Krane Quantum Statistics Example (p. 307)

(a) Five identical but distinguishable particles sharing 6 units of energy → Classical Maxwell-Boltzmann particles ①

10.2 Classical and Quantum Statistics 307

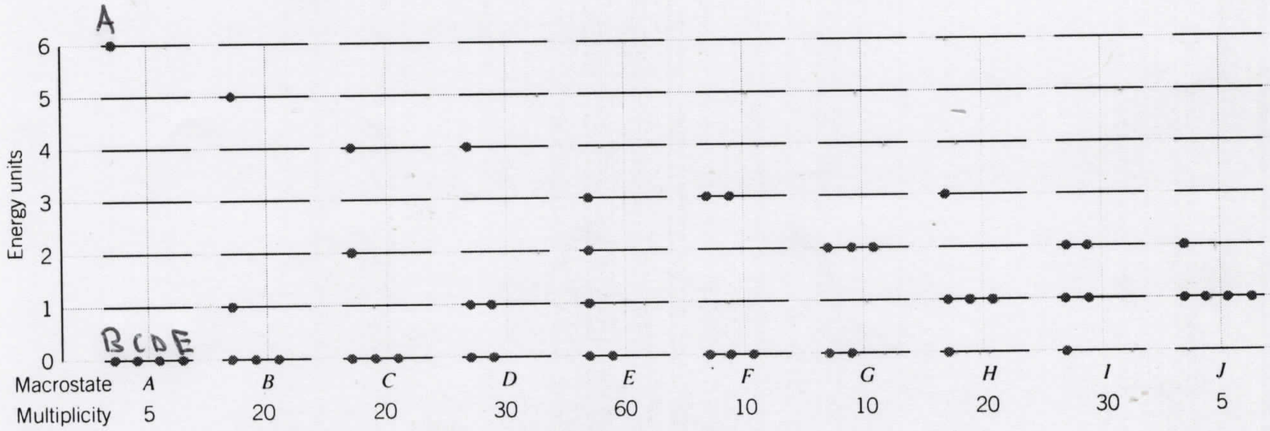


FIGURE 10.3 The macrostates of a system in which five identical particles share 6 units of energy.

To find the # of microstates for each macrostate, determine the # of ways 5 identical particles can share 6 units of energy and divide by the # of permutations that give the same microstate:

A: $\frac{5!}{4!} = 5$

B: $\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3!}{3!} = 20$

C: $\frac{5!}{3!} = 20$

D: $\frac{5!}{2!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 2} = 30$

E: $\frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2!}{2!} = 60$

F: $\frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 2} = 10$

G: $\frac{5!}{3!2!} = 10$

H: $\frac{5!}{3!} = 20$

$I: \dfrac{5!}{2!2!} = 30$ (2)

$J: \dfrac{5!}{4!} = 5$

The total # of microstates is then

$$\#\text{ microstates} = 5+20+20+30+60+10+10+20+30+5$$
$$= 210$$

The probability that a particle has En units of energy is then:

$$P(E=6) = \frac{1\cdot 5}{210}$$

$$P(E=5) = \frac{1\cdot 20}{210}$$

$$P(E=4) = \frac{1\cdot 20}{210} + \frac{1\cdot 30}{210} = \frac{50}{210}$$

$$P(E=3) = \frac{1\cdot 60}{210} + \frac{2\cdot 10}{210} + \frac{1\cdot 20}{210} = \frac{100}{210}$$

$$P(E=2) = \frac{1\cdot 20}{210} + \frac{1\cdot 60}{210} + \frac{3\cdot 10}{210} + \frac{2\cdot 30}{210} + \frac{1\cdot 5}{210} = \frac{175}{210}$$

$$P(E=1) = \frac{1\cdot 20}{210} + \frac{2\cdot 30}{210} + \frac{1\cdot 60}{210} + \frac{3\cdot 20}{210} + \frac{2\cdot 30}{210} + \frac{4\cdot 5}{210} = \frac{280}{210}$$

$$P(E=0) = \frac{4\cdot 5}{210} + \frac{3\cdot 20}{210} + \frac{3\cdot 20}{210} + \frac{2\cdot 30}{210} + \frac{2\cdot 60}{210} + \frac{3\cdot 10}{210} + \frac{2\cdot 10}{210} + \frac{1\cdot 20}{210} + \frac{1\cdot 30}{210} = \frac{420}{210}$$

