NORTHERN ILLINOIS UNIVERSITY

PHYSICS DEPARTMENT

Physics 283 – Modern Physics   Fall 2023

Quiz #2

INSTRUCTIONS:

(1) No graphic calculators are allowed for this quiz.
(2) Do all your work in the examination blue booklet
(3) You may find the following formulas useful:

$\lambda = h/p$	4.1	Statistical momentum uncertainty	$\Delta p_x = \sqrt{(p_x^2)_{\text{av}} - (p_{x,\text{av}})^2}$	4.4
$a \sin\theta = n\lambda \quad n = 1, 2, 3, \ldots$	4.2	Wave packet (discrete k)	$y(x) = \sum A_i \cos k_i x$	4.5
$\Delta x \Delta\lambda \sim \varepsilon\lambda^2$	4.3	Wave packet (continuous k)	$y(x) = \int A(k) \cos kx \, dk$	4.5
$\Delta f \Delta t \sim \varepsilon$	4.3	Group speed of wave packet	$v_{\text{group}} = \dfrac{d\omega}{dk}$	4.6
$\boxed{\Delta x \Delta p \geq \hbar/2}$	4.4			
$\boxed{\Delta E \Delta t \geq \hbar/2}$	4.4			

Chapter Summary

		Section			Section
Time-independent Schrödinger equation	$-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$	5.3	Infinite potential energy well	$\psi_n(x) = \sqrt{\dfrac{2}{L}} \sin\dfrac{n\pi x}{L}$, $E_n = \dfrac{h^2 n^2}{8mL^2}$ $(n = 1, 2, 3, \ldots)$	5.4
Time-dependent Schrödinger equation	$\Psi(x, t) = \psi(x) e^{-i\omega t}$	5.3	Two-dimensional infinite well	$\psi(x, y) = \dfrac{2}{L} \sin\dfrac{n_x \pi x}{L} \sin\dfrac{n_y \pi y}{L}$, $E = \dfrac{h^2}{8mL^2}(n_x^2 + n_y^2)$	5.4
Probability density	$P(x) = \lvert\psi(x)\rvert^2$	5.3	Simple harmonic oscillator ground state	$\psi(x) = (m\omega_0/\hbar\pi)^{1/4} e^{-(\sqrt{km}/2\hbar)x^2}$	5.5
Normalization condition	$\int_{-\infty}^{+\infty} \lvert\psi(x)\rvert^2 dx = 1$	5.3	Simple harmonic oscillator energies	$E_n = (n + \frac{1}{2})\hbar\omega_0$ $(n = 0, 1, 2, \ldots)$	5.5
Probability in interval x_1 to x_2	$P(x_1 : x_2) = \int_{x_1}^{x_2} \lvert\psi(x)\rvert^2 dx$	5.3	Potential energy step, $E > U_0$	$\psi_0(x < 0) = A \sin k_0 x + B \cos k_0 x$, $\psi_1(x > 0) = C \sin k_1 x + D \cos k_1 x$	5.6
Average or expectation value of $f(x)$	$\left\langle f(x) \right\rangle = \int_{-\infty}^{+\infty} \lvert\psi(x)\rvert^2 f(x) \, dx$	5.3	Potential energy step, $E < U_0$	$\psi_0(x < 0) = A \sin k_0 x + B \cos k_0 x$, $\psi_1(x > 0) = Ce^{k_1 x} + De^{-k_1 x}$	5.6
Constant potential energy, $E > U_0$	$\psi(x) = A \sin kx + B \cos kx$, $k = \sqrt{2m(E - U_0)/\hbar^2}$	5.4			
Constant potential energy, $E < U_0$	$\psi(x) = Ae^{k'x} + Be^{-k'x}$, $k' = \sqrt{2m(U_0 - E)/\hbar^2}$	5.4			

Hydrogen quantum numbers	$n = 1, 2, 3, \ldots$ $l = 0, 1, 2, \ldots, n-1$ $m_l = 0, \pm 1, \pm 2, \ldots, \pm l$	7.3
Hydrogen energy levels	$E_n = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n^2}$	7.3
Hydrogen wave functions	$\psi_{n,l,m_l}(r,\theta,\phi) = R_{n,l}(r)\Theta_{l,m_l}(\theta)\Phi_{m_l}(\phi)$	7.3
Radial probability density	$P(r) = r^2 \lvert R_{n,l}(r)\rvert^2$	7.4

