NORTHERN ILLINOIS UNIVERSITY

PHYSICS DEPARTMENT

Physics 374 – Junior Physics Lab

An Introduction to Error Analysis

Physics is an experimental science. The physicist uses mathematics but the models he constructs aren't abstract fantasies—they must describe the real world. All physical theories are inspired by experimental observations of nature and must ultimately agree with these observations to survive. The interplay between theory and experiment is the essence of modern science.

The task of constructing a theory, inherently difficult, is compounded by the fact that the observations are never perfect. Because instruments and experimenters depart from the ideal, measurements are always slightly uncertain. This uncertainty appears as variations between successive measurements of the same quantity.

With better instrumentation and greater care, these fluctuations can be reduced but they can never be completely eliminated *. One can, however, estimate how large the uncertainty is likely to be and to what extent one can trust the measurements. Over the years powerful statistical methods have been devised to do this. In an imperfect world we can expect no more; and once we understand the limitations of an experiment we can, if we feel the need, try to improve on it.

In the laboratory you, like any scientist, will have to decide just how far you can trust your observations. Here is a brief description of experimental uncertainties and their analysis. Your TA will indicate how far you need to carry the analysis in each experiment.

* In certain instances, there is a fundamental limit imposed on this process of improvement. Quantum mechanics, which deals with very small systems such as atoms and nuclei, suggests that there is an inherent uncertainty in nature that even perfect instrumentation cannot overcome. Luckily, most experiments do not confront this ultimate barrier.

I. Definitions

The difference between the observed value of a quantity and the true value is called the *error* of the measurement. This term is misleading; it does not necessarily imply that the experimenter has made a mistake. "Uncertainty" is a better term but it is not as commonly used.

Errors may be conveniently classified into three types:

A. Illegitimate Errors: These are the true mistakes or blunders either in measurement or in computation. Reading the wrong scale or misplacing a decimal point in multiplying are examples. These errors usually stand out if the data is examined critically. You can correct such errors when you find them by eliminating their cause and possibly by repeating the measurement. If it's too late for that, you can at least guess where the mistake is likely to lie.

B. Systematic Errors: These errors arise from faulty calibration of equipment, biased observers, or other undetected disturbances that cause the measured values to deviate from the true value—always in the same direction. The bathroom scale that read –3 lbs before anyone steps on it exhibits a systematic error. These errors cannot be adequately treated by statistical methods. They must be estimated and, if possible, corrected from an understanding of the experimental techniques used.

Systematic errors affect the *accuracy* of the experiment; that is, how closely the measurements agree with the true value.

C. Random Errors: These are the unpredictable fluctuations about the average or "true" value that cannot be reduced except by redesign of the experiment. These errors must be tolerated although we can estimate their size. Random errors affect the *precision* of an experiment; that is, how closely the results of successive measurements are grouped.

An experiment may be accurate but not precise—or precise but not accurate.

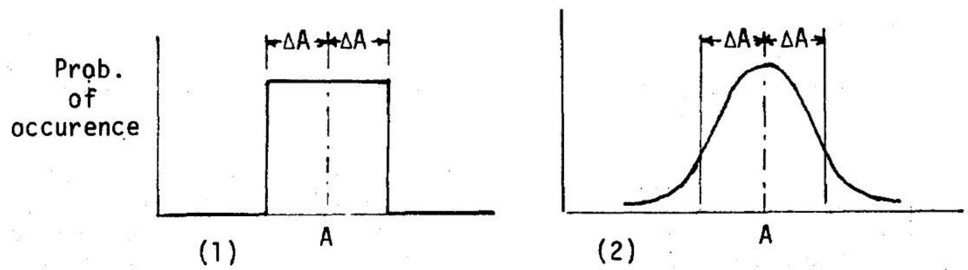
The concepts of error analysis that are introduced below are strictly applicable only to random errors.