To normalize the probabilities, divide by the # of particles: 3

$f(E=6) = \frac{5}{210} \cdot \frac{1}{5} = 0.00476$

$f(E=5) = \frac{20}{210} \cdot \frac{1}{5} = 0.01905$

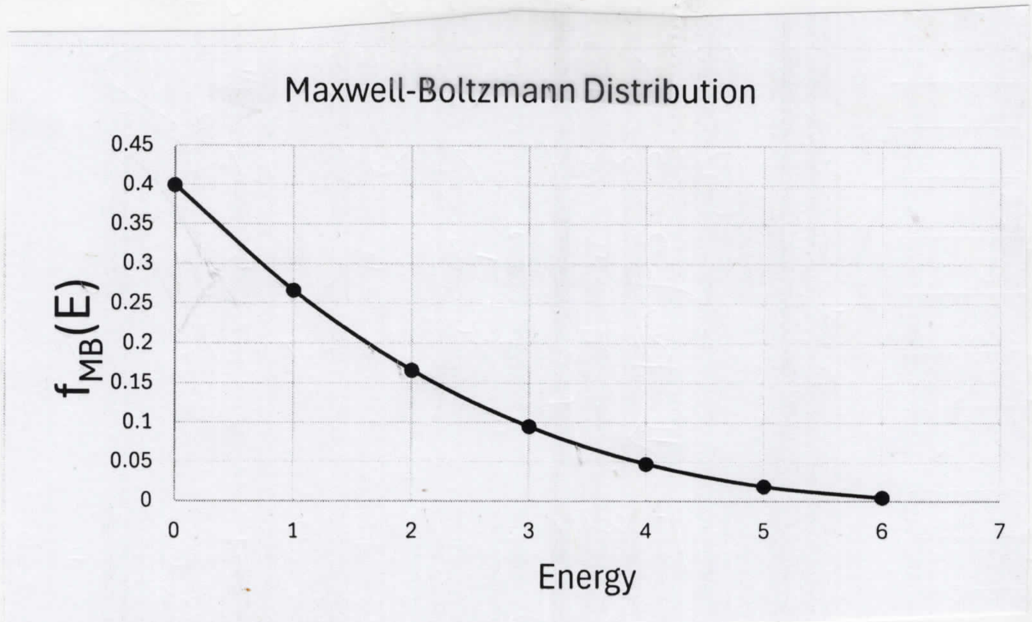
$f(E=4) = \frac{50}{210} \cdot \frac{1}{5} = 0.04762$

$f(E=3) = \frac{100}{210} \cdot \frac{1}{5} = 0.09524$

$f(E=2) = \frac{175}{210} \cdot \frac{1}{5} = 0.16667$

$f(E=1) = \frac{280}{210} \cdot \frac{1}{5} = 0.26667$

$f(E=0) = \frac{420}{210} \cdot \frac{1}{5} = 0.40000$



(b) Five identical but indistinguishable particles sharing 6 units of energy → Quantum Bose-Einstein particles

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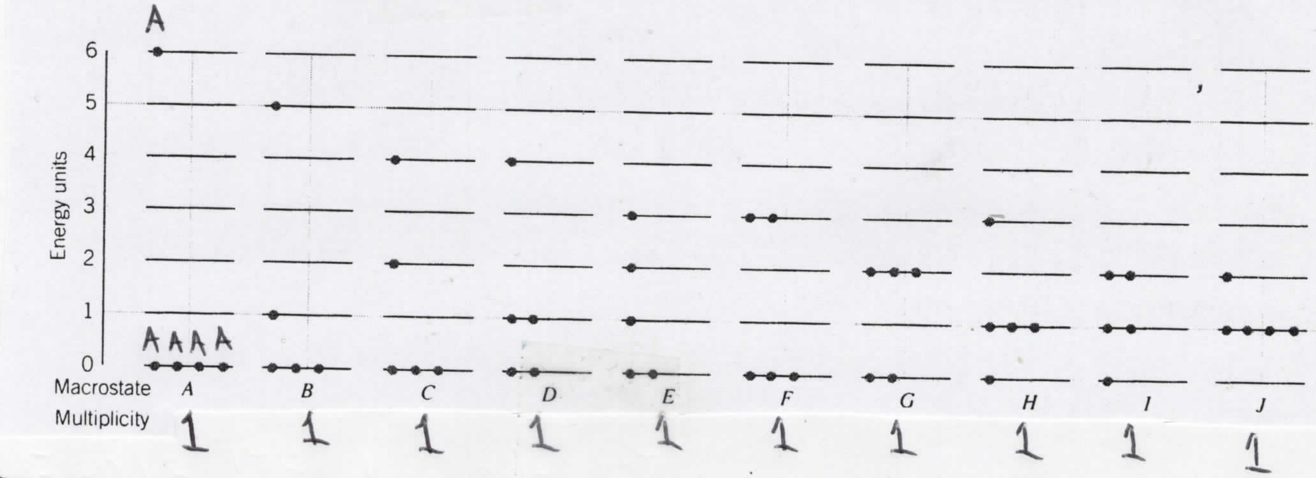
10.2 Classical and Quantum Statistics 307



