{a_1 + a_2(1-\omega)}, \quad c = \frac{-b_2(1-\omega)}{a_1 + a_2(1-\omega)}, \quad d_1 = \frac{a_1}{a_1 + a_2(1-\omega)},$$
$$d_2 = \frac{b_1}{a_1 + a_2(1-\omega)}$$

Expressing this equation in terms of lag operators:

$$[1 - bL + cL^2]\hat{W}_t = d_1\hat{\theta}_t + d_2\hat{\theta}_{t-1}.$$

The roots of this equation are

$$\mu_1 = \frac{b + \sqrt{b^2 - 4c}}{2}, \qquad \mu_2 = \frac{b - \sqrt{b^2 - 4c}}{2}$$

Thus,

$$\hat{W}_t = \frac{1}{(1-\mu_1 L)(1-\mu_2 L)}[d_1\hat{\theta}_t + d_2\hat{\theta}_{t-1}]$$

Under reasonable parameter values it can be demonstrated that

$$0 < \mu_2 < 1$$

$$\mu_1 > 1.$$

As a result,

$$\hat{W}_t = \mu_2 \hat{W}_{t-1} - d_1 \sum_{j=1}^{\infty} \left(\frac{1}{\mu_1}\right)^j \hat{\theta}_{t+j} - d_2 \sum_{j=1}^{\infty} \left(\frac{1}{\mu_1}\right)^j \hat{\theta}_{t+j-1}$$

Suppose that $\hat{\theta}_t$=0 for $t$≤0, and that for $t$≥1 there are random shocks to $\hat{\theta}_t$, so that $\hat{\theta}_t = \sum_{i=0}^{t} \varepsilon_i$ .

$$\hat{W}_t = \mu_2 \hat{W}_{t-1} - (d_1 + d_2) \sum_{j=1}^{\infty} \left(\frac{1}{\mu_1}\right)^j \hat{\theta}_t \qquad \text{since } E_t \hat{\theta}_{t+j} = \hat{\theta}_t$$

$$\hat{W}_t = \mu_2 \hat{W}_{t-1} - \frac{d_1 + d_2}{\mu_1 - 1} \hat{\theta}_t$$

$$\hat{W}_t = \frac{d_1 + d_2}{1 - \mu_1} [\theta_t + \mu_2 \theta_{t-1} + \mu_2^2 \theta_{t-2} + \mu_2^3 \theta_{t-3} + \cdots + \mu_2^{t-1} \theta_1]$$

$$\hat{W}_t = \frac{d_1 + d_2}{1 - \mu_1} \left[ \sum_{i=1}^{t} \varepsilon_i + \mu_2 \sum_{i=1}^{t-1} \varepsilon_i + \mu_2^2 \sum_{i=1}^{t-2} \varepsilon_i + \mu_2^3 \sum_{i=1}^{t-3} \varepsilon_i + \cdots + \mu_2^{t-1} \varepsilon_1 \right]$$

$$\hat{W}_t = \frac{d_1 + d_2}{1 - \mu_1} \left[ \varepsilon_t \frac{1 - \mu_2}{1 - \mu_2} + \varepsilon_{t-1} \frac{1 - \mu_2^2}{1 - \mu_2} + \varepsilon_{t-2} \frac{1 - \mu_2^3}{1 - \mu_2} + \cdots + \varepsilon_1 \frac{1 - \mu_2^t}{1 - \mu_2} \right]$$

$$\hat{W}_t = \frac{d_1 + d_2}{(1 - \mu_1)(1 - \mu_2)} \sum_{i=1}^{t} \varepsilon_i (1 - \mu_2^{t-i+1})$$

It can be demonstrated that $\frac{d_1 + d_2}{(1 - \mu_1)(1 - \mu_2)} = 1$ .

Thus,

$$\hat{W}_t = \sum_{i=1}^{t} \varepsilon_i (1 - \mu_2^{t-i+1})$$

From equation (4), the relationship $du_t = -s_L \hat{L}_t$ , and the relationship $\hat{\Omega}_t = \omega \hat{W}_t + (1 - \omega) \hat{W}_{t-1}$ ,

$$du_t = -s_L \frac{[\gamma - \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})]\hat{W}_t - \gamma\hat{\theta}_t - \phi(\gamma-1)\hat{A}_t + \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})\hat{W}_{t-1}}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u}$$

In response to a series of stochastic demand shocks,

$$du_t = -s_L \frac{[\gamma - \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})]\sum_{i=1}^{t}\varepsilon_i(1-\mu_2^{t-i+1}) - \gamma\sum_{i=1}^{t}\varepsilon_i + \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})\sum_{i=1}^{t-1}\varepsilon_i(1-\mu_2^{t-i})}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u}$$

$$du_t = -s_L \frac{-\gamma\sum_{i=1}^{t}\varepsilon_i\mu_2^{t-i+1} - \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})\varepsilon_t + \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})\sum_{i=1}^{t}\varepsilon_i\mu_2^{t-i+1} - \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})\sum_{i=1}^{t-1}\varepsilon_i\mu_2^{t-i}}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u}$$

$$du_t = -s_L \frac{-\gamma\mu_2\sum_{i=1}^{t}\varepsilon_i\mu_2^{t-i} + \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})(\mu_2 - 1)\sum_{i=1}^{t}\varepsilon_i\mu_2^{t-i}}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u}$$

$$du_t = -s_L \frac{-\gamma\mu_2 + \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})(\mu_2 - 1)}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u}\sum_{i=1}^{t}\varepsilon_i\mu_2^{t-i}$$

$$du_t = z\sum_{i=1}^{t}\varepsilon_i\mu_2^{t-i}$$

where

$$z = -s_L \frac{-\gamma\mu_2 + \phi(\gamma-1)(1-\omega)e^{-1}(e_W + e_{\Delta W})(\mu_2 - 1)}{\phi\gamma - \phi - \gamma - \phi(\gamma-1)s_L e^{-1} e_u}$$

If a regression of the form $\hat{W}_t = \hat{\beta}_1 du_t + \hat{\beta}_2 \hat{W}_{t-1} + \varepsilon_t$ is estimated, it can be demonstrated that

$$\hat{\beta}_1 = \frac{1-\mu_2}{z}$$

$$\hat{\beta}_2 = 1.$$

These coefficients can be demonstrated to be correct from the following derivation:

$$\begin{aligned}
\hat{\beta}_1 du_t + \hat{\beta}_2 \hat{W}_{t-1} &= \hat{\beta}_1 z \sum_{i=1}^{t} \varepsilon_i \mu_2^{t-i} + \hat{\beta}_2 \sum_{i=1}^{t-1} \varepsilon_i (1-\mu_2^{t-i}) \\
&= (1-\mu_2) \sum_{i=1}^{t} \varepsilon_i \mu_2^{t-i} + \sum_{i=1}^{t-1} \varepsilon_i (1-\mu_2^{t-i}) \\
&= \sum_{i=1}^{t} \varepsilon_i \mu_2^{t-i} - \sum_{i=1}^{t} \varepsilon_i \mu_2^{t-i+1} + \sum_{i=1}^{t-1} \varepsilon_i - \sum_{i=1}^{t-1} \varepsilon_i \mu_2^{t-i} \\
&= \sum_{i=1}^{t} \varepsilon_i (1-\mu_2^{t-i+1}) \\
&= \hat{W}_t
\end{aligned}$$