A Model of the Natural Rate of Unemployment and the Phillips Curve with Efficiency Wages, Bargaining, and Search and Matching

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Abstract

This paper develops a model of wage setting that combines efficiency wages, bargaining, and search and matching. The model's equilibrium conditions determine the economy's natural rate of unemployment. It is demonstrated that the natural rate depends positively on workers' bargaining power, the responsiveness of workers' efficiency to their wages, unemployment benefits, the cost of maintaining a vacancy, and the separation rate, and that it depends negatively on the efficiency of matching. Expanding the model around its steady state yields an equation for the wage-wage Phillips curve, in which wage inflation depends negatively on current unemployment, positively on lagged unemployment (with the sum of coefficients on current and lagged unemployment being negative), positively on expected future wage inflation, and positively on lagged wage inflation. The model also yields an expression for the upward-sloping counterpart to the Phillips curve, referred to as the Dynamic Labor Demand (DLD) curve. In the DLD curve, wage inflation depends positively on the changes in the unemployment rate, the money supply, expected future real demand, real interest rates, and expected price inflation. In addition, wage inflation depends negatively on the change in technology in the DLD curve. (Wage inflation also depends on the lagged change in the unemployment rate, but the effect is ambiguous.) Shifts of the Phillips curve and DLD curve show the path of wage inflation and unemployment in response to macroeconomic shocks (e.g., shocks to real aggregate demand, monetary growth, and technology). Empirical tests with U.S. data find strong support for the model's predictions, as all the coefficients in the Phillips curve and DLD curve have the predicted sign, and most are significant at the 1% level.

Keywords: Natural rate; Phillips curve; Efficiency wage; Bargaining; Search and matching
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1. Introduction

This study develops a model of the labor market that combines efficiency wage setting, bargaining, and search and matching. From the model's steady-state conditions, an equation is derived that determines the economy's natural rate of unemployment. This equation shows how the natural rate is affected by changes in workers' bargaining power, the responsiveness of workers' efficiency to their wages, unemployment benefits, the probability of a separation, the cost of maintaining a vacancy, and the efficiency of the matching process.

The model is then expanded around its steady-state equilibrium, yielding an equation for the wage Phillips curve, in which wage inflation depends positively on expected future wage inflation and lagged wage inflation, depends negatively on the current unemployment rate, and depends positively on lagged unemployment. In the Phillips curve, the sum of the coefficients on current and lagged unemployment is negative. The slope of the Phillips curve depends on the model's microeconomic parameters, such as worker's bargaining power, the separation rate, the cost of a vacancy, the efficiency of matching, and the probability that a firm can adjust its wage. In addition, the same framework is used to derive the upward-sloping counterpart to the Phillips curve, referred to as the Dynamic Labor Demand (DLD) curve. In the DLD curve, wage inflation depends positively on the changes the money supply, expected future real demand, real interest rates, and expected price inflation, and wage inflation depends negatively on the change in technology. In addition, wage inflation depends positively on the current change in the unemployment rate, and it also depends on the lagged change in the unemployment rate, although the effect is ambiguous.

The intersection of the Phillips curve and the DLD curve determines the economy's unemployment rate and rate of wage inflation. The Phillips curve – DLD framework can be used to show the adjustment path the economy follows in the transition from its initial equilibrium to its new equilibrium in response to shocks to productivity, real demand, and monetary policy.

The Phillips curve and DLD curve are simultaneously estimated with U.S. macroeconomic data, and the empirical results strongly support the predictions of the theoretical model. The coefficients in the Phillips curve and the DLD curve always have the expected sign and are usually significant at the 1% level. In the DLD curve, 90% of the coefficients are significant at the 1% level. In the Phillips curve, the coefficients on current unemployment, expected future wage inflation, and lagged wage inflation are always significant at the 10% level, and 67% are significant at the 1% level. In addition, the empirical results support the model's predictions that the sum of the coefficients on current and lagged unemployment is negative in the Phillips curve, but equals 0 in the DLD curve. Also, the model predicts that the coefficient on the money supply in the DLD curve should equal the ratio over the sample period between changes in workers' hourly wages and nominal GDP per worker (approximately 0.85), and this prediction is supported by the empirical results.

2. Assumptions

In deriving the model, the following assumptions are made:

1. There are a fixed number of workers in the economy, each of whom inelastically supplies one unit of labor.

2. In each period a proportion, q, of workers separate from their jobs.

3. The utility of an individual worker (with subscript *i* denoting individual workers) when he or she is employed can be expressed as

$$U_{i,t}^{E} = W_{i,t} - G(e_{i,t}),$$

where $G(e_{i,t})$ represents the disutility of providing effort when working. Workers provide effort up to the point where the marginal disutility of providing effort equals the expected value of keeping their job. Thus, thus disutility of effort should be proportional to the average wage, so that utility can be expressed as,

$$U_{i,t}^{E} = W_{i,t} - g(e_{i,t})\overline{W}_{t}, \tag{1}$$

where $\overline{W}$ represents the average wage in the economy.

4. A worker's utility when unemployed is

$$U_{i,t}^{UN} = \Psi_{i,t} + B_{i,t},$$

where Ψ is the value of leisure and *B* represents unemployment benefits. Both are assumed to be proportional to the average wage,[1] so that

$$U_{i,t}^{UN} = (\psi + b^{*})\overline{W}_{t}. \tag{2}$$

5. Workers' efficiency is described by the effort model of Campbell (2006), so that efficiency can be expressed as

$$e = e[W_t / \overline{W}_t, h(u_t)] \qquad \text{with } e_W > 0, \quad e_h < 0, \quad e_{WW} < 0, \quad e_{Wh} > 0, \tag{3}$$

where *h* is the probability of an unemployed worker being hired, which depends negatively on the unemployment rate (*u*). In Campbell (2006), the unemployment rate affects effort indirectly through its effect on the hiring rate, since the cost of losing one's job depends on the probability of finding a new job if dismissed, which depends on the hiring rate rather than on the unemployment rate, *per se*. Also, in the turnover cost models of Stiglitz (1974), Schlicht (1978), and Salop (1979), the propensity to quit should depend on the probability of finding another job.[2] In addition, Section 3 shows that the equilibrium hiring rate is less affected by changes in the separation rate as compared to the equilibrium unemployment rate, making the equilibrium hiring rate more stable over time.

6. Firms produce output (Q) with the Cobb-Douglas production function,

$$Q_t = A_t^{\phi} L_t^{\phi} K_t^{1-\phi} e\left[W_t / \overline{W}_t, h(u_t)\right]^{\phi}, \tag{4}$$

where A represents technology (assumed to be labor augmenting), L is employment, K is the capital stock, and ϕ represents the elasticity of output with respect to labor input, which equals labor's share of national income.

7. Firms face a downward-sloping demand curves for the good they produce,

$$Q_t^D = Y_t \left(\frac{P_t}{\overline{P}_t} \right)^{-\gamma} \tag{5}$$

where Y is real demand per firm, P is the price of the firm's output, and $\overline{P}$ is the aggregate price level.

8. When a job opening is vacant, the firm incurs a cost of $c\overline{W}$ per unit time, where $\overline{W}$ is the average wage. It is assumed that the cost is proportional to the average wage, since most of the costs involved with recruiting new workers are labor costs.

9. The matching function can be expressed by the Cobb-Douglas function,

$$m_t = \kappa u_t^a v_t^{1-a},$$

where u is the unemployment rate, v is the vacancy rate, and κ measures the efficiency of matching.[3] Let the θ represent the ratio between vacancies and unemployment. Then, the probability of a firm filling a vacancy is

$$\eta_t = \frac{m_t}{v_t} = \kappa\theta_t^{-a},$$

and the probability of an unemployed worker being hired is

$$h_t = \frac{m_t}{u_t} = \kappa\theta_t^{1-a}. \quad (6)$$

10. Firms incur a fixed cost of operating. Thus, there are a finite number of firms, with the number of firms determined by this cost. It is assumed that N represents the ratio between workers and firms.

11. The workers at a firm are represented by a union, and the wage is determined from bargaining between firms and the union to maximize

$$\Omega = \beta(U_t^* - \overline{U}_t^*) + (1-\beta)(\Pi_t - \overline{\Pi}_t), \quad (7)$$

where β represents workers' bargaining power, U^* is the expected utility of workers in the union, $\overline{U}^*$ is the reservation utility of workers in the union, Π is the firm's profits, and $\overline{\Pi}$ is the reservation level of profits. The reservation level of profits is assumed to be 0. In addition, in line with Layard, Nickell, and Jackman (1991), it is assumed that the union's objective function is the fraction of the bargaining unit that is employed at the firm times the utility of these workers plus the fraction of workers who are not employed at the firm times the expected utility of these workers. Also, as in Layard, Nickell, and Jackman, it is assumed that the union's per-worker reservation level of utility equals the utility of an unemployed individual.

12. Following the derivation in Romer (2012), aggregate demand is determined from the New Keynesian IS-LM system described by

$$\text{IS:} \qquad Y_t = \beta^{-\frac{1}{\mu}}(1+r_t)^{-\frac{1}{\mu}} Y_{t+1}, \quad \text{and} \tag{8a}$$

$$\text{LM:} \qquad \frac{M_t}{\bar{P}_t} = Y_t \left(\frac{i_t}{1+i_t} \right)^{-\frac{1}{\mu}}, \tag{8b}$$ [4]

where Y is real output, β is the discount factor, r is the real interest rate, M is the money supply, and i is the nominal interest rate.

13. Wage setting follows a Calvo (1983) process in which each period a fraction, τ, of firms adjust their nominal wages to their optimal levels.

14. There is nominal inertia in wage setting. Two alternative ways to motivate this nominal inertia are to assume that firms not optimizing their wages index them to lagged wage inflation or to assume that firms' expectations of future wage inflation are a mixture of rational and adaptive expectations. While these assumptions are observationally equivalent, this study makes the latter assumption because it is supported by the findings of Fuhrer (1997), Roberts (1997), Pfajfar and Santoro (2010), and Levine et al. (2012), whereas there appears to be little empirical evidence for the assumption of non-optimizing firms to index their wages to lagged wage inflation. Accordingly, it is assumed that

$$\pi_{t+1}^{w,e} = \omega \pi_{t+1}^{w,ue} + (1-\omega)[\lambda_1 \pi_t^w + \lambda_2 \pi_{t-1}^w + \lambda_3 \pi_{t-2}^w + \cdots + \lambda_{T+1} \pi_{t-T}^w], \tag{9}$$

where $\pi_{t+1}^{w,ue}$ represents firms' unbiased expectations of future average wages, ω represents the degree of rational expectations, 1-ω represents the degree of adaptive expectations, and the λ's represent the weight placed on each lag in the adaptive component.

3. The Natural Rate of Unemployment

We first consider a steady state, in which all variables are constant. Because all variables are constant, the time subscripts can be suppressed.

Behavior of Firms

From (5), a firm's total revenues can be expressed as

$$PQ = Y^{\frac{1}{\gamma}} Q^{\frac{\gamma-1}{\gamma}} \overline{P} .$$

An equation for the present value of profits is determined from the framework of Pissarides (2000) for large firms. Accordingly, Π can be expressed as

$$\Pi = \int_0^{\infty} e^{-rt} \left[Y^{\frac{1}{\gamma}} \left(A^{\phi} L^{\phi} K^{1-\phi} e[W/\overline{W}, h(u)]^{\phi} \right)^{\frac{\gamma-1}{\gamma}} \overline{P} - WL - c\overline{W}V - P^K \dot{K} - \delta P^K K \right] dt \quad \text{(10a)}$$

subject to the constraint,

$$\dot{L} = \kappa \theta^{-a} V - qL .^{5} \quad \text{(10b)}$$

In the above equation, r is the real interest rate, P^K is the price of capital goods, and δ is the depreciation rate of capital.

The Appendix demonstrates that if the derivative of the profit function with respect to K is set equal to zero, P^K is set equal to $\overline{P}$, and the resulting equation is solved for K, the optimal capital stock is,

$$K = \left(\frac{\gamma}{(1-\phi)(\gamma-1)} \right)^{\frac{\gamma}{\phi-\phi\gamma-1}} (\delta + r)^{\frac{\gamma}{\phi-\phi\gamma-1}} Y^{-\frac{1}{\phi-\phi\gamma-1}} A^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}} L^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}} e[W/\overline{W}, h(u)]^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}}. \quad (11)$$

Substituting this value of K back into the original profit function yields

$$\Pi = \int_0^\infty e^{-rt} \Bigg[Y^{\frac{1}{1-\phi+\phi\gamma}} A^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}} L^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}} e[W/\overline{W}, h(u)]^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}} \overline{P} (\delta + r)^{\frac{-(1-\phi)(\gamma-1)}{1-\phi+\phi\gamma}}$$

$$\times \left(\frac{\gamma}{(1-\phi)(\gamma-1)} \right)^{\frac{(1-\phi)(\gamma-1)}{1-\phi+\phi\gamma}} \left(\frac{\delta(1-\phi+\gamma\phi)+\gamma r}{\gamma} \right) - WL - c\overline{W}V \Bigg] dt \quad (12a)$$

$$\text{s.t. } \dot{L} = \kappa\theta^{-a} V - qL. \quad (12b)$$

Differentiating (12) with respect to L yields two first-order conditions,

$$e^{-rt} \left[\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma} Y^{\frac{1}{1-\phi+\phi\gamma}} A^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}} L^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}-1} e[W/\overline{W}, h(u)]^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}} \overline{P} (\delta + r)^{\frac{-(1-\phi)(\gamma-1)}{1-\phi+\phi\gamma}} D - W \right]$$

$$- qx + \frac{dx}{dt} = 0 \quad (13a)$$

and

$$- e^{-rt} c\overline{W} + \kappa\theta^{-a} x = 0, \quad (13b)$$

where

$$D = \left(\frac{\gamma}{(1-\phi)(\gamma-1)} \right)^{\frac{(1-\phi)(\gamma-1)}{1-\phi+\phi\gamma}} \left(\frac{\delta(1-\phi+\gamma\phi)+\gamma r}{\gamma} \right).$$

From (13a) and (13b), the Appendix demonstrates that employment can be expressed as,

$$L = \left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}\left(W+\frac{r+q}{\kappa}c\theta^{a}\overline{W}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma} YA^{\phi(\gamma-1)} e[W/\overline{W}, h(u)]^{\phi(\gamma-1)} \times \overline{P}^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}. \tag{14}$$

The Appendix also demonstrates that substituting (14) into (12a) results in the following expression for the profit function:

$$\Pi = \frac{1}{r}\left\{YA^{\phi(\gamma-1}e[W/\overline{W}, h(u)]^{\phi(\gamma-1)}\overline{P}^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma} \times\left[\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{-\phi(\gamma-1)} - \left(W+\frac{q}{\kappa\theta^{-a}}c\overline{W}\right)\times\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{\phi-\phi\gamma-1}\right]\right\}. \tag{15}$$