For indistinguishable particles, all microstates condense to just 1 microstate. All arrangement of particles yields the same state. The total # of microstates is then:

$$\#\text{ microstates} = 1+1+1+1+1+1+1+1+1+1 = 10$$

Since there are 10 macrostates. Then

$$P(E=6) = 1 \cdot \frac{1}{10}$$

$$P(E=5) = 1 \cdot \frac{1}{10}$$

$$P(E=4) = 1 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10} = \frac{2}{20}$$

$$P(E=3) = 1 \cdot \frac{1}{10} + 2 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10} = \frac{4}{10}$$

$$P(E=2) = 1\cdot\frac{1}{10} + 1\cdot\frac{1}{10} + 3\cdot\frac{1}{10} + 2\cdot\frac{1}{10} + 1\cdot\frac{1}{10} = \frac{8}{10}$$

$$P(E=1) = 1\cdot\frac{1}{10} + 2\cdot\frac{1}{10} + 1\cdot\frac{1}{10} + 3\cdot\frac{1}{10} + 2\cdot\frac{1}{10} + 4\cdot\frac{1}{10} = \frac{13}{10}$$

$$P(E=0) = 4\cdot\frac{1}{10} + 3\cdot\frac{1}{10} + 3\cdot\frac{1}{10} + 2\cdot\frac{1}{10} + 2\cdot\frac{1}{10} + 3\cdot\frac{1}{10} + 2\cdot\frac{1}{10} + 1\cdot\frac{1}{10} + 1\cdot\frac{1}{10} = \frac{21}{10}$$

The normalized probabilities are then:

$$f(E=6) = \frac{1}{10}\cdot\frac{1}{5} = 0.02$$

$$f(E=5) = \frac{1}{10}\cdot\frac{1}{5} = 0.02$$

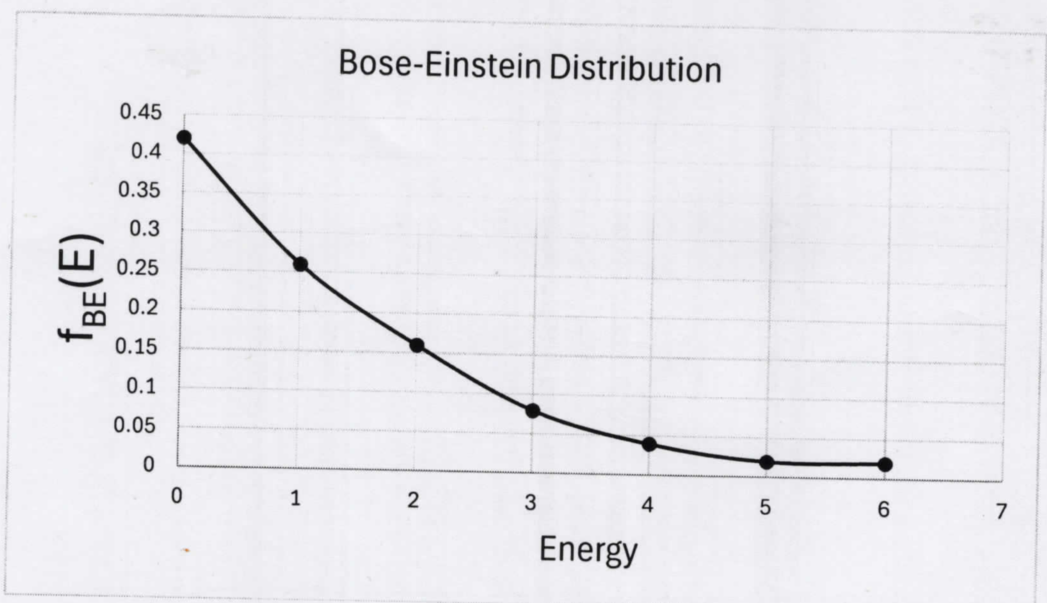
$$f(E=4) = \frac{2}{10}\cdot\frac{1}{5} = 0.04$$