II. Practices and Concepts

The result of a measurement is usually recorded as $A \pm \Delta A$ where A is the best guess for the value and ΔA means "the change in A". The interpretation is that the true value of A lies between $A + \Delta A$ and $A - \Delta A$. The length of a wooded block recorded as 5.2 ± 0.3 cm would be expected to lie between 4.9 and 5.5 cm.

A. **Significant Figures**: The number of figures in a result A is, by itself, often indicative of the uncertainty. In the example above, the value of 15.2 cm implies that the error will be at most several tenths of a cm since it isn't physically realistic to compute an error to more than one (occasionally two) significant figures. Accordingly, it wouldn't make sense to write the result as 15 ± 0.3 cm since writing 15 implies that we know nothing about tenths of centimeters. Likewise, writing 15.2 ± 1 cm is inconsistent. The result and the error estimate must always be in agreement concerning the least uncertain decimal place.

B. **Interpretation, Best Value**: When a result is quoted with an error estimate, the error value has, ideally, a rather precise meaning. The interpretation that the true value must be within the error limits and that any value within these limits (as shown in Fig. 1) is unrealistic. A more accurate picture is given by the familiar "bell shaped curve" (Fig. 2), which is the Gaussian distribution. This is the most frequently occurring distribution of sampling theory.



The interpretation of this distribution is that the most likely value is the best guess A; nearby values are only slightly less likely; but there remains a small chance that even a faraway value might be the correct one. For a careful experiment in which the errors have been properly analyzed, the value of ΔA gives an estimate of the precision expected of the best guess A. There is a *68% chance* that the true value lies within the limits $A - \Delta A$ and $A + \Delta A$, and one can predict from the Gaussian, or normal, distribution the probability for larger or smaller limits.

C. **Estimating Errors:** How can one estimate the size of experimental uncertainties? The first and simplest way is to use the least count of the measuring instrument. For instance, a ruler graduated in millimeters would certainly allow you to determine lengths to the nearest mm. Probably by interpolating between graduations, you could estimate the length to 0.3 or even 0.2 mm. You couldn't convincingly estimate the length to within 0.1 mm. Thus the precision of the instrument introduces an uncertainty whose magnitude you know.

As a minimal effort whenever you are recording data, you should always include such an error estimate for every bit of data recorded. You may not subsequently use this information and its magnitude may turn out to be unrealistic. But the practice you receive in considering such uncertainties is very important.

D. **<u>Repeated Measurements</u>:** When there is time or when the piece of data is crucial, you can repeat the measurement several times. The scattering of values about the average shows how large the random error must be. For instance, in the set of measurements 1.0, 1.1, 1.3, 0.9, 1.3 cm, a reasonable guess for the best value is the average, 1.1 cm, and the uncertainty in each measurement seems to be about 0.2 cm.

1. **<u>Best Value</u>:** Statistics does show that the average, or mean, of a set of measurements provides the best estimate of the true value. This is simply the sum of the measurements divided by the number taken:

$$x_{ave} \equiv \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

where $\bar{x}$ is the mean of the N measurements of the quantity x, labeled x_i where the index i runs from 1 to N.

2. **<u>Standard Deviation</u>:** The bests estimate of the error in this mean is shown by statistics to be derived from the standard deviation. The sample standard deviation is found by taking the differences of each x_i value from the mean, squaring these differences and added them, dividing by $N-1$, and then taking the square root:

$$\sigma_S = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2}$$

The factor $(N-1)$ implies that for $N=1$, $\sigma_S = 0/0$: the standard deviation is indeterminate. This is what is needed: from only one measurement it is impossible to say anything about the experimental error.