Let *S* be defined as

$$S = \frac{W+\dfrac{q}{\kappa\theta^{-a}}c\overline{W}}{W+\dfrac{r+q}{\kappa\theta^{-a}}c\overline{W}},$$

where S=1 if r=0. Substituting this expression into (15) enables profits to be expressed as

$$\Pi = \frac{1}{r}YA^{\phi(\gamma-1}e[W/\overline{W}, h(u)]^{\phi(\gamma-1)}\overline{P}^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma} \times\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{-\phi(\gamma-1)}\left(\frac{1+\phi(\gamma-1)(1-S)}{\phi(\gamma-1)}\right). \tag{16}$$

Behavior of Workers

Let V^{EC} represent the present value of utility for a worker employed at his or her current firm, V^{U} represent the present value of utility for an unemployed worker, and V^{EO} represent the present value of utility for a worker employed at another firm. Equilibrium is described by the following flow conditions:

$$rV^{EC} = W - g(e)\overline{W} - q[V^{EC} - V^{U}], \tag{17a}$$

$$rV^{EO} = \overline{W} - g(e)\overline{W} - q[V^{EO} - V^{U}], \tag{17b}$$

and

$$rV^{U} = (\psi + b^{*})\overline{W} + h(u)[V^{EO} - V^{U}]. \tag{17c}$$

The Appendix demonstrates that the expected utility of a worker employed at his or her current firm is

$$V^{EC} = \frac{W}{r+q} - \frac{\chi\overline{W}}{r+q} + \frac{q}{r+q}V^{U}, \tag{18}$$

where χ is the equilibrium value of $g(e)$, and that the expected utility of an unemployed worker is

$$V^{U} = \frac{(\psi + b^{*})r + (\psi + b^{*})q + h(u)(1-\chi)}{r[r+q+h(u)]}\overline{W}. \tag{19}$$

From Assumption 11, the union's expected utility is

$$U^{*} = \frac{L}{N}V^{EC} + \frac{N-L}{N}V^{U},$$

where L/N is the fraction of the bargaining unit that is employed at the firm, and $(N-L)/N$ is the fraction of workers who are not employed at the firm. The reservation level of utility is assumed to equal the utility of an unemployed individual, so that $\overline{U}^{*} = V^{U}$. Thus, the difference between expected utility and reservation utility can be expressed as,

$$U^{*} - \overline{U}^{*} = \frac{L}{N}V^{EC} + \frac{N-L}{N}V^{U} - V^{U} = \frac{L}{N}[V^{EC} - V^{U}]. \tag{20}$$

The Appendix demonstrates that substituting (18) and (19) into (20), and then substituting (14) into the resulting expression yields

$$U^{*}-\overline{U}^{*}=\frac{1}{N}\left[\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{\phi-\phi\gamma-1}D^{1-\phi+\phi\gamma}YA^{\phi(\gamma-1}e[W/\overline{W},u]^{\phi(\gamma-1)}\right.$$
$$\left.\times\overline{P}^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}\left(\frac{W}{r+q}-\frac{br+bq+h(u)}{(r+q)[r+q+h(u)]}\overline{W}\right)\right], \tag{21}$$

where $b=\psi+b^{*}+\chi$.

From Assumption 11, the wage is chosen to maximize equation (7). The Appendix demonstrates that assuming that $\overline{\Pi}=0$ and substituting (16) and (21) into (7) and maximizing the resulting expression yields the equation,

$$e_{W}[W/\overline{W},h(u)]e[W/\overline{W},h(u)]^{-1}=\frac{\frac{W}{\overline{W}}-\frac{br+bq+h(u)}{r+q+h(u)}-\frac{\beta}{\phi(\gamma-1)}\left[\frac{br+bq+h(u)}{r+q+h(u)}+\frac{r+q}{\kappa\theta^{-a}}c\right]}{\left(\frac{W}{\overline{W}}+\frac{r+q}{\kappa\theta^{-a}}c\right)\left[\frac{W}{\overline{W}}-\frac{br+bq+h(u)}{r+q+h(u)}\right]}. \tag{22}$$

In equilibrium, the condition that separations equal new hires implies that

$$h(u)=\frac{q(1-u)}{u}. \tag{23}$$

In addition, (6) and (23) imply that the equilibrium value of θ is

$$\theta=\kappa^{-\frac{1}{1-a}}h^{\frac{1}{1-a}}=\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{1}{1-a}}. \tag{24}$$

The Appendix demonstrates that setting W equal to $\overline{W}$ in (22) and substituting (23) and (24) into the resulting expression yields the equilibrium condition:

$$e_W\left[1,\frac{q(1-u)}{u}\right]e\left[1,\frac{q(1-u)}{u}\right]^{-1}$$

$$=\frac{1-\dfrac{\beta}{\phi(\gamma-1)(1-b)}\left[b+\dfrac{\frac{q(1-u)}{u}}{r+q}+\left(r+q+\dfrac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\dfrac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right]}{\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\dfrac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)}. \tag{25}$$

Solving (25) for u determines the economy's natural rate of unemployment. From this equilibrium condition, the effects of the model's parameters on the natural rate can be calculated. Let

$$\begin{aligned}
Q=\frac{q}{u^2}\Bigg[&\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e[1,h]^{-1}e_{Wh}[1,h]\\
&-\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e_W[1,h]e[1,h]^{-2}e_h[1,h]\\
&+e_W[1,h]e[1,h]^{-1}(r+q)c\kappa^{-\frac{1}{1-a}}\frac{a}{1-a}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\\
&+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{1}{r+q}-\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\\
&+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{a}{1-a}\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\Bigg]>0,
\end{aligned}$$

$$R = \Bigg[e_W[1,h]e[1,h]^{-1} c\kappa^{-\frac{1}{1-a}} \left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}$$

$$+ e_W[1,h]e[1,h]^{-1} \frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}} \left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}} \frac{(1-u)}{u}$$

$$+ \left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right) e[1,h]^{-1} e_{Wh}[1,h] \frac{(1-u)}{u}$$

$$- \left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right) e_W[1,h]e[1,h]^{-2} e_h[1,h] \frac{(1-u)}{u}$$

$$+ \frac{\beta}{\phi(\gamma-1)(1-b)} \frac{r(1-u)}{(r+q)^2 u} + \frac{\beta}{\phi(\gamma-1)(1-b)} c\kappa^{-\frac{1}{1-a}} \left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}$$

$$+ \frac{\beta}{\phi(\gamma-1)(1-b)} c\kappa^{-\frac{1}{1-a}} \left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}} \frac{(1-u)}{u}$$

$$+ \frac{\beta}{\phi(\gamma-1)(1-b)} \frac{a}{1-a} \left(r+q+\frac{q(1-u)}{u}\right) c\kappa^{-\frac{1}{1-a}} \left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}} \frac{(1-u)}{u} \Bigg] > 0,$$

and

$$Z = b + \frac{q(1-u)}{(r+q)u} + \left(r+q+\frac{q(1-u)}{u}\right) c\kappa^{-\frac{1}{1-a}} \left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}} > 0\,.$$

Then, the Appendix demonstrates that

$$\frac{du}{d\beta} = \frac{\dfrac{Z}{\phi(\gamma-1)(1-b)}}{Q} > 0\,,$$

$$\frac{du}{db} = \frac{\dfrac{Z\beta + \beta(1-b)}{\phi(\gamma-1)(1-b)^2}}{Q} > 0,$$

$$\frac{du}{dc} = \frac{\left[e_W[1,h]e[1,h]^{-1}(r+q) + \frac{\beta}{\phi(\gamma-1)(1-b)}\left(r+q+\frac{q(1-u)}{u} \right) \right] \kappa^{-\frac{1}{1-a}} \left(\frac{q(1-u)}{u} \right)^{\frac{a}{1-a}}}{Q} > 0,$$

$$\frac{du}{d\kappa} = \frac{-\left[e_W[1,h]e[1,h]^{-1}\frac{a}{1-a}(r+q) + \frac{1}{1-a}\frac{\beta\left(r+q+\frac{q(1-u)}{u} \right)}{\phi(\gamma-1)(1-b)} \right] c\kappa^{\frac{a-2}{1-a}} \left(\frac{q(1-u)}{u} \right)^{\frac{a}{1-a}}}{Q} < 0,$$

and

$$\frac{du}{dq} = \frac{R}{Q} > 0 .$$

In addition, the effect of e_W (the derivative of efficiency with respect of the relative wage) on the natural rate can also be calculated. Suppose efficiency is expressed as

$$e[W/\overline{W}, h(u), z] \;, \quad \text{with } \; e_{Wz} > 0, \; e_z = 0 .$$

Thus, z is a shift variable that raises the derivative of efficiency with respect to the relative wage, but that has a neutral effect on efficiency itself. Then the Appendix demonstrates that

$$\frac{du}{dz} = \frac{e_{Wz}[1,h,z]e[1,h,z]^{-1}}{Q} > 0 .$$

Thus, as workers' efficiency becomes more responsive to their wages, the economy's natural rate increases. In efficiency wage models, efficiency is generally treated as a function of workers' effort and quit propensities. Thus, the above relationship implies that equilibrium unemployment should be higher for subgroups of workers and for countries in which effort and quit propensities are highly responsive to relative wages.

Overall, the above comparative statics show that the natural rate depends positively on workers' bargaining power (β), the ratio between unemployment benefits and wages (b), the cost of maintaining a vacancy (c), the separation rate (q), and the derivative of efficiency with respect to the wage (e_W), and it depends negatively on the efficiency of matching (κ).

Suppose that the economy is characterized by a pure efficiency wage model, with no bargaining or search and matching, so that c=0 and β=0. In this case,

$$\frac{du}{dq} = \frac{e[1,h]^{-1} e_{Wh}[1,h] \frac{(1-u)}{u} - e_W[1,h] e[1,h]^{-2} e_h[1,h] \frac{(1-u)}{u}}{\frac{q}{u^2}\left[e[1,h]^{-1} e_{Wh}[1,h] - e_W[1,h] e[1,h]^{-2} e_h[1,h]\right]},$$

resulting in the relationship,

$$\frac{du}{dq}\frac{q}{u} = 1-u. \qquad (26)$$

Thus, in a pure efficiency wage model, the elasticity of the unemployment rate with respect to the separation rate equals 1 minus the natural rate of unemployment, so that the natural rate moves almost one-for-one with the separation rate. Campbell and Duca (2008) demonstrate that the U.S. natural rate does, in fact, appear to behave as predicted by (26). Shimer (2007) constructs data on the separation rate, and Campbell and Duca (2008) adjust Shimer's data for business cycle fluctuations, and then smooth it with a Hodrick-Prescott filter, producing a cyclically-adjusted and smoothed separation rate. They then calibrate a model with U.S. data from 1960-1970, so that the average value of the model's natural rate over this period is equal to the average value of the Congressional Budget Office's (CBO) estimate of the U.S. natural rate.

Using the relationship in (26), they use the model to simulate the economy's natural rate from 1960-2005, based on data on the percentage of workers under the age of 35 and the cyclically-adjusted and smoothed separation rate. In addition, they also simulate the ratio between the average duration of unemployment and the unemployment rate over the same period. The simulation demonstrates that the model's predicted natural rate and duration-unemployment ratio closely track both the actual natural rate (based on the CBO's estimates) and the duration-unemployment ratio. They also find that, over the 1960-1991 period, changes in the separation rate based on demographics alone can explain these variables. However, after 1991 there was a decrease in the separation rate that cannot be explained by demographics alone, and the model needs to incorporate this unexplained decrease in the separation rate to accurately track the natural rate and the duration-unemployment ratio after 1991.[6]

The assumption that efficiency depends on the hiring rate, which is a function of the unemployment rate, means that changes in the separation rate will have a much greater impact on the equilibrium unemployment rate than on the equilibrium hiring rate, a prediction that is supported by U.S. data. Equation (23) shows that a change in the economy's equilibrium separation rate changes results in a change in equilibrium unemployment, the equilibrium hiring rate, or some combination of the two. Comparing the 1970-1990 period with the 1991-2007 period indicates that changes in the separation rate have a much greater impact on the unemployment rate than on the hiring rate. Between the 1970-1990 period and the 1990-2007 period, the U.S. economy's separation rate fell by 21%. In response, the unemployment rate declined 19%, and the hiring rate declined 2%. (The reason for not also considering the 1960's is that actual GDP appears to have been above potential GDP for most of this decade, while the latter decades were characterized by alternating inflationary and recessionary gaps.) Thus, it appears that almost all of the effect of the fall in the separation rate was to change equilibrium unemployment, leaving the equilibrium hiring rate relatively unaffected. Thus, it seems reasonable to model efficiency as a function of the hiring rate instead of the unemployment rate itself.

4. The Phillips Curve and the Dynamic Labor Demand (DLD) Curve

The Phillips curve can be obtained from approximating the equation for the natural rate around its steady-state equilibrium by totally differentiating this equation and dividing by the original equation. While the time subscripts are dropped in deriving the economy's natural rate, they are needed to obtain an expression for the Phillips curve, since the Phillips curve is a relationship over time between inflation and unemployment. With time subscripts, (22) can be expressed as

$$e_W[W_t/\overline{W}_t, h_t]e[W_t/\overline{W}_t, h_t]^{-1} = \frac{\dfrac{W_t}{\overline{W}_t} - \dfrac{br+bq+h_t}{r+q+h_t} - \dfrac{\beta}{\phi(\gamma-1)}\left[\dfrac{br+bq+h_t}{r+q+h_t} + \dfrac{r+q}{\kappa\theta^{-a}}c\right]}{\left(\dfrac{W_t}{\overline{W}_t} + \dfrac{r+q}{\kappa\theta^{-a}}c\right)\left[\dfrac{W_t}{\overline{W}_t} - \dfrac{br+bq+h_t}{r+q+h_t}\right]}. \tag{27}$$

When modeling the natural rate, it is assumed that hires are equal to separations. However, this is no long true when the economy is away from its long-run equilibrium. In this case, the difference between hires and separations equals the change in employment, so that

$$h_t u_t N - q(1-u_t)N = (1-u_t)N - (1-u_{t-1})N .$$

Thus, the probability of a hire can be expressed as

$$h_t = \frac{q(1-u_t) + u_{t-1} - u_t}{u_t}. \tag{28}$$

Also, the vacancy-unemployment ratio (θ) is related to the hiring rate through the expression,

$$h_t = \kappa\theta_t^{1-a}.$$

Solving the above equation for θ and substituting (28) into the resulting expression enables the vacancy-unemployment ratio to be expressed as,

$$\theta_t = \kappa^{-\frac{1}{1-a}} h_t^{\frac{1}{1-a}} = \kappa^{-\frac{1}{1-a}} \left[\frac{q(1-u_t) + u_{t-1} - u_t}{u_t} \right]^{\frac{1}{1-a}}. \tag{29}$$

The Appendix demonstrates that by substituting (28) and (29) into (27), taking total derivatives, dividing by the original equation, assuming staggered wage setting (from assumption 13), and assuming mixed rational and adaptive expectations (from assumption 14), the Phillips curve can be expressed as

$$\pi_t^w = \psi_1 \pi_{t+1}^{w,ue} + \psi_2 \sum_{j=1}^{T} \lambda_{j+1} \pi_{t-j}^w - \psi_3 du_t + \psi_4 du_{t-1}, \tag{30}$$

where
$$\psi_1 = \frac{\beta\omega}{1-(1-\omega)\lambda_1} > 0,$$

$$\psi_2 = \frac{\beta(1-\omega)}{1-(1-\omega)\lambda_1} > 0,$$

$$\psi_3 = \frac{\tau[1-\beta(1-\tau)]}{(1-\tau)[1-(1-\omega)\lambda_1]} E > 0,$$

$$\psi_4 = \frac{\tau[1-\beta(1-\tau)]}{(1-\tau)[1-(1-\omega)\lambda_1]} F > 0.$$

In (30), π_t^w represents the percentage change in nominal wages, and du_t represents the percentage-point difference between the unemployment rate and the natural rate. The expressions for E and F

are very complex and are reported in the Appendix. In the Phillips curve, wage inflation depends on expected future wage inflation and current wage inflation, with the sum approximately equal to 1, and exactly equal to 1 if β=1. Wage inflation also depends negatively on the level of current unemployment and depends positively on the level of lagged unemployment. The Appendix demonstrates that $\psi_3 > \psi_4$, so the sum of the coefficients on current and lagged unemployment is unambiguously negative. In addition, while the model incorporates technology shocks, these shocks do not appear in the Phillips curve, so there is no need to control for technology shocks in estimating the Phillips curve. However, as discussed below, technology shocks do appear in the DLD curve, so these shocks result in a movement along the Phillips curve.

