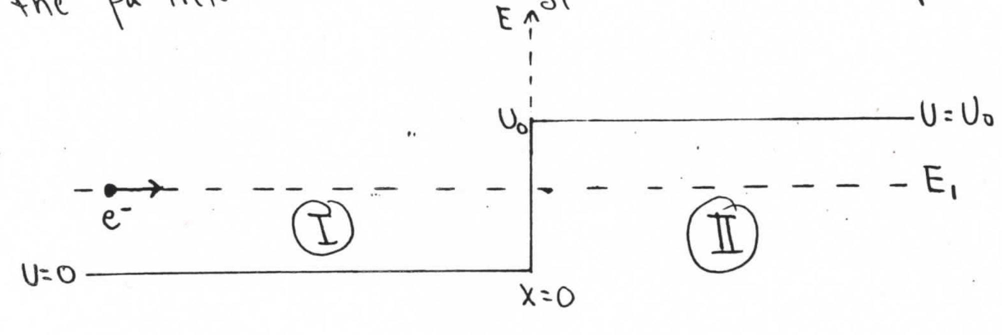
Physics 360 - Lecture Notes # 2

(III) Scattering at a Step Potential

Let's look at the problem of a particle scattering from a step when the particle has less energy than the step height:



Since the e^- does not have enough energy to make it over the step, then Region (II) is a classically forbidden region. Let's write down and solve the Schrodinger equation in regions (I) + (II).

In Region (I):

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_I}{dx^2} + \underbrace{U(x)}_{0}\Psi_I = E\Psi_I \Rightarrow \frac{d^2\Psi_I}{dx^2} = \frac{-2mE}{\hbar^2}\Psi_I = -K_I^2\Psi_I$$

The solution should be quite familiar by now:

$$\Psi_I(x) = Ae^{iK_Ix} + Be^{-iK_Ix} \quad , \quad K_I = \sqrt{\frac{2mE}{\hbar^2}}$$

In Region (II):

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_{II}}{dx^2} + U(x)\Psi_{II} = E\Psi_{II} \Rightarrow \frac{d^2\Psi_{II}}{dx^2} = -\frac{2m}{\hbar^2}(E-U_0)\Psi_{II} = -\alpha_{II}^2\Psi_{II}$$

Note that $E-U_0 < 0$, then $\alpha_{II} = \sqrt{\frac{2m}{\hbar^2}(E-U_0)} = \sqrt{\frac{2m}{\hbar^2}(-1)(U_0-E)}$

$$= \sqrt{-1}\sqrt{\frac{2m}{\hbar^2}(U_0-E)}$$

$$= i\sqrt{\frac{2m}{\hbar^2}(U_0-E)}$$

$$= iK_{II}$$

The solution is

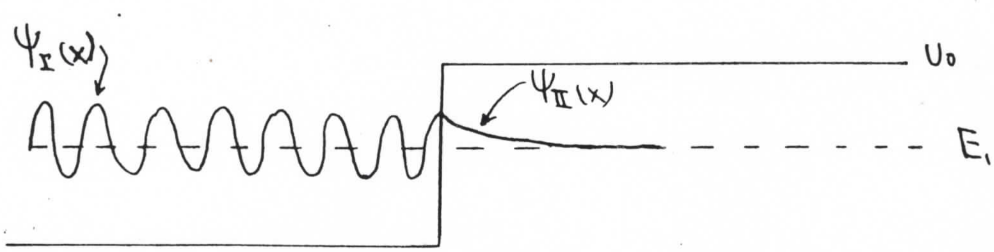
$$\psi_{II}(x) = Ce^{-i\alpha_{II}} + De^{i\alpha_{II}} = Ce^{-i(iK_{II})x} + De^{i(iK_{II})x}$$
$$= Ce^{K_{II}x} + De^{-K_{II}x}$$

which are growing and dying exponentials.

But, $\psi_{II}(x)$ must always be finite, thus the term $Ce^{K_{II}x}$ is not allowed since $\lim_{x\to\infty} Ce^{K_{II}x} \to \infty$. Thus, to prevent $\psi_{II}(x)$ from becoming infinite, we must set $C = 0$. Thus, in Region II:

$$\psi_{II}(x) = De^{-K_{II}x} \quad \text{(a dying exponential)}$$

Surprisingly, there is a finite probability for the particle to exist in the classically forbidden region. A plot of ψ_I, ψ_{II} follows:



The depth of penetration of the particle into the classically forbidden region is characterized by a (1/e) length, that is, the value of x at which the probability drops by 1/e:

$$\text{Probability} = |\psi_{II}|^2 = e^{-2K_{II}x_e} = \frac{1}{e} = e^{-1}$$

$$2K_{II}x_e = 1$$

$$x_e = \frac{1}{2K_{II}} = \frac{\hbar}{2\sqrt{2m(U_0-E)}} = \text{penetration depth}$$

Such behavior allows particles to penetrate barriers, as we will see in the next scattering problem.