Physics 283

Quiz # 2 Solutions

1

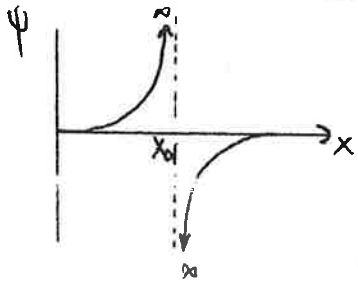
$n\lambda = 2d\sin\theta$, $n = 1$

$\lambda = 2d\sin\theta = 2(0.252\,nm)\sin(18.1^\circ) = 0.156\,nm$

(2)

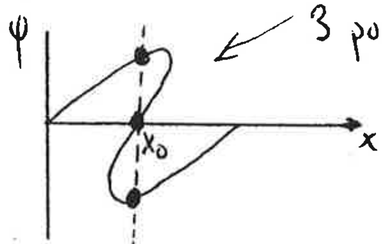
(a), (b), (d), & (f) represent functions that have physical significance

(c) This function cannot have physical significance because it is infinite at $x = x_0$ and discontinuous there



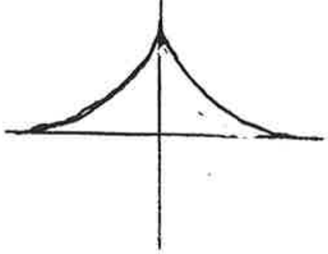
(e) This function cannot have physical significance because it is multivalued at $x = x_0$

3 possible values at $x = x_0$



(g) Since $\lim_{x \to \infty} x^2 = \infty$, x^2 blows up at $x = \infty$ and thus cannot be a solution of Schrodinger's equation.

(h)



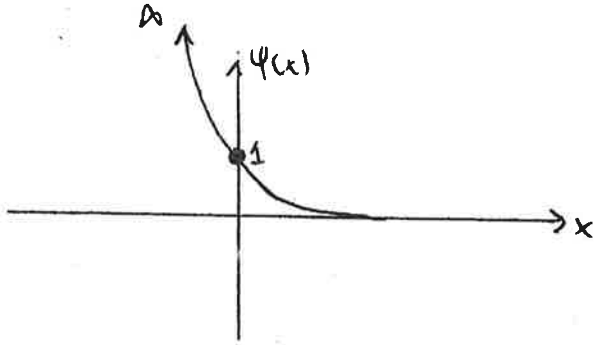
The slope of $\psi(x) = e^{-|x|}$ is discontinuous at $x = 0$ since

$$\left.\frac{d\psi}{dx}\right]_{x>0} < 0 \quad , \quad \left.\frac{d\psi}{dx}\right]_{x<0} > 0$$

The slope changes abruptly and discontinuously at $x = 0$

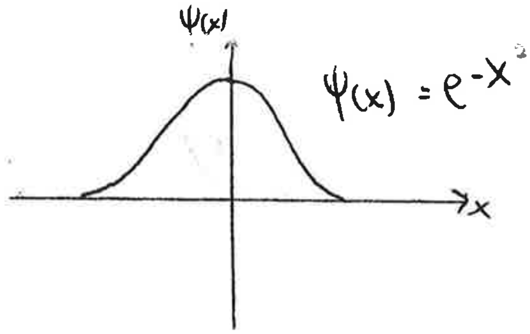
Thus, $e^{-|x|}$ cannot be a solution of Schrodinger's equation.

(i)



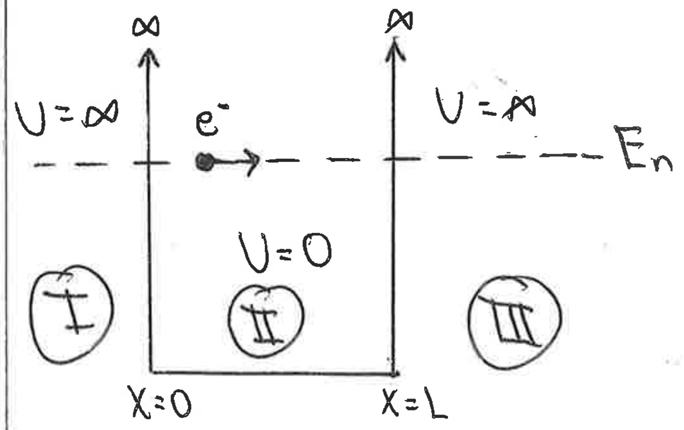
Since $\lim_{x\to -\infty} e^{-x} = \infty$, e^{-x} blows up at $x=-\infty$, and it cannot be a solution of Schrodinger's Eqn.

(j)



This is a well-behaved function (a Gaussian) and thus it can be a solution of Schrodingers equation for all values of x.