Since current wage inflation is a function of unemployment and of both expected future and lagged wage inflation, (30) is an equation for the wage-wage Phillips curve. However, when economists estimate Phillips curves, the right-hand side variable is generally expected price inflation rather than expected wage inflation. While expected price inflation is the independent variable in the vast majority of Phillips curve studies, the right-hand side variable in Phelps's (1968) seminal paper is expected wage inflation, resulting in a wage-wage Phillips curve.[7] In addition, Perry (1978) and Campbell (2017) regress the change in average hourly earnings (*AHE*) on the unemployment rate and on lagged values of either *AHE*, the change in the consumer price index (*CPI*), or the change in private nonfarm GDP deflator (*GDPD*). Both studies find a higher R^2 and a sum of coefficients on lagged inflation that is closer to 1 with lagged *AHE* than with lagged *CPI* or *GDPD*, suggesting that the wage inflation process may be described more accurately by a wage-wage Phillips curve than by a wage-price Phillips curve.

The model developed in this study can also be used to derive the upward-sloping counterpart to the Phillips curve. The DLD curve is derived in the Appendix, and it is demonstrated that

$$\begin{aligned}\pi_t^w &= \sigma_1(du_t - du_{t-1}) + \sigma_2(du_{t-1} - du_{t-2}) - \xi(\hat{A}_t - \hat{A}_{t-1}) + (\hat{M}_t - \hat{M}_{t-1}) \\ &\quad + \xi(\hat{Y}_{t+1} - \hat{Y}_t) + \frac{(1-\phi)\xi s_{r\delta}}{\phi}(dr_t - dr_{t-1}) + s_i(d\pi_t^e - d\pi_{t-1}^e).\end{aligned} \tag{31}$$

In (31), variables with "^'s" over them (e.g., $\hat{M}_t$) represent percentage deviations from steady-state values, while variables preceded by "*d*" (e.g., dr_t) represent percentage-point deviations from steady-state values. Also, s_i is the absolute value of the semi-elasticity of money demand with respect to nominal interest rates, ξ is the ratio between s_i and the absolute value of the semi-elasticity of aggregate spending with respect to real interest rates, and $s_{r\delta} = [1/(r+\delta)]$. Also, σ_1 and σ_2 are given by the expressions,

$$\sigma_1 = \frac{\dfrac{a(1-s_W)(q+u)}{1-a}}{u^2} - \frac{e^{-1}e_h(q+u)}{u^2} + s_L^{-1}(1+\xi)\left(1 + \frac{e_h e^{-1} s_L(q+u)}{u^2}\right) > 0,$$

$$\sigma_2 = -\left[\xi\frac{e_h e^{-1}}{u} + \frac{(1-s_W)[a/(1-a)]}{u}\right],$$

where e_h is the derivative of efficiency with respect to the hiring rate, s_L represents the relationship between percentage deviations in aggregate hours worked and percentage-point deviations in the unemployment rate from their steady-state values, and s_W is defined in the Appendix. In (31), the sign on σ_2 is theoretically ambiguous.

In the DLD curve, wage inflation depends positively on the current change in the unemployment rate, the change in the money supply, the expected future change in demand, the change in the real interest rate, and the change in expected price inflation. In addition, wage inflation depends negatively on the change in technology and depends either positively or negatively on the lagged change in the unemployment rate. While the prediction that wage inflation depends negatively on technology shocks (which also means that technology shocks shift the DLD to the right and thus raise unemployment) seems counterintuitive, these predictions are consistent with Basu, Fernald, and Kimball's (2006) findings that technological improvements initially result in lower employment and slightly lower nominal wages. While the direct effect of positive technology shocks is to initially reduce employment and nominal wages, these shocks should also increase future demand growth ($\hat{Y}_{t+1} - \hat{Y}_t$) by raising permanent income and the marginal product of capital, which will positively affect future employment and nominal wages.

In the DLD curve, wage inflation depends on the current change in the unemployment rate and on the lagged change in the unemployment rate. Thus, if u_t, u_{t-1}, and u_{t-2} are entered as separate variables, the sum of coefficients on these variables should equal 0, a hypothesis that is tested in Section 5.

In (30) and (31), there are two relationships, the Phillips curve and the DLD curve, that simultaneously determine the economy's unemployment rate and the rate of wage inflation. The Phillips curve is shifted by lagged wage inflation, expected future wage inflation, and lagged unemployment. The DLD curve is shifted by changes in technology, the money supply, expected future real demand, real interest rates, and expected inflation, along with the first and second lagged values of the unemployment rate. The Phillips curve – DLD framework can be used to show the transition path over time of unemployment and wage inflation in response to aggregate shocks, most importantly shocks to technology, the money supply, and real demand.

5. Empirical Estimation of the Phillips Curve and the DLD Curve

This section presents estimates of the Phillips curve and the DLD curve. The growth rate in wages (π_t^w) is measured by the percentage change in Average Hourly Earnings (*AHE*), a series available since 1964, and du_t is the deviation of the unemployment rate from the Congressional Budget Office's estimate of the natural rate. Because expected future wage inflation ($\pi_{t+1}^{w,ue}$) may be correlated with the error term in the Phillips curve, this variable is instrumented. To instrument $\pi_{t+1}^{w,ue}$, the DLD curve in levels[8] in (31) is solved for du_t, and the resulting expression is substituted into (30), which produces a difference equation. The solution to the difference equation shows that wage inflation depends on the current, lagged, and future values of variables in the DLD, except for the unemployment rate. Thus, the set of instruments are the changes in the money supply (along with changes in trend velocity), technology, real GDP, and real interest rates.[9]

In the DLD curve, technology is measured with the utilization-adjusted series on total factor productivity maintained by the Federal Reserve Bank of San Francisco, based on the method of Basu, Fernald, and Kimball (2006) as updated in Basu, Fernald, Fisher, and Kimball (2013). However, these estimates exhibit a great deal of volatility on a quarter-to-quarter basis, so they might not be an accurate measure of true technological change. The real cost of imported crude oil (*OilPrice*) is also included to capture the effect of an additional type of supply shock. The measure of the money supply is M2 per worker. To account for long-term changes in money demand, the trend change in velocity, estimated with a Hodrick-Prescott filter with a smoothing parameter of 1600, is also included in the regressions. Expected future real GDP growth is unobserved, but is instrumented with real net wealth per worker, the University of Michigan's Index of Consumer Expectations, and technology.[10]

The real interest rate is measured by the 3-month Treasury bill rate[11] minus the percentage change in the GDP deflator. Expected price inflation is measured by the Livingston series of expectations of the GDP deflator, which is available since 1971. Including this variable may be problematic since it is probably highly correlated with past and expected future wage inflation and since it is not available for the entire sample period. In addition, it enters (31) only because it represents the difference between nominal and real interest rates. Because of these potential issues, regressions are estimated both with and without this variable.

From (31), the DLD depends positively on the change in unemployment from period *t*-1 to *t* and on the change in unemployment from period *t*-2 to *t*-1. To examine whether the DLD does, in fact, depend on the change in unemployment (as predicted), the regressions include u_t, u_{t-1}, and u_{t-2} as separate variables (as opposed to the regressions including $u_t - u_{t-1}$ and $u_{t-1} - u_{t-2}$). If the DLD does depend on the change in the unemployment rate, then the sum of the coefficients on these three values of unemployment should equal 0.

Because future wage inflation and future aggregate demand are instrumented, the equations are estimated with full information maximum likelihood (FIML). An alternative way to estimate these equations is with generalized methods of moments (GMM). However, Campbell (2017) shows that the coefficients in a Phillips curve regression are much more stable with FIML than with GMM estimation. In addition, Lindé (2005) simulates macroeconomic data with known parameters and estimates Phillips curves with this simulated data, and he finds that FIML estimation results in coefficient estimates that are much closer to the true parameter values as compared with GMM.

Equations are estimated both over the entire 1967:2-2017:1 sample period and with data from 1967:2-2007:4, which excludes the Great Recession and the subsequent recovery. The reason for estimating equations that exclude post-2007 data is that the adjustment to the 2008 recession would likely have entailed notional nominal wage reductions for a significant number of workers. If these reductions did not occur because of downward nominal wage rigidity, the estimated coefficients in the Phillips curve and DLD curve may differ from what they would have been in the absence of wage rigidity.

The results are presented in Table 1, with estimates over the 1967:2-2017:1 sample period in the first two columns and estimates over the 1967:2-2007:4 sample period in the last two columns. Twelve lags of wage inflation are included in the Phillips curve, and Table 1 reports the value of the first lag, the sum of the 2^{nd} through 4^{th} lag, the sum of the 5^{th} through 8^{th} lag, and the sum of the 9^{th} through 12^{th} lag. The equation for the DLD curve includes the current and two lagged values of the unemployment rate. Eq. (31) predicts that the coefficient on the current unemployment rate should be positive and that the sum of the coefficients on the current, first lagged, and second lagged values should equal 0. Also included in the DLD equation are the current and three lagged values of the changes in technology, the M2 money supply, trend velocity, oil prices, instrumented future real GDP, real interest rates, and expected inflation (in two columns), and the sum of these coefficients is reported in Table 1.[12]

In the Phillips curve, the coefficient on the current unemployment rate is always negative and significant at the 10% level, and this coefficient is always significant at the 1% level with data over the entire sample period. The coefficient on lagged unemployment is always positive, and is significant at the 5% level with equations estimated over the entire sample period. As predicted, the sum of the coefficients on current and lagged unemployment is negative, and the sum is significantly different from 0 at the 5% level in all four columns and is significant at the 1% level in three of the four columns.

In the regression with pre-2008 data, the sum of coefficients on the unemployment rate lies between –0.0675 and –0.0722. These estimates are roughly consistent with Galí's (2011) estimates of the new Keynesian wage Phillips curve with quarterly data through the end of 2007 (with lagged year-to-year price inflation as an independent variable), in which the sum of coefficients is –0.096 or –0.099, depending on the measure of wages.

In the DLD curve, equation (31) predicts that the coefficient on current unemployment should be positive, although the signs on unemployment lagged 1 or 2 periods are ambiguous. In addition, (31) predicts that the coefficients on oil prices, the money supply (as well as trend velocity), future real GDP, real interest rates, and expected inflation should be positive, while the coefficient on technology should be negative. In Table 1, all 30 coefficients have the predicted sign, all are significant at the 10% level, and 27 are significant at the 1% level. Thus, the predictions of the theoretical model are strongly supported by the empirical results with U.S. macroeconomic data.

In addition, (31) implies three specific predictions concerning the coefficients in the DLD. First, the model predicts that the sum of the coefficients on current unemployment and lagged unemployment should equal 0. Second, the coefficient on the change in technology is predicted to equal the negative of the coefficient on the change in real demand. Third, the coefficient on the growth in the money supply (adjusted for changes in trend velocity) should equal 1. However, labor's share of national income fell over both the full sample period and the pre-2008 sample period, so nominal wages grew more slowly than nominal GDP per worker. On average, the ratio between the average rise in *AHE* and the average rise in nominal GDP per worker was 0.842 over the pre-2008 sample period and 0.855 over the full sample period. If ν represents this ratio, then the sum of the coefficients on monetary growth and trend velocity should equal ν.

The last three rows of Table 1 report Likelihood Ratio tests of these restrictions. The test of the restriction that $u_t + u_{t-1} + u_{t-2} = 0$ can be rejected at the 10% level in only one column, and the coefficient on current unemployment is always within 3% of the sum of coefficients of the first and second lag of unemployment. The restriction that *A*=-*Y* cannot be rejected at even the 10% level with pre-2008 data, but can be rejected at the 1% level over the entire sample period. However, true technological change is unobservable, and the technology data used in the regressions are quite volatile and may not reflect true exogenous changes in technology. Thus, the rejection of this restriction over the entire sample period may be a result of the way technological growth was estimated.

The restriction that the sum of coefficients on M2 growth and trend velocity equals ν can never be rejected at even the 10% level. These results indicate that monetary growth (adjusted for trend velocity) has the expected effect on nominal wage growth. Thus, in addition to all the coefficients having the predicted signs, two of the three restrictions on the coefficients implied by the theoretical model are supported by the empirical estimation, and the restriction that is rejected over the entire sample period may be a result of measurement error.

The coefficient on future wage inflation lies between 0.304 and 0.417 in the regressions in Table 1. These estimates of the degree of forward-looking expectations are similar to the findings of Lindé (2005), which estimates a coefficient on forward-looking inflation of 0.282 to 0.457 in the Phillips curve, and to those of Levine *et al*. (2012), which estimates the degree of rational vs. adaptive expectations with, finding (with their best specification) that 17-25% of firms and 30-34% of households have rational expectations.

6. Possible Extensions to the Model

With U.S. quarterly data from 1951-2003, Shimer (2005) calculates the standard deviation of the unemployment rate, the vacancy rate, the vacancy-unemployment ratio, the job finding rate, the separation rate, and the growth in labor productivity, and also calculates the correlations between these variables. With a canonical search and matching model, he then simulates shocks to productivity (which are assumed to represent demand shocks) and shocks to the separation rate. He finds that, in response to these shocks, the standard deviations of these variables and the correlations between these variables generally do not come close to matching their values with actual data, and these statistical measures are often off by a factor of 10-20. The failure of the canonical search and matching model to replicate U.S. labor market data has become known as the "Shimer puzzle."