3. **<u>Standard Error of the Mean</u>:** The standard deviation is not quite the error estimate that is needed. The sample standard deviation, σ_S, is the estimated uncertainty of each individual measurement x_i. The value σ_S fluctuates as more samples are taken, but it doesn't systematically get smaller. Haven taken N repeated measurements, our estimate of the true value is the mean, $\bar{x}$. The *standard error* (or *standard deviation*) of the mean is the estimate of the error in this value. The standard error of the mean is related to the standard deviation by a simple expression:

$$\sigma_m = \frac{\sigma_S}{\sqrt{N}}$$

This important result will be derived (and understood) when we examine another source of error, propagation errors, in the next computer lab. However, notice that the uncertainty of the mean is reduced by a factor of $\sqrt{N}$ from the uncertainty in an individual measurement. This is the reason for repeatedly measuring the same quantity. Note, however, that the reduction is only by $\sqrt{N}$. Making 100 measurements instead of 10 only reduces the uncertainty by a factor of 3!

Calculating the standard deviation can be tedious. However there are programs for computers and programmable calculators which give the mean and standard deviation of a series of entries. Some scientific calculators have this program built in.

Example: Suppose we measured the length of a line four times and obtained values of 5.5, 5.3, 4.9, 4.7 cm. These values sum to 20.4 cm, and our best estimate of the length is the mean value $\bar{x} = \frac{1}{4}(20.4) = 5.1$ cm. The error estimate associated with each measurement is the standard deviation:

$$\sigma_S = \sqrt{\frac{1}{3}\left[(5.5-5.1)^2 + (5.3-5.1)^2 + (4.9-5.1)^2 + (4.7-5.1)^2\right]} = 0.37 \text{ cm}$$

The set of four measurements taken together gives us an estimated error σ_m of the mean:

$$\sigma_m = \frac{0.37}{\sqrt{4}} = 0.18 \, \text{cm}$$

The final result then for the length of the line is: $5.1 \pm 0.2 \, \text{cm}$.

III. Significant Figures

There is a natural shorthand for the estimate of errors and their propagation in the use of a definite number of figures to represent in a measurement. These figures are called *significant figures*. The measurement 15.23 cm, for example, has four significant figures; it is uncertain to a few one-hundredths of a centimeter (the exact uncertainty is deliberately left vague). It is different from 15 cm, 15.2 cm, 15.230 cm, and 15.2300 cm; those numbers might all represent the same measurement, but the uncertainty is different in each case.

It makes no difference, however, whether we write 15.23 cm, 0.l523 m, or 152.3 mm; each number has four significant figures. A number with m significant figures has a fractional uncertainty of 1 part in 10^m.

In adding or subtracting numbers, the largest uncertainty will dominate. This belongs to the number whose least significant figure is farthest to the left (that is, the last significant digit of the most imprecise number); the sum (or differences) will have no significant figures beyond this point. For instance, adding the numbers below gives:

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  15.2304
    2.13
489.5
  62
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569
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For multiplication and division, the largest fractional error will dominate. It occurs in the number with the _*fewest*_ significant figures. Hence the result can have no more significant digits than the least accurate of the factors. As an example:

$$\frac{15.23 \times 471}{19} = 380$$

The answer is not 377 or 377.5437. We could be even more definite by writing 3.8×10^2.

IV. Propagation of Errors

In the laboratory we seldom measure directly the quantities or results of interest. Instead we must measure others from which the results are derived. For example, to measure the volume of a rectangular solid, we measure the three sides and multiply these values. In the course of such a calculation, the errors in the measured quantities travel or "propagate" through the computation to affect the result.

The basic equation which describes the propagation of errors is best expressed in terms of partial derivatives. Suppose that we require a quantity P that is a function of a set of variables a, b, c, $\cdots$ that we actually can measure. P may be expressed as

$$P = f(a,b,c,\cdots)$$

For the rectangular solid, V is a function of the length, ℓ, width w, and height h, and we could write:

$$V = f\left(\ell,w,h\right) = \ell \times w \times h$$

The mean values of a, b, c, $\cdots$ are substituted into f to give a value for P. The relation between the error in P (call it σ_P) and the errors in a, b, c, $\cdots$ (σ_a, σ_b, σ_c, $\cdots$) are given by the following equation:

$$\sigma_P^2 = \left(\frac{\partial f}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial f}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial f}{\partial c}\right)^2 \sigma_c^2 + \cdots \qquad (1)$$

Consult the following references if you want to see how this result is obtained:

P. R. Bevington, *Data Reduction and Error Analysis for the Physics Sciences*
H. Young, *Statistical Treatment of Experimental Data*
Y. Beers, *Introduction to the Theory of Error*

The partial derivative of f with respect to a is written as: $\frac{\partial f}{\partial a}$. It means: take the derivative of f with respect to a keeping all the other variables b, c, $\cdots$ constant.