③ Particle in a Box



$$U(x) = \begin{cases} 0 & \text{for } 0 \le x \le L \\ \infty & \text{for } x<0,\ x>L \end{cases}$$

(a) Since $U = \infty$ in Region I : $\Psi_I(x) = 0$

(b) Since $U = \infty$ in Region III : $\Psi_{III}(x) = 0$

electron cannot be found outside the box

(c) $\Psi_{II}(x) = A e^{ikx} + B e^{-ikx}$

Continuity at $x = 0$ gives:

$$\Psi_I(x=0) = \Psi_{II}(x=0)$$

$$0 = Ae^{0} + Be^{-0} = A + B \Rightarrow \boxed{A = -B}$$

Continuity at $x = L$ gives:

$$\Psi_{II}(x=L) = \Psi_{III}(x=L)$$

$$Ae^{ikL} + Be^{-ikL} = 0$$

$$A\left(e^{ikL} - e^{-ikL}\right) = 0 \qquad (\text{since } B = -A)$$

$$2iA\sin(kL) = 0 \qquad \left(\text{using Euler identity } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right)$$

This is valid only if: $kL = n\pi \Rightarrow \boxed{k_n = \frac{n\pi}{L}}$

Then, $\psi_{II}(x) = 2iA\sin(k_n x) = 2iA\sin\left(\frac{n\pi x}{L}\right)$

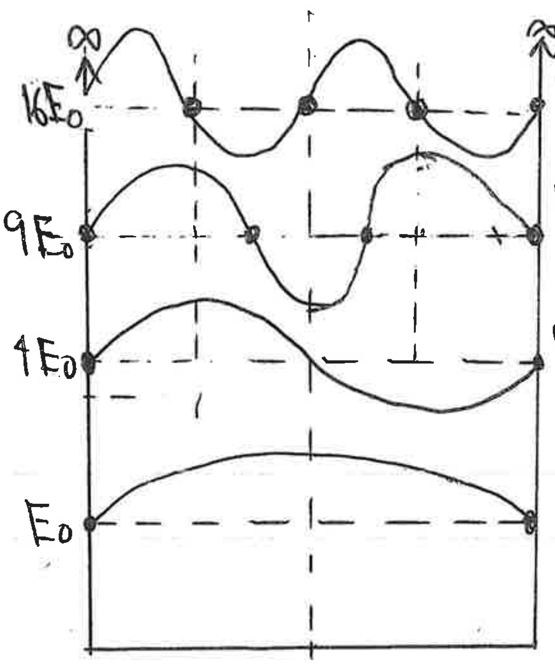
(d) Plugging $\psi_{II}(x)$ back into the Schrödinger eqn. gives the allowed energies

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} + \cancel{U(x)}^{\,0}\psi_{II} = E\psi_{II}$$

$$-\frac{\hbar^2}{2m}\underbrace{\left(\frac{n\pi}{L}\right)\left(\frac{-n\pi}{L}\right)\left(2iA\sin\left(\frac{n\pi x}{L}\right)\right)}_{\frac{d^2\psi_{II}}{dx^2}} = E\left(2iA\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$+\frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 = E_n \Rightarrow E_n = \left(\frac{\hbar^2\pi^2}{2mL^2}\right)\cdot n^2 = E_0 n^2$$

(e) & (f)



$\psi_{n=4} \Rightarrow$ 4 humps

$\psi_{n=3} \Rightarrow$ 3 humps

$\psi_{n=2} \Rightarrow$ 2 humps

$\psi_{n=1} \Rightarrow$ 1 hump

(g) Normalization:

$$\int_{-\infty}^{\infty} |\Psi(x)|^2 \, dx = 1 = \int_0^L 4A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx$$

Note: $\cos 2A = 1 - \sin^2 A \Rightarrow \sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$$4A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 4A^2 \int_0^L \frac{1}{2}\left(1 - \cos\left[\frac{2n\pi x}{L}\right]\right) dx$$

$$= 2A^2 \int_0^L \left(1 - \cos\left[\frac{2n\pi x}{L}\right]\right) dx = 2A^2 \int_0^L dx - 2A^2 \int_0^L \cos\left(\frac{2n\pi x}{L}\right) dx$$

$$\int_0^L dx = x\Big]_0^L = L - 0 = L$$

$$\int_0^L \cos\left(\frac{2n\pi x}{L}\right) dx = \left(\frac{L}{2n\pi}\right) \sin\left(\frac{2n\pi x}{L}\right)\Big]_0^L = \frac{L}{2n\pi}\left(\cancelto{0}{\sin(2n\pi)} - \cancelto{0}{\sin(0)}\right)$$

$$= 0$$

$$\Rightarrow 4A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{4A^2}{2}(L) = 2A^2 L = 1$$

$$A = \frac{1}{\sqrt{2L}}$$

$$\Rightarrow \Psi_{II}(x) = 2iA \sin\left(\frac{n\pi x}{L}\right) = \frac{2i}{\sqrt{2L}} \sin\left(\frac{n\pi x}{L}\right) = i\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