The present study considers the joint determination of unemployment and wage inflation. The model can also be extended to simultaneously consider the behavior of wage inflation, unemployment, vacancies, and the hiring rate. The empirical results in Section 5 demonstrate that the model does a good job of explaining the behavior of wage inflation and unemployment. In addition, Campbell (2017) develops a pure efficiency wage model that is similar to the model in the present study, but which does not incorporate bargaining and search and matching considerations. It is demonstrated that this more parsimonious model can generate large and persistent fluctuations in the unemployment rate in response to demand shocks. For example, a 5% decrease in aggregate demand gradually raises the unemployment rate until it is 2.7% above the natural rate eight quarters after the shock, with the unemployment rate then declining until it is approximately equal to the natural rate after 20 quarters. In actual U.S. recessions over the sample period, the time for unemployment to reach is peak value ranges from 3-10 quarters, and the time for it to return to the natural rate is between 17-24 quarters, consistent with the predictions of the model.

Given the model's ability to explain unemployment and wage inflation dynamics, it is possible that using the model to simulate the response of unemployment, vacancies, the vacancy-unemployment ratio, and the hiring rate to exogenous shocks may result in predicted standard deviations and correlations in these variables that are reasonably close to their actual standard deviations and correlations, at least in comparison to the canonical search and matching model. Also, in Shimer (2005) aggregate demand shocks are proxied by the change in productivity. In the present study, aggregate demand and productivity shocks appear separately in the DLD curve, so the labor market's response to both types of shocks can be analyzed separately, as well as the response to separation rate shocks. Thus, simulating the behavior of unemployment, vacancies, the vacancy-unemployment ratio, and the hiring rate in the model developed in this study may be a promising approach to try to replicate the statistical properties of labor market data.

7. Conclusion

This study develops a model of the economy that incorporates efficiency wages, bargaining, and search and matching. An equation is derived that determines the economy's equilibrium unemployment rate (i.e., the natural rate of unemployment), and comparative statics show how changes in workers' bargaining power, the responsiveness of workers' efficiency to wages, the level of unemployment benefits, the cost of a vacancy, the efficiency of the matching process, and the separation rate affect the natural rate.

Approximating the model around its steady-state equilibrium results in two dynamic macroeconomic relationships. One is the wage-wage Phillips curve, in which wage inflation depends on expected future wage inflation, lagged wage inflation, the current unemployment rate, and the lagged unemployment rate, with the overall effect of unemployment being negative. The second is the Dynamic Labor Demand (DLD) curve. In this relationship, wage inflation depends on the current and lagged changes in the unemployment rate and on changes in technology, the money supply, expected future real aggregate demand, the real interest rate, and expected price inflation. The joint Phillips curve – DLD curve system can be used to show the behavior of wage inflation and unemployment in the transition from the economy's initial equilibrium to its new equilibrium in response to exogenous aggregate shocks.

Empirical tests of the model with U.S. macroeconomic data show strong support for the model's predictions. All the coefficients in both the Phillips curve and the DLD curve have the predicted sign, almost all are significant at the 10% level, and most are significant at the 1% level. In addition, support is found for two specific predictions about the magnitude of the coefficients in the DLD. The model predicts that the sum of coefficients on the current and lagged unemployment rates equals 0 and that the coefficient on the money supply equals the ratio over the sample period between changes in workers' hourly wages and nominal GDP per worker, and these restrictions generally cannot be rejected at even the 10% level.

Appendix

Derivation of (11):

Setting the derivative of (10a) with respect to K equal to 0 yields

$$e^{-rt}\left[\frac{(1-\phi)(\gamma-1)}{\gamma}Y^{\frac{1}{\gamma}}A^{\frac{\phi(\gamma-1)}{\gamma}}L^{\frac{\phi(\gamma-1)}{\gamma}}K^{\frac{(1-\phi)(\gamma-1)}{\gamma}-1}e[W/\overline{W},h(u)]^{\frac{\phi(\gamma-1)}{\gamma}}\overline{P}-WL-c\overline{W}V-\delta P^{K}\right] \\ -P^{K}\frac{d}{dt}\left(-e^{-rt}\right)=0 \tag{A1}$$

Solving (A1) for the optimal capital stock yields

$$K=\left(\frac{\gamma}{(1-\phi)(\gamma-1)}\right)^{\frac{\gamma}{\phi-\phi\gamma-1}}(\delta+r)^{\frac{\gamma}{\phi-\phi\gamma-1}}Y^{-\frac{1}{\phi-\phi\gamma-1}}A^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}}L^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}}e[W/\overline{W},h(u)]^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}}. \tag{A2}$$

Derivation of (14):

From (13a) and (13b),

$$\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}Y^{\frac{1}{1-\phi+\phi\gamma}}A^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}}L^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}-1}e[W/\overline{W},h(u)]^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}}\overline{P}(\delta+r)^{\frac{-(1-\phi)(\gamma-1)}{1-\phi+\phi\gamma}}D-W-\frac{r+q}{\kappa\theta^{-a}}c\overline{W}=0$$

$$L=\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}\left(W+\frac{r+q}{\kappa}c\theta^{a}\overline{W}\right)^{\phi-\phi\gamma-1}D^{1-\phi+\phi\gamma}YA^{\phi(\gamma-1)}e[W/\overline{W},h(u)]^{\phi(\gamma-1)} \\ \times\overline{P}^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}.$$

Derivation of (15):

$$\Pi = \int_0^\infty e^{-rt}\Bigg[Y_t^{\frac{1}{1-\phi+\phi\gamma}} A_t^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}} \left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{-\phi(\gamma-1)} \left(W_t + \frac{r+q}{\kappa\theta^{-a}} c\overline{W}\right)^{-\phi(\gamma-1)} D^{\phi(\gamma-1)} Y_t^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}} A_t^{\frac{\phi^2(\gamma-1)^2}{1-\phi+\phi\gamma}}$$

$$\times e[W_t/\overline{W}_t^{\,e}, u_t]^{\frac{\phi^2(\gamma-1)^2}{1-\phi+\phi\gamma})} \overline{P}_t^{\,\phi(\gamma-1)} (\delta+r)^{\frac{-(1-\phi)(\gamma-1)^2\phi}{1-\phi+\phi\gamma}} e[W_t/\overline{W}_t^{\,e}, u_t]^{\frac{\phi(\gamma-1)}{1-\phi+\phi\gamma}} \overline{P}_t (\delta+r)^{\frac{-(1-\phi)(\gamma-1)}{1-\phi+\phi\gamma}} D$$

$$-\left(W_t + c\overline{W}_t \frac{q\theta^a}{\kappa}\right)\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}$$

$$\times\left(W_t + \frac{r+q}{\kappa\theta^{-a}} c\overline{W}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma} Y_t A_t^{\phi(\gamma-1)} e[W_t/\overline{W}_t^{\,e}, u_t]^{\phi(\gamma-1)} \overline{P}_t^{\,1-\phi+\phi\gamma} (\delta+r)^{-(1-\phi)(\gamma-1)} \Bigg] dt$$

$$\Pi = \int_0^\infty e^{-rt} Y A_t^{\phi(\gamma-1)} e[W_t/\overline{W}_t^{\,e}, u_t]^{\phi(\gamma-1)} \overline{P}_t^{\,1-\phi+\phi\gamma} (\delta+r)^{-(1-\phi)(\gamma-1)} \left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma}$$

$$\times\Bigg[\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)\left(W_t + \frac{r+q}{\kappa\theta^{-a}} c\overline{W}\right)^{-\phi(\gamma-1)}$$

$$-\left(W_t + c\overline{W}_t \frac{q\theta^a}{\kappa}\right)\times\left(W_t + \frac{r+q}{\kappa\theta^{-a}} c\overline{W}\right)^{\phi-\phi\gamma-1}\Bigg] dt$$

$$\Pi = \frac{1}{r}\Bigg\{ Y A^{\phi(\gamma-1} e[W/\overline{W}, h(u)]^{\phi(\gamma-1)} \overline{P}^{\,1-\phi+\phi\gamma} (\delta+r)^{-(1-\phi)(\gamma-1)} \left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma}$$

$$\times\Bigg[\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)\left(W + \frac{r+q}{\kappa\theta^{-a}} c\overline{W}\right)^{-\phi(\gamma-1)} - \left(W + c\overline{W}\frac{q\theta^a}{\kappa}\right)\times\left(W + \frac{r+q}{\kappa\theta^{-a}} c\overline{W}\right)^{\phi-\phi\gamma-1}\Bigg]\Bigg\}$$

Derivation of (18) and (19):

From (17b),

$$(r+q)V^{EO} = \overline{W} - g(e)\overline{W} + qV^{U}$$

$$V^{EO} = \frac{(1-g(e))\overline{W}}{r+q} + \frac{q}{r+q}V^{U}.$$

Substituting this expression into (17c) yields

$$rV^{U} = (\psi + b^{*})\overline{W} + h(u)\left[\frac{(1-g(e))\overline{W}}{r+q} + \frac{q}{r+q}V^{U} - V^{U}\right]$$

$$\left[r - \frac{h(u)q}{r+q} + h(u)\right]V^{U} = (\psi + b^{*})\overline{W} + \frac{h(u)}{r+q}\overline{W} - \frac{h(u)g(e)}{r+q}\overline{W}$$

$$V^{U} = \frac{(\psi + b^{*})r + (\psi + b^{*})q + h(u)(1-g(e))}{r[r+q+h(u)]}\overline{W}.$$

From (17a),

$$V^{EC}[r+q] = W - g(e)\overline{W} + qV^{U}$$

$$V^{EC} = \frac{W}{r+q} - \frac{g(e)\overline{W}}{r+q} + \frac{q}{r+q}V^{U}$$

Derivation of (21):

$$U^{*} - \overline{U}^{*} = \frac{L}{N}\left[\frac{W}{r+q} + \frac{q}{r+q}V^{U} - V^{U}\right]$$

$$U^{*} - \overline{U}^{*} = \frac{L}{N}\left[\frac{W}{r+q} - \frac{r}{r+q}V^{U}\right]$$

$$U-\overline{U}=\frac{L}{N}\left[\frac{W(t)}{r+q}-\frac{br+bq+h(u)}{(r+q)[r+q+h(u)]}\overline{W}\right]$$

$$U-\overline{U}=\frac{1}{N}\left[\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{\phi-\phi\gamma-1}D^{1-\phi+\phi\gamma}YA^{\phi(\gamma-1)}\right.$$
$$\left.\times e[W/\overline{W},h(u)]^{\phi(\gamma-1)}\overline{P}^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}\left(\frac{W}{r+q}-\frac{br+bq+h(u)}{(r+q)[r+q+h(u)]}\overline{W}\right)\right]$$

<u>Derivation of (22):</u>

$$0=\beta\Pi\frac{d(U-\overline{U})}{dW}+(1-\beta)(U-\overline{U})\frac{d\Pi}{dW}$$

$$\frac{d\Pi}{dW}=YA^{\phi(\gamma-1)}\overline{P}^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}D^{1-\phi+\phi\gamma}\left(\frac{1+\phi(\gamma-1)(1-S)}{\phi(\gamma-1)}\right)$$
$$\times\left[\phi(\gamma-1)e[W/\overline{W},h(u)]^{\phi(\gamma-1)-1}e_W[W/\overline{W},h(u)]\frac{1}{\overline{W}}\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{-\phi(\gamma-1)}\right.$$
$$\left.-\phi(\gamma-1)\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{-\phi(\gamma-1)-1}e[W/\overline{W},h(u)]^{\phi(\gamma-1)}\right]$$

$$\frac{d(U-\overline{U})}{dW}=\frac{\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}D^{1-\phi+\phi\gamma}YA_t^{\phi(\gamma-1)}\overline{P}_t^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}}{N}$$
$$\times\left[(\phi-\phi\gamma-1)\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{\phi-\phi\gamma-2}e[W/\overline{W},h(u)]^{\phi(\gamma-1)}\left(\frac{W}{r+q}-\frac{br+bq+h(u)}{(r+q)[r+q+h(u)]}\overline{W}\right)\right.$$
$$+\phi(\gamma-1)\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{\phi-\phi\gamma-1}e[W/\overline{W},h(u)]^{\phi(\gamma-1)-1}e_W[W/\overline{W},u]\frac{1}{\overline{W}}$$
$$\times\left(\frac{W}{r+q}-\frac{br+bq+h(u)}{(r+q)[r+q+h(u)]}\overline{W}\right)$$
$$\left.+\left(W+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{\phi-\phi\gamma-1}e[W/\overline{W},h(u)]^{\phi(\gamma-1)}\frac{1}{r+q}\right]$$

$$0=\beta\Pi_t \frac{\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma} Y_t A_t^{\phi(\gamma-1)} \overline{P}_t^{1-\phi+\phi\gamma} (\delta+r)^{-(1-\phi)(\gamma-1)}}{N}$$

$$\times\Bigg[(\phi-\phi\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-2} e[W_t/\overline{W}_t, h(u_t)]^{\phi(\gamma-1)}\left(\frac{W_t}{r+q}-\frac{br+bq+h(u_t)}{(r+q)[r+q+h(u_t)]}\overline{W}_t\right)$$

$$+\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-1} e[W_t/\overline{W}_t, h(u_t)]^{\phi(\gamma-1)-1} e_W[W_t/\overline{W}_t, u_t]\frac{1}{\overline{\overline{W}}_t}$$

$$\times\left(\frac{W_t}{r+q}-\frac{br+bq+h(u_t)}{(r+q)[r+q+h(u_t)]}\overline{W}_t\right)$$

$$+\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-1} e[W_t/\overline{W}_t, h(u_t)]^{\phi(\gamma-1)}\frac{1}{r+q}\Bigg]$$

$$+(1-\beta)(U_t-\overline{U}_t)Y_t A_t^{\phi(\gamma-1)}\overline{P}_t^{1-\phi+\phi\gamma}(\delta+r)^{-(1-\phi)(\gamma-1)}\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma}\left(\frac{1+\phi(\gamma-1)(1-B)}{\phi(\gamma-1)}\right)$$

$$\times\Bigg[\phi(\gamma-1)e[W_t/\overline{W}_t, h(u_t)]^{\phi(\gamma-1)-1} e_W[W_t/\overline{W}_t, h(u_t)]\frac{1}{\overline{\overline{W}}_t}\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{-\phi(\gamma-1)}$$

$$-\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{-\phi(\gamma-1)-1} e[W_t/\overline{W}_t, h(u_t)]^{\phi(\gamma-1)}\Bigg]$$

$$0=\beta\Pi_t\frac{1}{N}\times\Bigg[(\phi-\phi\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-2} e[W_t/\overline{W}_t, h(u_t)]\left(\frac{W_t}{r+q}-\frac{br+bq+h(u_t)}{(r+q)[r+q+h(u_t)]}\overline{W}_t\right)$$

$$+\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-1} e_W[W_t/\overline{W}_t, h(u_t)]\frac{1}{\overline{\overline{W}}_t}\left(\frac{W_t}{r+q}-\frac{br+bq+h(u_t)}{(r+q)[r+q+h(u_t)]}\overline{W}_t\right)$$