$$f(E=3) = \frac{4}{10}\cdot\frac{1}{5} = 0.08$$

$$f(E=2) = \frac{8}{10}\cdot\frac{1}{5} = 0.16$$

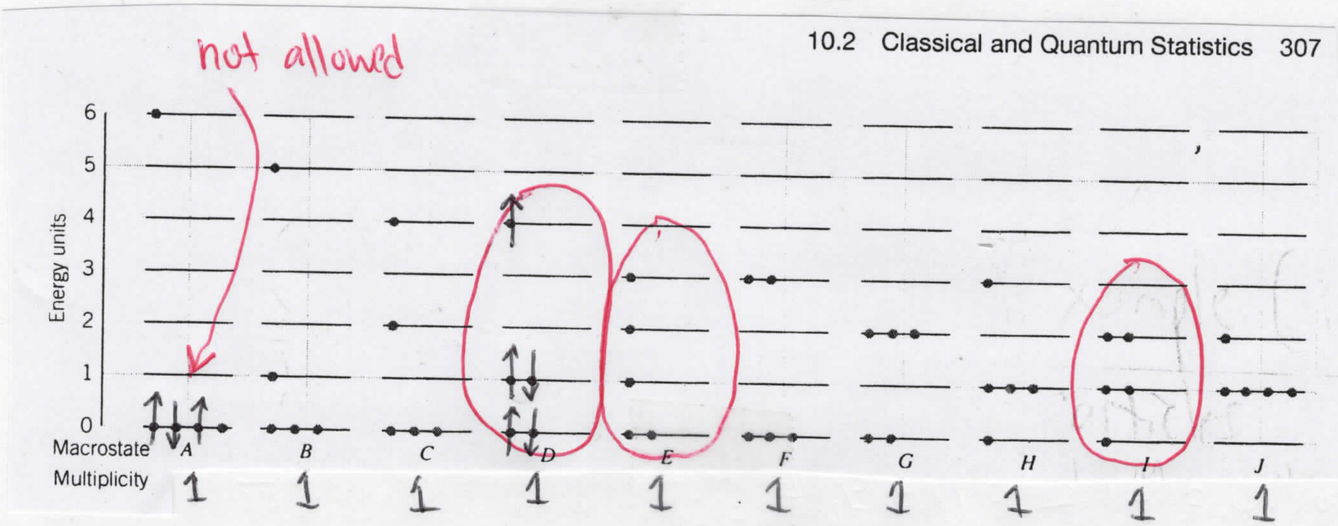
$$f(E=1) = \frac{13}{10}\cdot\frac{1}{5} = 0.26$$

$$f(E=0) = \frac{21}{10}\cdot\frac{1}{5} = 0.42$$



(c) Five identical but indistinguishable particles sharing 6 units of energy obeying the Pauli Exclusion Principle

→ Quantum Fermi-Dirac particles



All microstates condense to just 1 microstate since all particles are indistinguishable. However, due to the Pauli Exclusion Principle, only 3 macrostates survive: they are circled. The total # of microstates is then:

$$\# \text{ microstates} = 1 + 1 + 1 = 3$$

Then:

$P(E=6) = 0$

$P(E=5) = 0$

$P(E=4) = 1 \cdot \frac{1}{3}$

$P(E=3) = 1 \cdot \frac{1}{3}$

$P(E=2) = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = \frac{3}{3}$

$P(E=1) = 2 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = \frac{5}{3}$

$P(E=0) = 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{5}{3}$

The normalized probabilities are then:

$f(E=6) = 0$

$f(E=5) = 0$

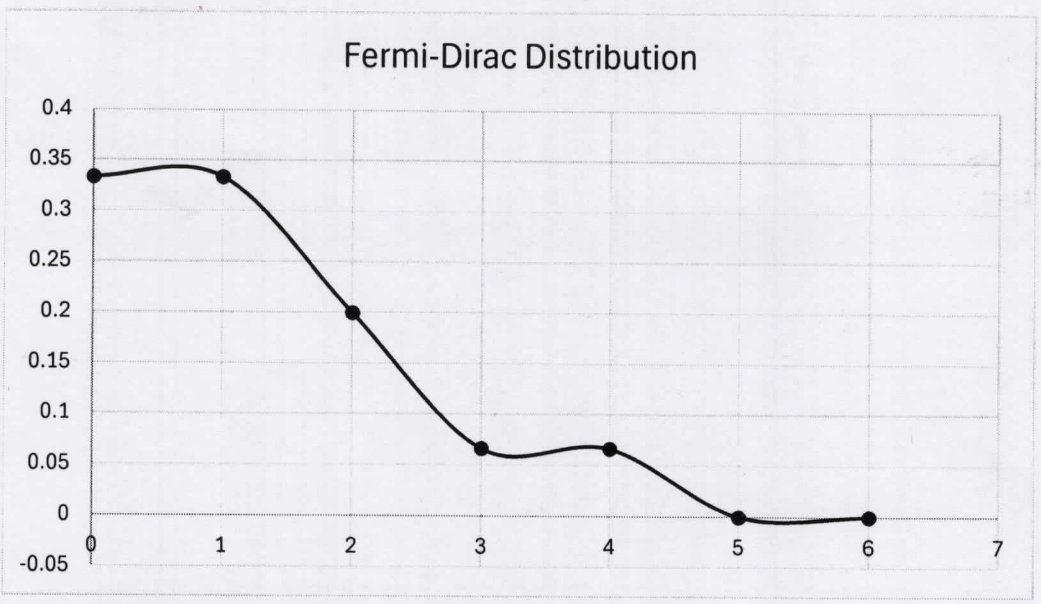
$f(E=4) = \frac{1}{3} \cdot \frac{1}{5} = 0.06667$

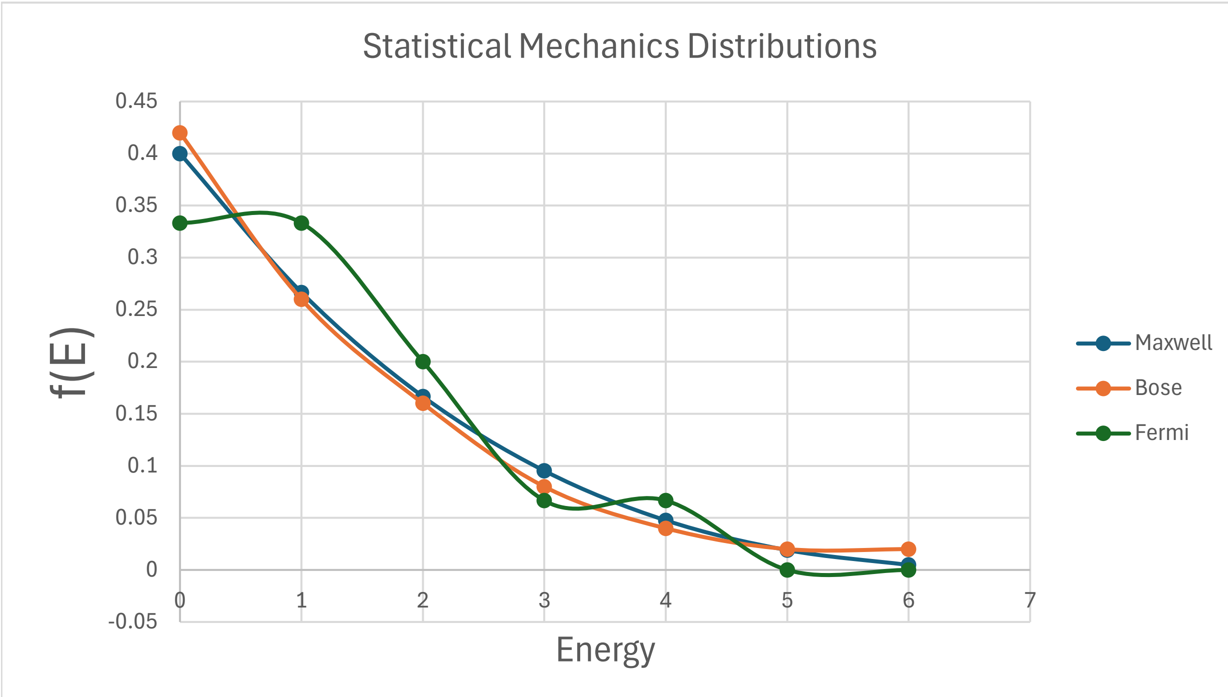
$f(E=3) = \frac{1}{3} \cdot \frac{1}{5} = 0.06667$

$f(E=2) = \frac{3}{3} \cdot \frac{1}{5} = 0.2$

$f(E=1) = \frac{5}{3} \cdot \frac{1}{5} = 0.33333$

$f(E=0) = \frac{5}{3} \cdot \frac{1}{5} = 0.33333$




Statistical Mechanics Distributions
f(E)
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
1
2
3
4
5
6
7
Energy
Maxwell
Bose
Fermi