TABLE 7.1 Some Hydrogen Atom Wave Functions

n	l	m_l	$R(r)$	$\Theta(\theta)$	$\Phi(\phi)$
1	0	0	$\frac{2}{a_0^{3/2}}e^{-r/a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
2	0	0	$\frac{1}{(2a_0)^{3/2}}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
2	1	0	$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$	$\sqrt{\frac{3}{2}}\cos\theta$	$\frac{1}{\sqrt{2\pi}}$
2	1	± 1	$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$	$\mp\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$
3	0	0	$\frac{2}{(3a_0)^{3/2}}\left(1 - \frac{2r}{3a_0} + \frac{2r^2}{27a_0^2}\right)e^{-r/3a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
3	1	0	$\frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0} - \frac{r^2}{6a_0^2}\right)e^{-r/3a_0}$	$\sqrt{\frac{3}{2}}\cos\theta$	$\frac{1}{\sqrt{2\pi}}$
3	1	± 1	$\frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0} - \frac{r^2}{6a_0^2}\right)e^{-r/3a_0}$	$\mp\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$
3	2	0	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\sqrt{\frac{5}{8}}(3\cos^2\theta - 1)$	$\frac{1}{\sqrt{2\pi}}$
3	2	± 1	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\mp\sqrt{\frac{15}{4}}\sin\theta\cos\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$
3	2	± 2	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm 2i\phi}$

$$a \sin\theta = m\lambda \text{ for } m = \pm1, \pm2, \pm3, \ldots$$

$$\beta = \frac{\phi}{2} = \frac{\pi a \sin\theta}{\lambda}$$

$$E = N\Delta E_0 \frac{\sin\beta}{\beta}$$

$$I = I_0\left(\frac{\sin\beta}{\beta}\right)^2$$

$$\theta = 1.22\frac{\lambda}{D}$$

$$m\lambda = 2d\sin\theta,\ m = 1, 2, 3\ldots$$

$$c = 2.99792458 \times 10^8 \text{ m/s} \approx 3.00 \times 10^8 \text{ m/s}$$

$$n = \frac{c}{v}$$

$$\theta_r = \theta_i$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \quad \text{for } n_1 > n_2$$

$$I = I_0 \cos^2\theta$$

$$\tan\theta_b = \frac{n_2}{n_1}$$

Chapter Summary

		Section
Scattering impact parameter	$b = \frac{zZ}{2K}\frac{e^2}{4\pi\varepsilon_0}\cot\frac{1}{2}\theta$	6.3
Fraction scattered at angles $> \theta$	$f_{>\theta} = nt\pi b^2$	6.3
Rutherford scattering formula	$N(\theta) = \frac{nt}{4r^2}\left(\frac{zZ}{2K}\right)^2\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\frac{1}{\sin^4\frac{1}{2}\theta}$	6.3
Distance of closest approach	$d = \frac{1}{4\pi\varepsilon_0}\frac{zZe^2}{K}$	6.3
Balmer formula	$\lambda = (364.5\text{ nm})\frac{n^2}{n^2-4}$ $(n = 3, 4, 5, \ldots)$	6.4
Radii of Bohr orbits in hydrogen	$r_n = \frac{4\pi\varepsilon_0\hbar^2}{me^2}n^2 = a_0n^2$ $(n = 1, 2, 3, \ldots)$	6.5
Energies of Bohr orbits in hydrogen	$E_n = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n^2} = \frac{-13.60\text{ eV}}{n^2}$ $(n = 1, 2, 3, \ldots)$	6.5

		Section
Excitation energy of level n	$E_n - E_1$	6.5
Binding (or ionization) energy of level n	$\lvert E_n \rvert$	6.5
Hydrogen wavelengths in Bohr model	$\lambda = \frac{64\pi^3\varepsilon_0^2\hbar^3c}{me^4}\left(\frac{n_1^2n_2^2}{n_1^2-n_2^2}\right) = \frac{1}{R_\infty}\left(\frac{n_1^2n_2^2}{n_1^2-n_2^2}\right)$	6.5
Single-electron atoms with $Z > 1$	$r_n = \frac{a_0n^2}{Z}$, $E_n = -(13.60\text{ eV})\frac{Z^2}{n^2}$	6.5
Reduced mass of proton-electron system	$m = \frac{m_em_p}{m_e + m_p}$	6.8

Euler Identies: $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$c = 3\times 10^{8} \text{ meters/sec}$$

$$e = 1.602\times 10^{-19}\,\mathrm{C}$$

$$h = 6.626\times 10^{-34}\,\mathrm{J}\cdot\mathrm{sec}$$

$$hc = 1240\,\mathrm{eV}\cdot\mathrm{nm}$$

$$1\ \mathring{\mathrm{A}} = 10^{-10} \text{ meters}$$

$$1\,\mathrm{eV} = 1.602\times 10^{-19}\,\mathrm{J}$$

Show all your work in the solution of each problem.