$$+\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-1} e[W_t/\overline{W}_t, h(u_t)]\frac{1}{r+q}\Bigg]$$

$$+(1-\beta)(U_t-\overline{U}_t)\left(\frac{1+\phi(\gamma-1)(1-B)}{\phi(\gamma-1)}\right)$$

$$\times\Bigg[\phi(\gamma-1)e_W[W_t/\overline{W}_t, h(u_t)]\frac{1}{\overline{\overline{W}}_t}\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{-\phi(\gamma-1)}-\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{-\phi(\gamma-1)-1} e[W_t/\overline{W}_t, h(u_t)]\Bigg]$$

$$0=\beta Y_t A_t^{\phi(\gamma-1)} e[W_t/\overline{W}_t^{\,e}, h(u_t)]^{\phi(\gamma-1)} \overline{P}_t^{\,1-\phi+\phi\gamma} (\delta+r)^{-(1-\phi)(\gamma-1)} \left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma}$$
$$\times\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{-\phi(\gamma-1)}\left(\frac{1+\phi(\gamma-1)(1-B)}{\phi(\gamma-1)}\right)$$
$$\times\frac{1}{N}\left[(\phi-\phi\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-2} e[W_t/\overline{W}_t, h(u_t)]\left(\frac{W_t}{r+q}-\frac{br+bq+h(u_t)}{(r+q)[r+q+h(u_t)]}\overline{W}_t\right)\right.$$
$$+\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-1} e_W[W_t/\overline{W}_t, h(u_t)]\frac{1}{\overline{\overline{W}}_t}\left(\frac{W_t}{r+q}-\frac{br+bq+h(u_t)}{(r+q)[r+q+h(u_t)]}\overline{W}_t\right)$$
$$\left.+\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{\phi-\phi\gamma-1} e[W_t/\overline{W}_t, h(u_t)]\frac{1}{r+q}\right]$$
$$+(1-\beta)\frac{1}{N}\left[\left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma} Y_t A_t^{\phi(\gamma-1)} e[W_t/\overline{W}_t, h(u_t)]^{\phi(\gamma-1)} \overline{P}_t^{\,1-\phi+\phi\gamma}\right.$$
$$\left.\times(\delta+r)^{-(1-\phi)(\gamma-1)}\left(\frac{W_t}{r+q}-\frac{br+bq+h(u_t)}{(r+q)[r+q+h(u_t)]}\overline{W}_t\right)\right]\left(\frac{1+\phi(\gamma-1)(1-B)}{\phi(\gamma-1)}\right)$$
$$\times\left[\phi(\gamma-1)e_W[W_t/\overline{W}_t, h(u_t)]\frac{1}{\overline{\overline{W}}_t}\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{-\phi(\gamma-1)}-\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)^{-\phi(\gamma-1)-1} e[W_t/\overline{W}_t, h(u_t)]\right]$$

$$0=\beta\left[(\phi-\phi\gamma-1)e[W_t/\overline{W}_t, h(u_t)]\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W}_t\right)\right.$$
$$\left.+\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)e_W[W_t/\overline{W}_t, h(u_t)]\frac{1}{\overline{\overline{W}}_t}\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W}_t\right)+\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)e[W_t/\overline{W}_t, h(u_t)]\right]$$
$$+(1-\beta)\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W}_t\right)\left[\phi(\gamma-1)e_W[W_t/\overline{W}_t, h(u_t)]\frac{1}{\overline{\overline{W}}_t}\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)-\phi(\gamma-1)e[W_t/\overline{W}_t, h(u_t)]\right]$$

$$\beta\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)e_W[W_t/\overline{W}_t, h(u_t)]e[W_t/\overline{W}_t, u_t]^{-1}\frac{1}{\overline{\overline{W}}_t}\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W}_t\right)$$
$$+(1-\beta)\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W}_t\right)\phi(\gamma-1)e_W[W_t/\overline{W}_t, h(u_t)]e[W_t/\overline{W}_t, h(u_t)]^{-1}\frac{1}{\overline{\overline{W}}_t}\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)$$
$$=-\beta(\phi-\phi\gamma-1)\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W}_t\right)-\beta\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W}_t\right)$$
$$+(1-\beta)\phi(\gamma-1)\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W}_t\right)$$

$$\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W_t}\right)e_W[W_t/\overline{W_t},h(u_t)]e[W_t/\overline{W_t},h(u_t)]^{-1}\frac{1}{\overline{W_t}}\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W_t}\right)$$
$$=\beta\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W_t}\right)-\beta\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W_t}\right)$$
$$+\phi(\gamma-1)\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W_t}\right)$$

$$e_W[W_t/\overline{W_t},h(u_t)]e[W_t/\overline{W_t},h(u_t)]^{-1}=\frac{\beta\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W_t}\right)-\beta\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W_t}\right)+\phi(\gamma-1)\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W_t}\right)}{\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W_t}\right)\frac{1}{\overline{W_t}}\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W_t}\right)}$$

$$e_W[W_t/\overline{W_t},h(u_t)]e[W_t/\overline{W_t},h(u_t)]^{-1}=\frac{\phi(\gamma-1)\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W_t}\right)-\beta\left(\frac{br+bq+h(u_t)}{r+q+h(u_t)}+\frac{r+q}{\kappa\theta^{-a}}c\right)\overline{W_t}}{\phi(\gamma-1)\left(W_t+\frac{r+q}{\kappa\theta^{-a}}c\overline{W_t}\right)\frac{1}{\overline{W_t}}\left(W_t-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\overline{W_t}\right)}$$

$$e_W[W_t/\overline{W_t},h(u_t)]e[W_t/\overline{W_t},h(u_t)]^{-1}=\frac{\phi(\gamma-1)\frac{W_t}{\overline{W_t}}-\beta\frac{br+bq+h(u_t)}{r+q+h(u_t)}-\phi(\gamma-1)\frac{br+bq+h(u_t)}{r+q+h(u_t)}-\beta\frac{r+q}{\kappa\theta^{-a}}c}{\phi(\gamma-1)\left(\frac{W_t}{\overline{W_t}}+\frac{r+q}{\kappa\theta^{-a}}c\right)\left[\frac{W_t}{\overline{W_t}}-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\right]}$$

$$e_W[W_t/\overline{W_t},h(u_t)]e[W_t/\overline{W_t},h(u_t)]^{-1}=\frac{\frac{W_t}{\overline{W_t}}-\frac{br+bq+h(u_t)}{r+q+h(u_t)}-\frac{\beta}{\phi(\gamma-1)}\left[\frac{br+bq+h(u_t)}{r+q+h(u_t)}+\frac{r+q}{\kappa\theta^{-a}}c\right]}{\left(\frac{W_t}{\overline{W_t}}+\frac{r+q}{\kappa\theta^{-a}}c\right)\left[\frac{W_t}{\overline{W_t}}-\frac{br+bq+h(u_t)}{r+q+h(u_t)}\right]}$$

<u>Derivation of (25):</u>

$$e_W[1,h(u)]e[1,h(u)]^{-1}=\frac{1-\dfrac{br+bq+h(u)}{r+q+h(u)}-\dfrac{\beta}{\phi(\gamma-1)}\left[\dfrac{br+bq+h(u)}{r+q+h(u)}+\dfrac{r+q}{\kappa\theta^{-a}}c\right]}{\left(1+\dfrac{r+q}{\kappa\theta^{-a}}c\right)\left[1-\dfrac{br+bq+h(u)}{r+q+h(u)}\right]}$$

$$e_W[1,h(u)]e[1,h(u)]^{-1}=\frac{\dfrac{(1-b)(r+q)}{r+q+h(u)}-\dfrac{\beta}{\phi(\gamma-1)}\left[\dfrac{br+bq+h(u)}{r+q+h(u)}+\dfrac{r+q}{\kappa\theta^{-a}}c\right]}{\left(1+\dfrac{r+q}{\kappa\theta^{-a}}c\right)\dfrac{(1-b)(r+q)}{r+q+h(u)}}$$

$$e_W[1,h(u)]e[1,h(u)]^{-1}=\frac{(1-b)(r+q)-\dfrac{\beta}{\phi(\gamma-1)}\left[b(r+q)+h(u)+\dfrac{(r+q)[r+q+h(u)]}{\kappa}\theta^{a}c\right]}{\left(1+\dfrac{r+q}{\kappa\theta^{-a}}c\right)(1-b)(r+q)}$$

$$e_W[1,h(u)]e[1,h(u)]^{-1}=\frac{1-\dfrac{\beta}{\phi(\gamma-1)(1-b)}\left[b+\dfrac{h(u)}{r+q}+\dfrac{r+q+h(u)}{\kappa}\theta^{a}c\right]}{\left(1+\dfrac{r+q}{\kappa}\theta^{a}c\right)}$$

$$e_W\left[1,\frac{q(1-u)}{u}\right]e\left[1,\frac{q(1-u)}{u}\right]^{-1}=\frac{1-\dfrac{\beta}{\phi(\gamma-1)(1-b)}\left[b+\dfrac{\frac{q(1-u)}{u}}{r+q}+\left(r+q+\dfrac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\dfrac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right]}{\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\dfrac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)}$$

<u>Derivations of $du/d\beta$, du/db, du/dc, $du/d\kappa$, and du/dq:</u>

$$e_W[1, \frac{q(1-u)}{u}]e[1, \frac{q(1-u)}{u}]^{-1}\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)$$
$$=1-\frac{\beta}{\phi(\gamma-1)(1-b)}\left[b+\frac{q(1-u)}{(r+q)u}+\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right] \quad \text{(A4)}$$

Let $Z=b+\frac{q(1-u)}{(r+q)u}+\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}>0$

Totally differentiating (A4) with respect to u, β, b, q, c, and κ yields:

$$\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e[1,h]^{-1}e_{Wh}[1,h]\frac{-qdu+(1-u)udq}{u^2}$$
$$-\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e_W[1,h]e[1,h]^{-2}e_h[1,h]\frac{-qdu+(1-u)udq}{u^2}$$
$$+e_W[1,h]e[1,h]^{-1}\left(c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}dq+(r+q)\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}dc\right.$$
$$\left.-\frac{a}{1-a}(r+q)c\kappa^{\frac{a-2}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}d\kappa+\frac{a}{1-a}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\frac{-qdu+(1-u)udq}{u^2}\right)$$
$$=\frac{-Zd\beta}{\phi(\gamma-1)(1-b)}-\frac{Z\beta db}{\phi(\gamma-1)(1-b)^2}-\frac{\beta db}{\phi(\gamma-1)(1-b)}-\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{ru(1-u)dq-q(r+q)du}{(r+q)^2u^2}$$
$$-\frac{\beta}{\phi(\gamma-1)(1-b)}\left(r+q+\frac{q(1-u)}{u}\right)\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}dc$$
$$+\frac{1}{1-a}\frac{\beta}{\phi(\gamma-1)(1-b)}\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{\frac{a-2}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}d\kappa-\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}dq$$
$$-\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\frac{-qdu+(1-u)udq}{u^2}$$
$$\left.-\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{a}{1-a}\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\frac{-qdu+(1-u)udq}{u^2}\right)$$

$$\frac{q}{u^2}\Bigg[\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e[1,\,h]^{-1}e_{Wh}[1,\,h]$$

$$-\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e_W[1,\,h]e[1,\,h]^{-2}e_h[1,\,h]+e_W[1,\,h]e[1,\,h]^{-1}(r+q)c\kappa^{-\frac{1}{1-a}}\frac{a}{1-a}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}$$

$$+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{1}{r+q}-\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}$$

$$+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{a}{1-a}\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\Bigg]du$$

$$=\frac{Z}{\phi(\gamma-1)(1-b)}d\beta+\frac{Z\beta+\beta(1-b)}{\phi(\gamma-1)(1-b)^2}db$$

$$+\left[e_W[1,\,h]e[1,\,h]^{-1}(r+q)+\frac{\beta}{\phi(\gamma-1)(1-b)}\left(r+q+\frac{q(1-u)}{u}\right)\right]\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}dc$$

$$-\left[e_W[1,\,h]e[1,\,h]^{-1}\frac{a}{1-a}(r+q)+\frac{1}{1-a}\frac{\beta}{\phi(\gamma-1)(1-b)}\left(r+q+\frac{q(1-u)}{u}\right)\right]c\kappa^{\frac{a-2}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}d\kappa$$

$$+\Bigg[e_W[1,\,h]e[1,\,h]^{-1}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}+e_W[1,\,h]e[1,\,h]^{-1}\frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\frac{(1-u)}{u}$$

$$+\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e[1,\,h]^{-1}e_{Wh}[1,\,h]\frac{(1-u)}{u}$$

$$-\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e_W[1,\,h]e[1,\,h]^{-2}e_h[1,\,h]\frac{(1-u)}{u}+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{r(1-u)}{(r+q)^2u}$$

$$+\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}+\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\frac{(1-u)}{u}$$

$$+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{a}{1-a}\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\frac{(1-u)}{u}\Bigg)\Bigg]dq$$

Let

$$Q=\frac{q}{u^{2}}\left[\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e[1,h]^{-1}e_{Wh}[1,h]\right.$$

$$-\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e_{W}[1,h]e[1,h]^{-2}e_{h}[1,h]+e_{W}[1,h]e[1,h]^{-1}(r+q)c\kappa^{-\frac{1}{1-a}}\frac{a}{1-a}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}$$

$$+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{1}{r+q}-\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}$$

$$\left.+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{a}{1-a}\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\right]>0$$

$$R=\left[e_{W}[1,h]e[1,h]^{-1}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}+e_{W}[1,h]e[1,h]^{-1}\frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\frac{(1-u)}{u}\right.$$

$$+\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e[1,h]^{-1}e_{Wh}[1,h]\frac{(1-u)}{u}$$

$$-\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)e_{W}[1,h]e[1,h]^{-2}e_{h}[1,h]\frac{(1-u)}{u}+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{r(1-u)}{(r+q)^{2}u}$$

$$+\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}+\frac{\beta}{\phi(\gamma-1)(1-b)}c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\frac{(1-u)}{u}$$

$$\left.+\frac{\beta}{\phi(\gamma-1)(1-b)}\frac{a}{1-a}\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{2a-1}{1-a}}\frac{(1-u)}{u}\right)\right]>0$$

Then,

$$\frac{du}{d\beta}=\frac{\overline{\frac{Z}{\phi(\gamma-1)(1-b)}}}{Q}>0,$$

$$\frac{du}{db} = \frac{\dfrac{Z\beta + \beta(1-b)}{\phi(\gamma-1)(1-b)^2}}{Q} > 0,$$

$$\frac{du}{dc} = \frac{\left[e_W[1,h]e[1,h]^{-1}(r+q) + \dfrac{\beta}{\phi(\gamma-1)(1-b)}\left(r+q+\dfrac{q(1-u)}{u}\right)\right]\kappa^{-\frac{1}{1-a}}\left(\dfrac{q(1-u)}{u}\right)^{\frac{a}{1-a}}}{Q} > 0,$$

$$\frac{du}{d\kappa} = \frac{-\left[e_W[1,h]e[1,h]^{-1}\dfrac{a}{1-a}(r+q) + \dfrac{1}{1-a}\dfrac{\beta\left(r+q+\dfrac{q(1-u)}{u}\right)}{\phi(\gamma-1)(1-b)}\right]c\kappa^{\frac{a-2}{1-a}}\left(\dfrac{q(1-u)}{u}\right)^{\frac{a}{1-a}}}{Q} < 0,$$

$$\frac{du}{dq}=\frac{R}{Q}>0\,.$$

<u>Derivation of *du*/*dz*:</u>

With the shift variable, *z*, for changes in e_W, (A4) can be expressed as

$$e_W[1,\frac{q(1-u)}{u},\,z]e[1,\frac{q(1-u)}{u},\,z]^{-1}\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right)$$

$$=1-\frac{\beta}{\phi(\gamma-1)(1-b)}\left[b+\frac{q(1-u)}{(r+q)u}+\left(r+q+\frac{q(1-u)}{u}\right)c\kappa^{-\frac{1}{1-a}}\left(\frac{q(1-u)}{u}\right)^{\frac{a}{1-a}}\right]$$

with $e_{Wz}>0$, $e_z=0$.