We will show how this general formula is applied to some specific cases of error propagation.

A. <u>Addition</u>: $y = a + b$

The estimated errors on the measured quantities a and b are σ_a and σ_b. What is the uncertainty in σ_y ?

Evaluate: $\frac{\partial y}{\partial a} = 1$; $\frac{\partial y}{\partial b} = 1$ and use Eq. (1) to find:

$\sigma_y^2 = \sigma_a^2 + \sigma_b^2$ The *errors are added in quadrature*.

B. <u>Subtraction</u>: $y = a - b$

You can check that: $\sigma_y^2 = \sigma_a^2 + \sigma_b^2$ The same result as in addition.

C. <u>Multiplication:</u> $y = a \times b$

Evaluate: $\frac{\partial y}{\partial a} = b$ (when b is kept constant); $\frac{\partial y}{\partial b} = a$ (when a is kept constant)

so that $\sigma_y^2 = b^2\sigma_a^2 + a^2\sigma_b^2$

This result is better expressed in terms of *fractional or relative errors*, σ_y / y , σ_a / a , σ_b / b :

$$\left(\frac{\sigma_y}{y}\right)^2 = \left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2$$

Thus the *fractional errors are added in quadrature*.

D. <u>Division</u>: $y = \frac{a}{b}$

We have: $\frac{\partial y}{\partial a} = \frac{1}{b}$; $\frac{\partial y}{\partial b} = -\frac{a}{b^2}$

so that: $\sigma_y^2 = \frac{1}{b^2}\sigma_a^2 + \frac{a^2}{b^4}\sigma_b^2$

By dividing each term by $y^2 = \left(\frac{a}{b}\right)^2$, we see that, again, the fractional errors are added in quadrature:

$$\left(\frac{\sigma_y}{y}\right)^2 = \left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2$$

E. <u>Power Law</u>: $y = a^n$

We find: $\frac{\partial y}{\partial a} = na^{n-1}$

so that: $\sigma_y = \left(na^{n-1}\right)\sigma_a$ or $\frac{\sigma_y}{y} = n\left(\frac{\sigma_a}{a}\right)$

Thus if $y = a^3$, the fractional error in y is three times the fractional error in a .

Example: Calculate the acceleration g due to gravity by dropping a ball a distance x in time t. You know that $x = \frac{1}{2}gt^2$. How do the uncertainties in x and t affect the uncertainty in g ?

Answer: Since $x = \frac{1}{2}gt^2$, then $g = \frac{2x}{t^2}$. Then $\frac{\partial g}{\partial x} = \frac{2}{t^2}$ and $\frac{\partial g}{\partial t} = -\frac{4x}{t^3}$, and therefore:

$$\sigma_g^2 = \left(\frac{2}{t^2}\right)^2 \sigma_x^2 + \left(\frac{4x}{t^3}\right)^2 \sigma_t^2$$

V. What Do We Expect from You in the Lab?

By now you may feel overwhelmed by the depth to which error analysis can be carried out. We do not want to obscure the physics involved, however. There would be no point in exhaustively discussing errors for an experiment that you do not really understand. What you should attempt to do is:

1. understand what sources of error are present in your experiment,

2. estimate how significant they are and how they influence your result, and

3. determine the deviation of the result that you find from the one you expect can be accounted for by these errors.

It is better to find an inaccurate result with large sources of error that you can explain than to "fudge" a right answer that cannot be justified by the method.