Writing down a number with no work shown gets no credit.

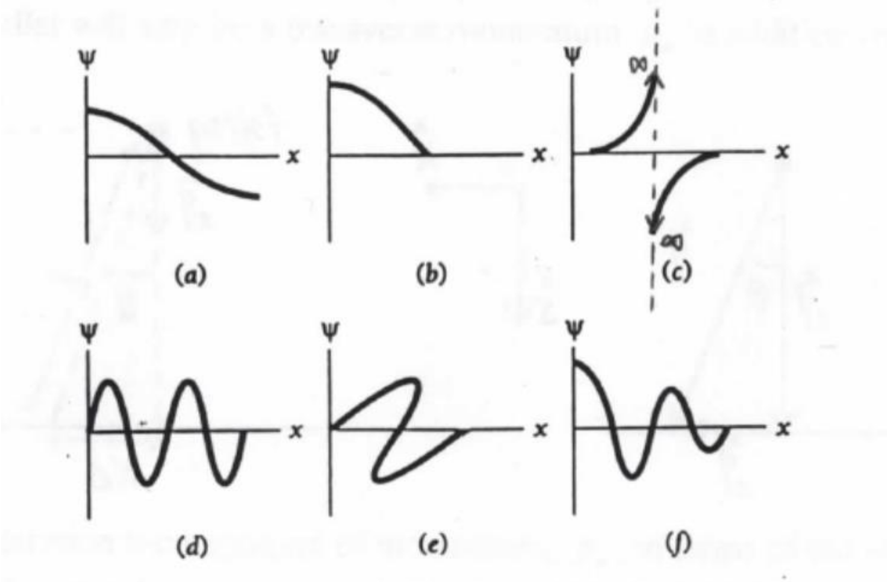
Put each problem on a separate page in the blue booklet.

Problem 1 (25 points)

Common table salt is composed mainly of NaCl crystals. In a NaCl crystal, there is a family of planes 0.252 nm apart. If the first-order maximum is observed at an incidence angle of 18.1°, what is the wavelength of the x-ray scattering from this crystal?

Problem 2 (25 points)

Which of the wave functions cannot have physical significance in the interval shown? For those that do not, explain why (very briefly).



Which of the following wave functions below cannot be solutions of Schrodinger's equation for all values of x? Why not?

(g) $\psi(x) = x^2$

(h) $\psi(x) = e^{-|x|}$

(i) $\psi(x) = e^{-x}$

(j) $\psi(x) = e^{-x^2}$

Problem 3 (25 points)

$$U(x)=\begin{cases}0 & \text{for} \quad 0\le x\le L\\ \infty & \text{for} \quad x<0, x>L\end{cases}$$

find the solution to the Schrodinger equation in the three regions:

(a) Region I: $x<0$

(b) Region III: $x>L$

(c) Region II: $0\le x\le L$.
Assume the general solution is $\psi(x)=Ae^{ikx}+Be^{-ikx}$ and apply the condition that $\psi(x)$ is continuous across any boundary.

(d) Find the allowed energies in the infinite square well potential (show all steps to the solution).

(e) Draw the energy level diagram for the 1st 4 allowed energies.

(f) Draw the wavefunctions for the 1st 4 energy states.

(g) Normalize the wavefunction (show all steps to the solution).

Problem 4 (25 points)

The Schrodinger equation for a hydrogen-like atom can be written in the form

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\frac{\ell(\ell+1)\hbar^2}{2mr^2}u-\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}u=Eu$$

Show that

$$u(r)=Ar^{\ell+1}e^{-br}$$

is a ground state solution to the Schrodinger equation above by finding the value of b in terms of a_0, Z, and ℓ ($a_0=4\pi\varepsilon_0\hbar^2/me^2$ = Bohr radius). Find the total energy, E, for this ground state. How does it differ from the typical Bohr energy

$$E_n=-\frac{1}{(4\pi\varepsilon_0)^2}\frac{mZ^2e^4}{2\hbar^2}\frac{1}{n^2}=-\frac{\hbar^2}{2ma_0^2}\frac{1}{n^2}$$