Totally differentiating the above equation with respect to *u* and *z* yields

$$e_{Wz}[1,\,h,\,z]e[1,\,h,\,z]^{-1}dz=Qdu$$

$$\frac{du}{dz}=\frac{e_{Wz}[1,\,h,\,z]e[1,\,h,\,z]^{-1}}{Q}>0$$

<u>Derivation of (30)</u>

$$e_W[W_t/\overline{W}_t,h_t]e[W_t/\overline{W}_t,h_t]^{-1}=\frac{\frac{W_t}{\overline{W}_t}-\frac{br+bq+h_t}{r+q+h_t}-\frac{\beta}{\phi(\gamma-1)}\left[\frac{br+bq+h(t)}{r+q+h(t)}+\frac{r+q}{\kappa\theta^{-a}}c\right]}{\left(\frac{W_t}{\overline{W}_t}+\frac{r+q}{\kappa\theta^{-a}}c\right)\left[\frac{W_t}{\overline{W}_t}-\frac{br+bq+h(t)}{r+q+h(t)}\right]}$$

$$e_W\left[\frac{W_t}{\overline{\overline{W}}_t}, \frac{q(1-u_t)+u_{t-1}-u_t}{u_t}\right] e\left[\frac{W_t}{\overline{\overline{W}}_t}, u_t\right]^{-1}$$

$$= \frac{\frac{W_t}{\overline{\overline{W}}_t} - \frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}} - \frac{\beta}{\phi(\gamma-1)}\left[\frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}} + (r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}} u_t^{-\frac{a}{1-a}} c\right]}{\left(\frac{W_t}{\overline{\overline{W}}_t} + (r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}} u_t^{-\frac{a}{1-a}} c\right)\left[\frac{W_t}{\overline{\overline{W}}_t} - \frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}}\right]}$$

$$e_W\left[\frac{W_t}{\overline{\overline{W}}_t}, \frac{q(1-u_t)+u_{t-1}-u_t}{u_t}\right] e\left[\frac{W_t}{\overline{\overline{W}}_t}, u_t\right]^{-1}$$

$$= \left(\frac{W_t}{\overline{\overline{W}}_t} + (r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}} u_t^{-\frac{a}{1-a}} c\right)^{-1}$$
$$- \frac{\beta}{\phi(\gamma-1)}\left[\frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}} + (r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}} u_t^{-\frac{a}{1-a}} c\right]$$
$$\times \left(\frac{W_t}{\overline{\overline{W}}_t} + (r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}} u_t^{-\frac{a}{1-a}} c\right)^{-1}\left[\frac{W_t}{\overline{\overline{W}}_t} - \frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}}\right]^{-1}$$

$$e_W\left[\frac{W_t}{\overline{\overline{W}}_t}, \frac{q(1-u_t)+u_{t-1}-u_t}{u_t}\right] e\left[\frac{W_t}{\overline{\overline{W}}_t}, u_t\right]^{-1}\left(\frac{W_t}{\overline{\overline{W}}_t} + (r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}} u_t^{-\frac{a}{1-a}} c\right)$$

$$= 1 - \frac{\beta}{\phi(\gamma-1)}\left[\frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}} + (r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}} u_t^{-\frac{a}{1-a}} c\right]$$
$$\times \left[\frac{W_t}{\overline{\overline{W}}_t} - \frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}}\right]^{-1}$$

$$e_W\left[\frac{W_t}{\overline{\overline{W}}_t}, \frac{q(1-u_t)+u_{t-1}-u_t}{u_t}\right]e\left[\frac{W_t}{\overline{\overline{W}}_t}, u_t\right]^{-1}\left(\frac{W_t}{\overline{\overline{W}}_t}+(r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}}u_t^{-\frac{a}{1-a}}c\right)$$

$$+\frac{\beta}{\phi(\gamma-1)}\left[\frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}}+(r+q)\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}}u_t^{-\frac{a}{1-a}}c\right]\left[\frac{W_t}{\overline{\overline{W}}_t}-\frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}}\right]^{-1}$$

$$=1$$

$$
\begin{aligned}
&\Big[e_{WW}[W_t/\overline{W}_t,h_t]e[W_t/\overline{W}_t,h_t]^{-1}(1/\overline{W}_t)dW_t - e_{WW}[W_t/\overline{W}_t,h_t]e[W_t/\overline{W}_t,h_t]^{-1}(W_t/\overline{W}_t^{\,2})d\overline{W}_t \\
&\quad + e_{Wh}[W_t/\overline{W}_t,h_t]e[W_t/\overline{W}_t,h_t]^{-1}\frac{udu_{t-1}-(u+q)du_t}{u^2} - e_W[W_t/\overline{W}_t,h_t]^2 e[W_t/\overline{W}_t,h_t]^{-2}(1/\overline{W}_t)dW_t \\
&\quad + e_W[W_t/\overline{W}_t,h_t]^2 e[W_t/\overline{W}_t,h_t]^{-2}(W_t/\overline{W}_t^{\,2})d\overline{W}_t - e_W[W_t/\overline{W}_t,h_t]e[W_t/\overline{W}_t,h_t]^{-2}e_h[W_t/\overline{W}_t,h_t]\frac{udu_{t-1}-(u+q)du_t}{u^2}\Big] \\
&\qquad \times\left(\frac{W_t}{\overline{W}_t}+(r+q)c\kappa^{-\frac{1}{1-a}}[q-(1+q)u_t+u_{t-1}]^{\frac{a}{1-a}}u_t^{-\frac{a}{1-a}}\right) \\
&\quad +\Bigg[\frac{1}{\overline{W}_t}dW_t-\frac{W_t}{\overline{W}_t^{\,2}}d\overline{W}_t-\frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}}[q-(1+q)u_t+u_{t-1}]^{\frac{2a-1}{1-a}}(1+q)u_t^{-\frac{a}{1-a}}du_t \\
&\qquad -\frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}}[q-(1+q)u_t+u_{t-1}]^{\frac{a}{1-a}}u_t^{-\frac{1}{1-a}}du_t+\frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}}[q-(1+q)u_t+u_{t-1}]^{\frac{2a-1}{1-a}}du_{t-1}\Bigg] \\
&\qquad\qquad \times e_W[W_t/\overline{W}_t,h_t]e[W_t/\overline{W}_t,h_t]^{-1} \\
&\quad -\frac{\beta}{\phi(\gamma-1)}\left[\frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}}+(r+q)c\kappa^{-\frac{1}{1-a}}[q(1-u_t)+u_{t-1}-u_t]^{\frac{a}{1-a}}u_t^{-\frac{a}{1-a}}\right] \\
&\qquad \times\left[\frac{W_t}{\overline{W}_t}-\frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}}\right]^{-2} \\
&\qquad \times\left[\frac{1}{\overline{W}_t}dW_t-\frac{W_t}{\overline{W}_t^{\,2}}d\overline{W}_t+\frac{(1-b)(q+r)(q+u_{t-1})}{[q-u_t(1-r)+u_{t-1}]^2}du_t-\frac{(1-b)(q+r)u_t}{[q-u_t(1-r)+u_{t-1}]^2}du_{t-1}\right] \\
&\quad +\frac{\beta}{\phi(\gamma-1)}\Bigg[-\frac{(1-b)(q+r)(q+u_{t-1})}{[q-u_t(1-r)+u_{t-1}]^2}du_t+\frac{(1-b)(q+r)u_t}{[q-u_t(1-r)+u_{t-1}]^2}du_{t-1} \\
&\qquad -\frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}}[q-(1+q)u_t+u_{t-1}]^{\frac{2a-1}{1-a}}u_t^{-\frac{a}{1-a}}(1+q)du_t-\frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}}[q-(1+q)u_t+u_{t-1}]^{\frac{a}{1-a}}u_t^{-\frac{1}{1-a}}du_t \\
&\qquad +\frac{a}{1-a}(r+q)c\kappa^{-\frac{1}{1-a}}[q-(1+q)u_t+u_{t-1}]^{\frac{2a-1}{1-a}}u_t^{-\frac{a}{1-a}}du_{t-1}\Bigg] \\
&\quad \times\left[\frac{W_t}{\overline{W}_t}-\frac{[br+bq-q-1]u_t+q+u_{t-1}}{q-u_t(1-r)+u_{t-1}}\right]^{-1} \\
&=0
\end{aligned}
$$

Let

$$s_1 = e_W[1, h]e[1, h]^{-1}\left(1+(r+q)c\kappa^{-\frac{1}{1-a}}[q(1-u)]^{\frac{a}{1-a}}u^{-\frac{a}{1-a}}\right)$$

$$1-s_1 = \frac{\beta}{\phi(\gamma-1)}\left[\frac{[br+bq-q]u+q}{q+ru} + (r+q)\kappa^{-\frac{1}{1-a}}[q(1-u)]^{\frac{a}{1-a}}u_t^{-\frac{a}{1-a}}c\right]\left[\frac{(1-b)(r+q)}{q+ru}\right]^{-1}$$

$$s_2 = \frac{\frac{[br+bq-q]u+q}{q+ru}}{\frac{[br+bq-q]u+q}{q+ru} + (r+q)c\kappa^{-\frac{1}{1-a}}[q(1-u)]^{\frac{a}{1-a}}u^{-\frac{a}{1-a}}}$$

$$1-s_2 = \frac{(r+q)c\kappa^{-\frac{1}{1-a}}[q(1-u)]^{\frac{a}{1-a}}u^{-\frac{a}{1-a}}}{\frac{[br+bq-q]u+q}{q+ru} + (r+q)c\kappa^{-\frac{1}{1-a}}[q(1-u)]^{\frac{a}{1-a}}u^{-\frac{a}{1-a}}}$$

$$s_3 = \frac{1}{1+(r+q)c\kappa^{-\frac{1}{1-a}}[q(1-u)]^{\frac{a}{1-a}}u^{-\frac{a}{1-a}}}$$

$$1-s_3 = \frac{r+q)c\kappa^{-\frac{1}{1-a}}[q(1-u)]^{\frac{a}{1-a}}u^{-\frac{a}{1-a}}}{1+(r+q)c\kappa^{-\frac{1}{1-a}}[q(1-u)]^{\frac{a}{1-a}}u^{-\frac{a}{1-a}}}$$

$$s_4 = \frac{1}{1-\frac{[br+bq-q]u+q}{q+ru}} > 1$$

$$1-s_4 = \frac{-\frac{[br+bq-q]u+q}{q+ru}}{1-\frac{[br+bq-q]u+q}{q+ru}} < 0$$

Note that $s_1 = \dfrac{e_W[1,h]e[1,h]^{-1}}{s_3}$

$$s_1 e_{WW} e_W^{-1}\hat{W}_t - s_1 e_{WW} e_W^{-1}\hat{\overline{W}}_t - s_1 e_{Wh} e_W^{-1}\frac{u+q}{u^2}du_t + \frac{s_1 e_{Wh} e_W^{-1}}{u}du_{t-1} - s_1 e_W e^{-1}\hat{W}_t + s_1 e_W e^{-1}\hat{\overline{W}}_t + s_1 e^{-1} e_h \frac{u+q}{u^2}du_t - \frac{s_1 e^{-1} e_h}{u}du_{t-1}$$
$$+ s_1 s_3 \hat{W}_t - s_1 s_3 \hat{\overline{W}}_t - s_1(1-s_3)\frac{a}{1-a}[q(1-u)]^{-1}(1+q)du_t - s_1(1-s_3)\frac{a}{1-a}u_t^{-1}du_t + s_1(1-s_3)\frac{a}{1-a}[q(1-u)]^{-1}du_{t-1}$$
$$-(1-s_1)s_4\hat{W}_t + (1-s_1)s_4\hat{\overline{W}}_t + (1-s_1)(1-s_4)\frac{(1-b)(q+r)(q+u)}{[q+ru][bru+bqu-qu+q]}du_t$$
$$-(1-s_1)(1-s_4)\frac{(1-b)(q+r)u}{[q+ru][bru+bqu-qu+q]}du_{t-1}$$
$$-(1-s_1)s_2\frac{(1-b)(q+r)(q+u)}{[q+ru][bru+bqu-qu+q]}du_t + (1-s_1)s_2\frac{(1-b)(q+r)u}{[q+ru][bru+bqu-qu+q]}du_{t-1}$$
$$-(1-s_1)(1-s_2)\frac{a}{1-a}[q(1-u)]^{-1}u^{-\frac{a}{1-a}}(1+q)du_t - (1-s_1)(1-s_2)\frac{a}{1-a}u^{-1}du_t$$
$$+(1-s_1)(1-s_2)\frac{a}{1-a}[q(1-u)]^{-1}u^{-\frac{a}{1-a}}du_{t-1}$$
$$=0$$

$$
\begin{aligned}
&\left[s_1 e_{WW} e_W^{-1} - s_1 e_W e^{-1} + s_1 s_3 - (1-s_1) s_4\right]\hat{W}_t \\
&= \left[s_1 e_{WW} e_W^{-1} - s_1 e_W e^{-1} + s_1 s_3 - (1-s_1) s_4\right]\hat{\overline{W}}_t + \Bigg[s_1 e_{Wh} e_W^{-1} \frac{u+q}{u^2} - s_1 e^{-1} e_h \frac{u+q}{u^2} + s_1(1-s_3)\frac{a}{1-a}[q(1-u)]^{-1}(1+q) \\
&\quad + s_1(1-s_3)\frac{a}{1-a} u_t^{-1} - (1-s_1)(1-s_4)\frac{(1-b)(q+r)(q+u)}{[q+ru][bru+bqu-qu+q]} + (1-s_1)s_2 \frac{(1-b)(q+r)(q+u)}{[q+ru][bru+bqu-qu+q]} \\
&\quad + (1-s_1)(1-s_2)\frac{a}{1-a}[q(1-u)]^{-1} u^{-\frac{a}{1-a}}(1+q) + (1-s_1)(1-s_2)\frac{a}{1-a} u^{-1} \Bigg] du_t \\
&\quad + \Bigg[-\frac{s_1 e_{Wh} e_W^{-1}}{u} + \frac{s_1 e^{-1} e_h}{u} - s_1(1-s_3)\frac{a}{1-a}[q(1-u)]^{-1} + (1-s_1)(1-s_4)\frac{(1-b)(q+r)u}{[q+ru][bru+bqu-qu+q]} \\
&\quad - (1-s_1)s_2 \frac{(1-b)(q+r)u}{[q+ru][bru+bqu-qu+q]} - (1-s_1)(1-s_2)\frac{a}{1-a}[q(1-u)]^{-1} u^{-\frac{a}{1-a}} \Bigg] du_{t-1}
\end{aligned}
$$