④ The Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{l(l+1)\hbar^2}{2mr^2}u - \frac{1}{4\pi\epsilon_0}\cdot\frac{Ze^2}{r}u = Eu$$

For $u(r) = A r^{l+1} e^{-br}$

$$\frac{du}{dr} = (l+1)A r^{l} e^{-br} - bA r^{l+1} e^{-br}$$

$$\frac{d^2u}{dr^2} = l(l+1)A r^{l-1} e^{-br} - b(l+1)A r^{l} e^{-br} - b(l+1)A r^{l} e^{-br} + b^2 A r^{l+1} e^{-br}$$

Inserting this into the Schrodinger equation gives:

$$-\frac{\hbar^2}{2m}\left[l(l+1)r^{l-1} - 2b(l+1)r^{l} + b^2 r^{l+1}\right]\cancel{Ae^{-br}} + \frac{l(l+1)\hbar^2}{2m}\frac{r^{l+1}}{r^2}\cancel{Ae^{-br}}$$

$$-\left(\frac{1}{4\pi\epsilon_0}\right)Ze^2\frac{r^{l+1}}{r}\cancel{Ae^{-br}} = E r^{l+1}\cancel{Ae^{-br}}$$

Note that $\frac{r^{l+1}}{r^2} = r^{l+1-2} = r^{l-1}$

$\frac{r^{l+1}}{r} = r^{l+1-1} = r^{l}$

Rearranging the Schrodinger equation in powers of r gives

$$\underbrace{\left[-\frac{\hbar^2}{2m}l(l+1) + \frac{l(l+1)\hbar^2}{2m}\right]}_{0} r^{l-1} + \left[\frac{\hbar^2}{2m}(2b)(l+1) - \left(\frac{1}{4\pi\epsilon_0}\right)Ze^2\right]r^{l} + \left[-\frac{\hbar^2}{2m}b^2 - E\right] = 0$$

$$\Rightarrow \left[\frac{\hbar^2 b}{m}(l+1) - \left(\frac{1}{4\pi\epsilon_0}\right)Ze^2\right]r^{l} + \left[-\frac{\hbar^2 b^2}{2m} - E\right] = 0$$

or $A r^{l} + B = 0$

In order for this to be valid for all possible values of r, $A = B = 0$ always.

When $A=0$: $\frac{\hbar^2 b(l+1)}{m} - \left(\frac{1}{4\pi\epsilon_0}\right) Ze^2 = 0$

$$\boxed{b = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Ze^2 m}{\hbar^2 (l+1)} = \frac{Z}{a_0}\cdot\frac{1}{(l+1)}}$$

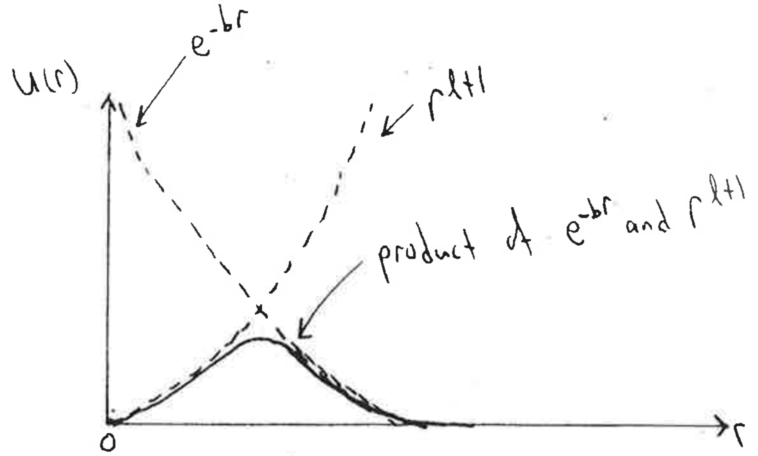
Setting the coefficient $B=0$ gives E:

$$-\frac{\hbar^2 b^2}{2m} - E = 0 \Rightarrow E = -\frac{\hbar^2 b^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{Z}{a_0}\cdot\frac{1}{l+1}\right)^2$$

Thus, $$\boxed{E = -\frac{\hbar^2 Z^2}{2ma_0^2}\cdot\frac{1}{(l+1)^2}}$$

The allowed energies are the Bohr energies if one sets $n = l+1$.

$u(r) = Ar^{l+1}e^{-br} =$



Since the wavefunction looks like a sinusoidal function of length ½ a wavelength (that is, it has 1 hump), we can safely assume that

$$u(r) = Ar^{l+1}e^{-br}$$

is a ground state wavefunction. The particle is in the ground State.