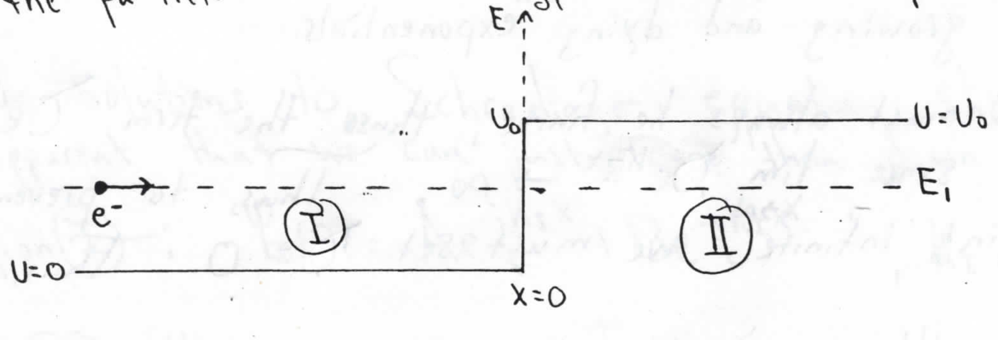
Physics 360 - Lecture Notes # 2

III Scattering at a Step Potential

Let's look at the problem of a particle scattering from a step when the particle has less energy than the step height:



Since the e- does not have enough energy to make it over the step, then Region II is a classically forbidden region. Let's write down and solve the Schrodinger equation in regions I + II.

In Region I:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I}{dx^2} + \underbrace{U(x)}_{0}\psi_I = E\psi_I \Rightarrow \frac{d^2\psi_I}{dx^2} = \frac{-2mE}{\hbar^2}\psi_I = -K_I^2\psi_I$$

The solution should be quite familiar by now:

$$\psi_I(x) = Ae^{iK_Ix} + Be^{-iK_Ix}, \quad K_I = \sqrt{\frac{2mE}{\hbar^2}}$$

In Region II:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} + U(x)\psi_{II} = E\psi_{II} \Rightarrow \frac{d^2\psi_{II}}{dx^2} = -\frac{2m}{\hbar^2}(E-U_0)\psi_{II} = -K_{II}^2\psi_{II}$$

Note that $E-U_0 < 0$, then $K_{II} = \sqrt{\frac{2m}{\hbar^2}(E-U_0)} = \sqrt{\frac{2m}{\hbar^2}(-1)(U_0-E)}$

$$= \sqrt{-1}\sqrt{\frac{2m}{\hbar^2}(U_0-E)}$$

$$= i\sqrt{\frac{2m}{\hbar^2}(U_0-E)}$$

$$= i\alpha_{II}$$

The solution is

$$\psi_{II}(x) = Ce^{iK_{II}} + De^{-iK_{II}} = Ce^{i(i\alpha_{II})x} + De^{-i(i\alpha_{II})x}$$

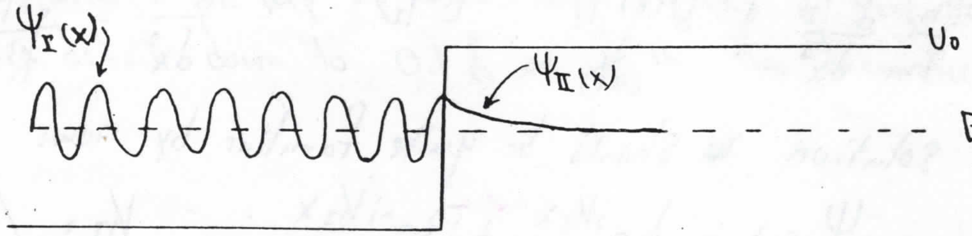
$$= Ce^{-\alpha_{II}x} + De^{\alpha_{II}x}$$

which are growing and dying exponentials.

But, $\psi_{II}(x)$ must always be finite, thus the term $De^{\alpha_{II}x}$ is not allowed since $\lim_{x\to\infty} De^{\alpha_{II}x} \to \infty$. Thus, to prevent $\psi_{II}(x)$ from becoming infinite, we must set $D = 0$. Thus, in Region (II):

$$\psi_{II}(x) = Ce^{-\alpha_{II}x} \quad \text{(a dying exponential)}$$

Surprisingly, there is a finite probability for the particle to exist in the classically forbidden region. A plot of ψ_I, ψ_{II} follows:



The depth of penetration of the particle into the classically forbidden region is characterized by a (1/e) length, that is, the value of x at which the probability drops by 1/e:

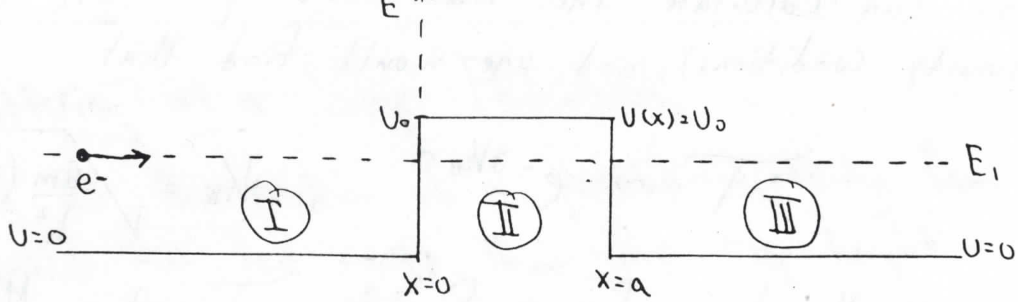
$$\text{Probability} = |\psi_{II}|^2 \sim e^{-2\alpha_{II}x_e} = \frac{1}{e} = e^{-1}$$

$$2\alpha_{II}x_e = 1$$

$$x_e = \frac{1}{2K_{II}} = \frac{\hbar}{2\sqrt{2m(U_0-E)}} = \text{penetration depth}$$

Such behavior allows particles to penetrate barriers, as we will see in the next scattering problem.

IV Scattering from a Potential Energy Barrier



The Solutions to Schrodinger's equation should now be so apparent that we can just write them down:

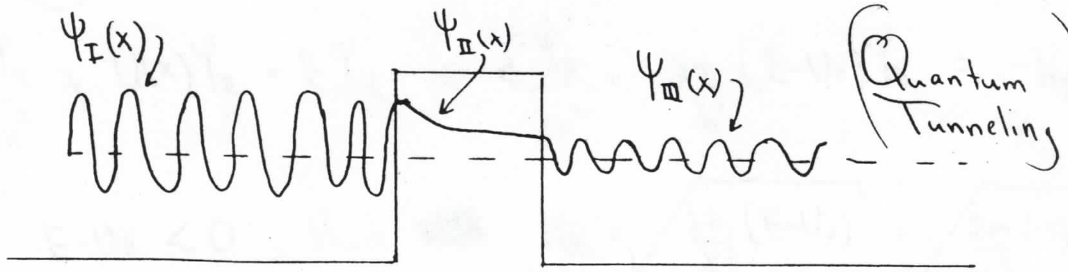
Region I : $\Psi_I(x) = Ae^{iK_I x} + Be^{-iK_I x}$, $K_I = \sqrt{\frac{2mE}{\hbar^2}}$

Region II $\Psi_{II}(x) = Ce^{-\alpha_{II} x} + De^{\alpha_{II} x}$, $\alpha_{II} = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$

Region III $\Psi_{III}(x) = Fe^{iK_{III} x} + Ge^{-iK_{III} x}$, $K_{III} = K_I = \sqrt{\frac{2mE}{\hbar^2}}$

In this case $D \neq 0$ Since this function $\Psi_{II}(x)$ only exists over the interval $0 \leq x \leq a$.

But $G = 0$ since there are no reflected waves (waves traveling to the left) in Region III. A sketch of these wavefunctions follows

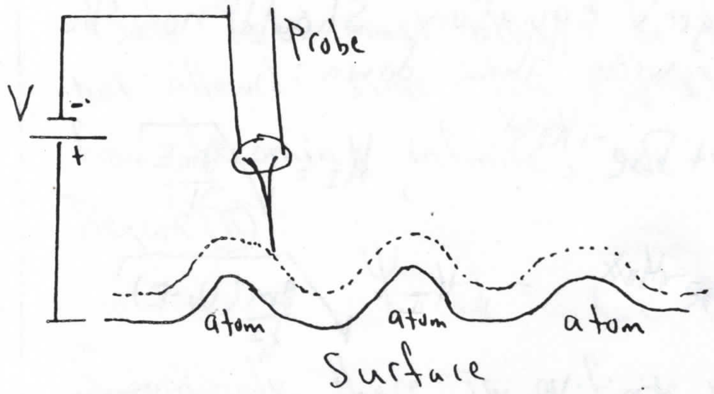


Classically, a particle can never appear in region $x > a$ since it does not have enough energy to overcome the barrier. Quantum mechanically, the particle can penetrate the barrier — this is called the tunneling effect.

One can calculate the transmission probability (using the continuity conditions), and one would find that

$$T \sim e^{-2K_{II}a} \quad , \quad K_{II} = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)}$$

This is the basis for a Scanning Tunneling Microscope



The metal probe has a tip so fine that it is a single atom. Electrons need several electron volts to escape the surface, but when only $V = 10\,mV$ is applied between probe and surface, electrons can tunnel through the air gap if the spacing is small enough (a nanometer or two).

Keeping current (which is sensitive to gap distance) constant by mechanically moving the tip up and down gives a map of the surface. One can see detail down to $0.1\,\text{Å}$ or $\frac{1}{10}$th the diameter of an atom.