$$
\begin{aligned}
&\left[s_1 e_{WW} e_W^{-1} + (1-s_1) e_W e^{-1} - (1-s_1) s_4\right]\hat{W}_t \\
&= \left[s_1 e_{WW} e_W^{-1} + (1-s_1) e_W e^{-1} - (1-s_1) s_4\right]\hat{\overline{W}}_t + \Bigg[s_1 e_{Wh} e_W^{-1} \frac{u+q}{u^2} - s_1 e^{-1} e_h \frac{u+q}{u^2} + s_1(1-s_3)\frac{a}{1-a}[q(1-u)]^{-1}(1+q) \\
&\quad + s_1(1-s_3)\frac{a}{1-a} u_t^{-1} - (1-s_1)(1-s_4)\frac{(1-b)(q+r)(q+u)}{[q+ru][bru+bqu-qu+q]} + (1-s_1)s_2 \frac{(1-b)(q+r)(q+u)}{[q+ru][bru+bqu-qu+q]} \\
&\quad + (1-s_1)(1-s_2)\frac{a}{1-a}[q(1-u)]^{-1} u^{-\frac{a}{1-a}}(1+q) + (1-s_1)(1-s_2)\frac{a}{1-a} u^{-1} \Bigg] du_t \\
&\quad + \Bigg[-\frac{s_1 e_{Wh} e_W^{-1}}{u} + \frac{s_1 e^{-1} e_h}{u} - s_1(1-s_3)\frac{a}{1-a}[q(1-u)]^{-1} + (1-s_1)(1-s_4)\frac{(1-b)(q+r)u}{[q+ru][bru+bqu-qu+q]} \\
&\quad - (1-s_1)s_2 \frac{(1-b)(q+r)u}{[q+ru][bru+bqu-qu+q]} - (1-s_1)(1-s_2)\frac{a}{1-a}[q(1-u)]^{-1} u^{-\frac{a}{1-a}} \Bigg] du_{t-1}
\end{aligned}
\tag{A5}
$$

Let

$$
E = \frac{\begin{aligned}&-\frac{s_1 e_{Wh} e_W^{-1}(u+q)}{u^2} + \frac{s_1 e^{-1} e_h (u+q)}{u^2} + (1-s_1)(1-s_4)\frac{(1-b)(q+r)(q+u)}{[q+ru][bru+bqu-qu+q]} \\ &-(1-s_1)s_2 \frac{(1-b)(q+r)(q+u)}{[q+ru][bru+bqu-qu+q]} - s_1(1-s_3)\frac{a}{1-a}[q(1-u)]^{-1}(1+q) \\ &-(1-s_1)(1-s_2)\frac{a}{1-a}[q(1-u)]^{-1} u^{-\frac{a}{1-a}}(1+q) - s_1(1-s_3)\frac{a}{1-a}u_t^{-1} - (1-s_1)(1-s_2)\frac{a}{1-a}u^{-1}\end{aligned}}{s_1 e_{WW} e_W^{-1} + (1-s_1)[e_W e^{-1} - s_4]} > 0
$$

and

$$F = \frac{\begin{gathered}-\frac{s_1 e_{Wh} e_W^{-1}}{u} + \frac{s_1 e^{-1} e_h}{u} + (1-s_1)(1-s_4)\frac{(1-b)(q+r)u}{[q+ru][bru+bqu-qu+q]} \\ -(1-s_1)s_2\frac{(1-b)(q+r)u}{[q+ru][bru+bqu-qu+q]} - s_1(1-s_3)\frac{a}{1-a}[q(1-u)]^{-1} - (1-s_1)(1-s_2)\frac{a}{1-a}[q(1-u)]^{-1}u^{-\frac{a}{1-a}}\end{gathered}}{s_1 e_{WW} e_W^{-1} + (1-s_1)[e_W e^{-1} - s_4]} > 0$$

The first four terms of E are greater than the first four terms of F by the factor $(q+u)/u$, and the fifth and sixth terms in E are greater than the fifth term in F by the factor $(1+q)$, thus implying that $E>F$. Given these expressions for E and F, (A5) be expressed as

$$\hat{W}_t = \hat{\overline{W}}_t - Edu_t + Fdu_{t-1}. \tag{A6}$$

Assume each period, a fraction, τ, of firms can adjust their wages. Then, following the derivation in Romer (2012) and Campbell (2017), wage inflation can be expressed as

$$\pi_t^w = \beta\pi_{t+1}^{w,e} - \frac{\tau[1-\beta(1-\tau)]}{1-\tau}Edu_t + \frac{\tau[1-\beta(1-\tau)]}{1-\tau}Fdu_{t-1}. \tag{A7}$$

From (9),

$$\pi_{t+1}^{w,e} = \omega\pi_{t+1}^{w,ue} + (1-\omega)[\lambda_1\pi_t^w + \lambda_2\pi_{t-1}^w + \lambda_3\pi_{t-2}^w + \cdots + \lambda_{T+1}\pi_{t-T}^w]. \tag{A8}$$

Substituting (A8) into (A7) yields

$$\pi_t^w = \beta\Big[\omega\pi_{t+1}^{w,ue} + (1-\omega)[\lambda_1\pi_t^w + \lambda_2\pi_{t-1}^w + \lambda_3\pi_{t-2}^w + \cdots + \lambda_{T+1}\pi_{t-T}^w]\Big] - \frac{\tau[1-\beta(1-\tau)]}{1-\tau}Edu_t + \frac{\tau[1-\beta(1-\tau)]}{1-\tau}Fdu_{t-1}$$

$$[1-(1-\omega)\lambda_1]\pi_t^w = \beta\Big[\omega\pi_{t+1}^{w,ue} + (1-\omega)[\lambda_2\pi_{t-1}^w + \lambda_3\pi_{t-2}^w + \cdots + \lambda_{T+1}\pi_{t-T}^w]\Big] - \frac{\tau[1-\beta(1-\tau)]}{1-\tau}Edu_t + \frac{\tau[1-\beta(1-\tau)]}{1-\tau}Fdu_{t-1}$$

$$\pi_t^w = \frac{\beta\omega\pi_{t+1}^{w,ue}}{1-(1-\omega)\lambda_1} + \frac{\beta(1-\omega)\sum_{j=1}^{T}\lambda_{j+1}\pi_{t-j}^w}{1-(1-\omega)\lambda_1} - \frac{\tau[1-\beta(1-\tau)]}{(1-\tau)[1-(1-\omega)\lambda_1]}Edu_t + \frac{\tau[1-\beta(1-\tau)]}{(1-\tau)[1-(1-\omega)\lambda_1]}Fdu_{t-1} \tag{A9}$$

Equation (A9) can be expressed as

$$\pi_t^w = \psi_1\pi_{t+1}^{w,ue} + \psi_2\sum_{j=1}^{T}\lambda_{j+1}\pi_{t-j}^w - \psi_3 du_t + \psi_4 du_{t-1}$$

where $\psi_1 = \frac{\beta\omega}{1-(1-\omega)\lambda_1} > 0$,

$$\psi_2 = \frac{\beta(1-\omega)}{1-(1-\omega)\lambda_1} > 0,$$

$$\psi_3 = \frac{\tau[1-\beta(1-\tau)]}{(1-\tau)[1-(1-\omega)\lambda_1]}E > 0,$$

$$\psi_4 = \frac{\tau[1-\beta(1-\tau)]}{(1-\tau)[1-(1-\omega)\lambda_1]}F > 0.$$

<u>Derivation of (31):</u>

From (14), (4), and (11), labor demand, output, and capital are:

$$L_t = \left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}\left(W_t + \frac{r+q}{\kappa}c\theta^a\overline{W}_t\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma} Y_t A_t^{\phi(\gamma-1)} e[W_t/\overline{W}_t, h(u_t)]^{\phi(\gamma-1)}$$
$$\times \overline{P}_t^{1-\phi+\phi\gamma}(\delta+r_t)^{-(1-\phi)(\gamma-1)}.$$

$$Y_t = A_t^{\phi} L_t^{\phi} K_t^{1-\phi} e\left[W_t/\overline{W}_t, h(u_t)\right]^{\phi}$$

$$K_t = \left(\frac{\gamma}{(1-\phi)(\gamma-1)}\right)^{\frac{\gamma}{\phi-\phi\gamma-1}} (\delta+r_t)^{\frac{\gamma}{\phi-\phi\gamma-1}} Y_t^{-\frac{1}{\phi-\phi\gamma-1}} A_t^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}} L^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}} e[W_t/\overline{W}_t, h(u_t)]^{-\frac{\phi(\gamma-1)}{\phi-\phi\gamma-1}}$$

Substituting (11) into (14) yields

$$L_t = \left(\frac{1-\phi+\phi\gamma}{\phi(\gamma-1)}\right)^{\phi-\phi\gamma-1}\left(W_t + \frac{r+q}{\kappa}c\kappa^{-\frac{a}{1-a}}\left[\frac{q(1-u_t)+u_{t-1}-u_t}{u_t}\right]^{\frac{a}{1-a}}\overline{W}_t\right)^{\phi-\phi\gamma-1} D^{1-\phi+\phi\gamma} Y_t \quad \text{(A10)}$$
$$\times A_t^{\phi(\gamma-1)} e[W_t/\overline{W}_t, h(u_t)]^{\phi(\gamma-1)} \overline{P}_t^{1-\phi+\phi\gamma}(\delta+r_t)^{-(1-\phi)(\gamma-1)}.$$

Substituting (11) into (A10) yields

$$Y_t = A_t^{\phi} L_t^{\phi}\left(\frac{\gamma}{(1-\phi)(\gamma-1)}\right)^{\frac{\gamma(1-\phi)}{\phi-\phi\gamma-1}} (\delta+r_t)^{\frac{\gamma(1-\phi)}{\phi-\phi\gamma-1}} Y_t^{-\frac{(1-\phi)}{\phi-\phi\gamma-1}} A_t^{-\frac{\phi(\gamma-1)(1-\phi)}{\phi-\phi\gamma-1}} L_t^{-\frac{\phi(\gamma-1)(1-\phi)}{\phi-\phi\gamma-1}}$$
$$\times e[W_t/\overline{W}_t, h_t]^{-\frac{\phi(\gamma-1)(1-\phi)}{\phi-\phi\gamma-1}} e\left[W_t/\overline{W}_t, h_t\right]^{\phi}$$

$$Y_t^{\frac{\phi\gamma}{1-\phi+\phi\gamma}} = \left(\frac{\gamma}{(1-\phi)(\gamma-1)}\right)^{\frac{\gamma(1-\phi)}{\phi-\phi\gamma-1}} A_t^{\frac{\phi\gamma}{1-\phi+\phi\gamma}} L_t^{\frac{\phi\gamma}{1-\phi+\phi\gamma}} (\delta+r_t)^{\frac{\gamma(1-\phi)}{\phi-\phi\gamma-1}} e[W_t/\overline{W}_t, h_t]^{\frac{\phi\gamma}{1-\phi+\phi\gamma}}$$

$$Y_t = \left(\frac{\gamma}{(1-\phi)(\gamma-1)}\right)^{-\frac{1-\phi}{\phi}} A_t L_t (\delta+r_t)^{-\frac{1-\phi}{\phi}} e\left[\frac{W_t}{\overline{W}_t}, \frac{q(1-u_t)+u_{t-1}-u_t}{u_t}\right] \quad \text{(A11)}$$

Totally differentiating (A11) and dividing by the original equation yields the following expression, in terms of deviations from steady-state values:

$$\hat{Y}_t = \hat{A}_t + \hat{L}_t - \frac{1-\phi}{\phi} s_{r\delta} dr_t + e_W e^{-1} \hat{W}_t - e_W e^{-1} \hat{\overline{W}}_t - \frac{e_h e^{-1}(q+u)}{u^2} du_t + \frac{e_h e^{-1}}{u} du_{t-1} \qquad \text{(A12)}$$

where $s_{r\delta} = \dfrac{1}{r+\delta}$.

Totally differentiating (8a) and dividing by the original equation yields

$$\hat{Y}_t = \hat{Y}_{t+1} - \frac{dr_t}{\mu(1+r_t)} .$$

Letting π_t^e represent expected price inflation, the IS curve can be expressed as

$$\hat{Y}_t = \hat{Y}_{t+1} - s_r(di_t - d\pi_t^e), \qquad \text{where } s_r = 1/[\mu(1+r)]. \qquad \text{(A13)}$$

Similarly, totally differentiating (8b) and dividing by the original expressions gives

$$\hat{M}_t = \hat{\overline{P}}_t + \hat{Y}_t - \frac{1}{\mu} \frac{\dfrac{(1+i_t)di_t - i_t di_t}{(1+i_t)^2}}{\dfrac{i_t}{1+i_t}} = \hat{\overline{P}}_t + \hat{Y}_t - \frac{1}{\mu i(1+i)} di_t ,$$

which can be approximated by

$$\hat{M}_t = \hat{\overline{P}}_t + \hat{Y}_t - s_i di_t , \qquad \text{where } s_i = 1/[\mu i(1+i)] . \qquad \text{(A14)}$$

An equation for the price level can be obtained by substituting di_t in (A13) into (A14), yielding

$$\hat{\overline{P}}_t = \hat{M}_t - \hat{Y}_t + \frac{s_i}{s_r}(\hat{Y}_{t+1} - \hat{Y}_t + s_r d\pi_t^e) ,$$

which can be expressed as

$$\hat{\bar{P}}_t = \hat{M}_t - (1+\xi)\hat{Y}_t + \xi\hat{Y}_{t+1} + s_i d\pi_t^e, \qquad \text{where } \xi = \frac{s_i}{s_r} > 0.$$

$$\hat{\bar{P}}_t = \hat{M}_t - (1+\xi)\left[\hat{A}_t + \hat{L}_t - \frac{1-\phi}{\phi} s_{r\delta} dr_t + e_W e^{-1}\hat{W}_t - e_W e^{-1}\hat{\bar{W}}_t - \frac{e_h e^{-1}(q+u)}{u^2} du_t + \frac{e_h e^{-1}}{u} du_{t-1}\right]$$
$$+ \xi\hat{Y}_{t+1} + s_i d\pi_t^e$$

$$\begin{aligned}\hat{\bar{P}}_t = {} & \hat{M}_t - (1+\xi)\hat{A}_t - (1+\xi)\hat{L}_t + (1+\xi)\frac{1-\phi}{\phi} s_{r\delta} dr_t - (1+\xi)e_W e^{-1}\hat{W}_t + (1+\xi)e_W e^{-1}\hat{\bar{W}}_t \\ & + (1+\xi)\frac{e_h e^{-1}(q+u)}{u^2} du_t - (1+\xi)\frac{e_h e^{-1}}{u} du_{t-1} + \xi\hat{Y}_{t+1} + s_i d\pi_t^e \end{aligned} \tag{A15}$$

Totally differentiating (A10) and dividing by the original equation yields

$$\begin{aligned}\hat{L}_t = {} & -(1-\phi+\phi\gamma)s_W\hat{W}_t - (1-\phi+\phi\gamma)(1-s_W)\hat{\bar{W}}_t + \frac{(1-\phi+\phi\gamma)(1-s_W)[a/(1-a)](q+u)}{u^2} du_t \\ & - \frac{(1-\phi+\phi\gamma)(1-s_W)[a/(1-a)]}{u} du_{t-1} \\ & + \hat{Y}_t + \phi(\gamma-1)\hat{A}_t + \phi(\gamma-1)e^{-1}e_W\hat{W}_t - \phi(\gamma-1)e^{-1}e_W\hat{\bar{W}}_t - \frac{\phi(\gamma-1)e^{-1}e_h(q+u)}{u^2} du_t \\ & + \frac{\phi(\gamma-1)e^{-1}e_h}{u} du_{t-1} + (1-\phi+\phi\gamma)\hat{\bar{P}}_t - (1-\phi)(\gamma-1)s_{r\delta} dr_t \end{aligned} \tag{A16}$$

where $s_W = \dfrac{1}{1+\dfrac{r+q}{\kappa}c\theta^a}$.

Substituting (A12) and (A15) into (A16) yields

$$\begin{aligned}\hat{L}_t &= -(1-\phi+\phi\gamma)s_W\hat{W}_t - (1-\phi+\phi\gamma)(1-s_W)\hat{\overline{W}}_t + \frac{(1-\phi+\phi\gamma)(1-s_W)[a/(1-a)](q+u)}{u^2}du_t \\ &\quad - \frac{(1-\phi+\phi\gamma)(1-s_W)[a/(1-a)]}{u}du_{t-1} \\ &\quad + \hat{A}_t + \hat{L}_t - \frac{1-\phi}{\phi}s_{r\delta}dr_t + e_W e^{-1}\hat{W}_t - e_W e^{-1}\hat{\overline{W}}_t - \frac{e_h e^{-1}(q+u)}{u^2}du_t + \frac{e_h e^{-1}}{u}du_{t-1} \\ &\quad + \phi(\gamma-1)\hat{A}_t \\ &\quad + \phi(\gamma-1)e^{-1}e_W\hat{W}_t - \phi(\gamma-1)e^{-1}e_W\hat{\overline{W}}_t - \frac{\phi(\gamma-1)e^{-1}e_h(q+u)}{u^2}du_t + \frac{\phi(\gamma-1)e^{-1}e_h}{u}du_{t-1} \\ &\quad - (1-\phi)(\gamma-1)s_{r\delta}dr_t + (1-\phi+\phi\gamma)\Bigg[\hat{M}_t - (1+\xi)\hat{A}_t - (1+\xi)\hat{L}_t + (1+\xi)\frac{1-\phi}{\phi}s_{r\delta}dr_t \\ &\quad - (1+\xi)e_W e^{-1}\hat{W}_t + (1+\xi)e_W e^{-1}\hat{\overline{W}}_t \\ &\quad + (1+\xi)\frac{e_h e^{-1}(q+u)}{u^2}du_t - (1+\xi)\frac{e_h e^{-1}}{u}du_{t-1} + \xi\hat{Y}_{t+1} + s_i d\pi_t^e\Bigg]\end{aligned}$$

The unemployment rate can be expressed as

$$u_t = \frac{N - L_t}{N}.$$

Differentiating this expression yields

$$du_t = \frac{-dL_t}{N} \approx -s_L\hat{L}_t,$$

where $s_L = \dfrac{L}{N}$.

Thus, $\hat{L}_t$ can be expressed as

$$\hat{L}_t = -s_L^{-1}du_t.$$

$$0 = -(1-\phi+\phi\gamma)\hat{\overline{W}}_t + \frac{(1-\phi+\phi\gamma)(1-s_W)[a/(1-a)](q+u)}{u^2}du_t$$
$$-\frac{(1-\phi+\phi\gamma)(1-s_W)[a/(1-a)]}{u}du_{t-1}$$
$$+\hat{A}_t - \frac{1-\phi}{\phi}s_{r\delta}dr_t - \frac{e_h e^{-1}(q+u)}{u^2}du_t + \frac{e_h e^{-1}}{u}du_{t-1} + \phi(\gamma-1)\hat{A}_t$$
$$-\frac{\phi(\gamma-1)e^{-1}e_h(q+u)}{u^2}du_t + \frac{\phi(\gamma-1)e^{-1}e_h}{u}du_{t-1}$$
$$-(1-\phi)(\gamma-1)s_{r\delta}dr_t + (1-\phi+\phi\gamma)\hat{M}_t - (1-\phi+\phi\gamma)(1+\xi)\hat{A}_t + (1-\phi+\phi\gamma)s_L^{-1}(1+\xi)du_t$$
$$+(1-\phi+\phi\gamma)(1+\xi)\frac{1-\phi}{\phi}s_{r\delta}dr_t + (1-\phi+\phi\gamma)(1+\xi)\frac{e_h e^{-1}(q+u)}{u^2}du_t$$
$$-(1-\phi+\phi\gamma)(1+\xi)\frac{e_h e^{-1}}{u}du_{t-1} + (1-\phi+\phi\gamma)\xi\hat{Y}_{t+1} + (1-\phi+\phi\gamma)s_i d\pi_t^e$$

$$(1-\phi+\phi\gamma)\hat{\overline{W}}_t = \left[\frac{(1-\phi+\phi\gamma)(1-s_W)[a/(1-a)](q+u)}{u^2} - \frac{(1-\phi+\phi\gamma)e^{-1}e_h(q+u)}{u^2}\right.$$
$$\left.+(1-\phi+\phi\gamma)s_L^{-1}(1+\xi)\left(1+\frac{e_h e^{-1}(q+u)}{u^2}\right)\right]du_t$$
$$-\left[(1-\phi+\phi\gamma)(1+\xi)\frac{e_h e^{-1}}{u} + \frac{(1-\phi+\phi\gamma)(1-s_W)[a/(1-a)]}{u}\right.$$
$$\left.-\frac{(1-\phi+\phi\gamma)e^{-1}e_h}{u}\right]du_{t-1}$$
$$+(1-\phi+\phi\gamma)\hat{M}_t - (1-\phi+\phi\gamma)\xi\hat{A}_t + (1-\phi+\phi\gamma)\xi\hat{Y}_{t+1} + (1-\phi+\phi\gamma)s_i d\pi_t^e$$
$$+\left[(1-\phi+\phi\gamma)(1+\xi)\frac{1-\phi}{\phi}s_{r\delta} - \frac{1-\phi}{\phi}s_{r\delta} - (1-\phi)(\gamma-1)s_{r\delta}\right]dr_t$$

$$\hat{\overline{W}}_t = \left[\frac{(1-s_W)[a/(1-a)](q+u)}{u^2} - \frac{e^{-1}e_h(q+u)}{u^2} + s_L^{-1}(1+\xi)\left(1+\frac{e_h e^{-1}s_L(q+u)}{u^2}\right)\right]du_t$$
$$-\left[\xi\frac{e_h e^{-1}}{u} + \frac{(1-s_W)[a/(1-a)]}{u}\right]du_{t-1} + \hat{M}_t - \xi\hat{A}_t + \xi\hat{Y}_{t+1} + s_i d\pi_t^e \qquad \text{(A17)}$$
$$+\frac{1-\phi}{\phi(1-\phi+\phi\gamma)}\left[(\xi-\xi\phi+\xi\phi\gamma-\phi+\phi\gamma)s_{r\delta} - \phi(\gamma-1)s_{r\delta}\right]dr_t$$

Subtracting the lag of (A17) from (A17) results in the following expression for wage inflation $(\pi_t^w = \hat{\bar{W}}_t - \hat{\bar{W}}_{t-1})$:

$$\pi_t^w = \left[\frac{(1-s_W)[a/(1-a)](q+u)}{u^2} - \frac{e^{-1}e_h(q+u)}{u^2} + s_L^{-1}(1+\xi)\left(1+\frac{e_h e^{-1} s_L (q+u)}{u^2}\right)\right](du_t - du_{t-1})$$
$$- \left[\xi \frac{e_h e^{-1}}{u} + \frac{(1-s_W)[a/(1-a)]}{u} \right](du_{t-1} - du_{t-2}) + (\hat{M}_t - \hat{M}_{t-1}) - \xi(\hat{A}_t - \hat{A}_{t-1})$$
$$+ \xi(\hat{Y}_{t+1} - \hat{Y}_t) + s_i(d\pi_t^e - d\pi_{t-1}^e) + \frac{(1-\phi)\xi s_{r\delta}}{\phi}(dr_t - dr_{t-1})$$

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Table 1

Estimates of the Phillips Curve and Dynamic Labor Demand Curve

	1967:2-2017:1		1967:2-2007:4	
	1	2	3	4
Phillips curve		w/ π^e		w/ π^e
u_t	-0.2371[a] (-3.01)	-0.2497[a] (-2.86)	-0.3017[b] (-2.25)	-0.3069[c] (-1.70)
u_{t-1}	0.1959[b] (2.26)	0.2126[b] (2.18)	0.2342[c] (1.69)	0.2347 (1.23)
π^w_{t+1}	0.325[a] (3.04)	0.304[b] (2.36)	0.403[a] (3.49)	0.417[a] (2.81)
π^w_{t-1}	0.199[a] (2.93)	0.203[a] (2.80)	0.201[a] (2.97)	0.194[c] (1.96)
π^w_{t-2} - π^w_{t-4}	0.323[a] (11.77)	0.353[a] (12.51)	0.279[a] (9.20)	0.275[a] (7.85)
π^w_{t-5} - π^w_{t-8}	0.038 (0.13)	0.028 (0.07)	0.162 (2.35)	0.152 (2.10)
π^w_{t-9} - π^w_{t-12}	0.138 (2.44)	0.134 (2.15)	-0.012 (-.01)	0.002 (0.00)
$\pi^w_{t+1} + \sum_{j=1}^{12} \pi^w_{t-j}$	1.024	1.021	1.032	1.040
$u_t + u_{t-1}$	-0.0411[a] (8.00)	-0.0371[b] (5.53)	-0.0675[a] (9.90)	-0.0722[a] (7.88)

Table 1 (continued)				
	1967:2-2017:1		1967:2-2007:4	
DLD	1	2	3	4
u_t	3.25[a] (2.92)	2.91[a] (2.89)	1.86[a] (3.17)	1.66[a] (3.02)
u_{t-1}	-3.73[a] (-2.72)	-3.19[a] (-2.63)	-2.05[a] (-2.89)	-1.84[b] (-2.52)
u_{t-2}	0.475 (1.19)	0.298 (0.85)	0.140 (0.51)	0.141 (0.47)
A	-0.545[a] (12.26)	-0.651[a] (17.11)	-0.183[c] (3.09)	-0.265[a] (7.02)
OilPrice	0.0217[a] (13.00)	0.0158[a] (7.22)	0.0194[a] (17.60)	0.0195[a] (16.89)
M	0.528[a] (16.80)	0.657[a] (26.37)	0.618[a] (27.38)	0.705[b] (36.05)
V^{trend}	0.150[a] (4.00)	0.161[a] (5.02)	0.173[a] (3.64)	0.186[a] (4.26)
Y	1.380[a] (41.45)	1.478[a] (42.56)	0.416[a] (7.48)	0.498[a] (11.87)
r	0.551[a] (12.47)	0.411[a] (9.54)	0.584[a] (23.93)	0.478[a] (20.09)
π^e		0.844[a] (8.21)		0.378[c] (3.54)
$u_t + u_{t-1} + u_{t-2}$	-0.0038	0.0238	-0.056	-0.035
$M + V^{trend}$	0.678	0.818	0.791	0.891
LR test: $u_t + u_{t-1} + u_{t-2} = 0$	0.01	0.50	2.95[c]	0.98
LR test: $A = -Y$	17.26[a]	17.58[a]	2.53	2.65
LR test: $M + V^{trend} = \nu$	2.57	0.15	0.30	0.36

t-statistic reported in parentheses for single coefficients; Likelihood Ratio statistic reported for sums of coefficients

[a] significant at the 1% level
[b] significant at the 5% level
[c] significant at the 10% level

Footnotes

[1] At the margin, workers supply labor up to the point where the marginal utility of leisure equals the wage. For a worker who is involuntarily unemployed, the marginal utility of leisure is less than the wage. All things being equal, as the average wage rises, the marginal utility of leisure for those who are unemployed will be higher. In addition, unemployment benefits generally rise as average wages increase.

[2] While many quits are into another job rather than into unemployment, it is likely that the probability of finding another job when employed is closely related to the probability of finding a job when unemployed. Thus, the probability of hire for an unemployed worker should be related to both quits into unemployment and quits into a different job.

[3] The Cobb-Douglas is a commonly used functional form for the matching function. See, for example, Romer (2012).

[4] It is assumed that the coefficient of relative risk aversion is the same in both the utility of consumption and in the utility of real money holdings.

[5] See Pissarides (2000), pp. 68-70 for an exposition of the search and matching model for large firms.

[6] See Figure 7 of Campbell and Duca (2008) for the simulation of the natural rate and see Figure 6 for the simulation of the duration-unemployment ratio. This paper is available at http://nzae.org.nz/wp-content/uploads/2011/08/nr1215387056.pdf.

[7] See equation 25 on p. 698 of Phelps (1968).

[8] In (31), the DLD in levels is $\hat{\bar{W}}_t = \sigma_1 du_t + \sigma_2 du_{t-1} + \hat{M}_t - \xi\hat{A}_t + \xi\hat{Y}_{t+1} + s_i d\pi_t^e + (1-\phi)\xi s_{r\delta}\phi^{-1} dr_t$.

[9] Another potential instrument from (31) is expected price inflation, but its inclusion reduces the sample size.

[10] Government defense spending was also considered for an instrument, but it had a very low significance level and often had a negative sign.

[11] The 3-month Treasury Bill rate is used in the regressions since both (8a) and (8b) suggest a short-term interest rate is the appropriate variable.

[12] Because the current and recent lagged values of trend velocity are highly correlated with one another, the sum of the current and three lagged changes in trend velocity is entered as a single variable.