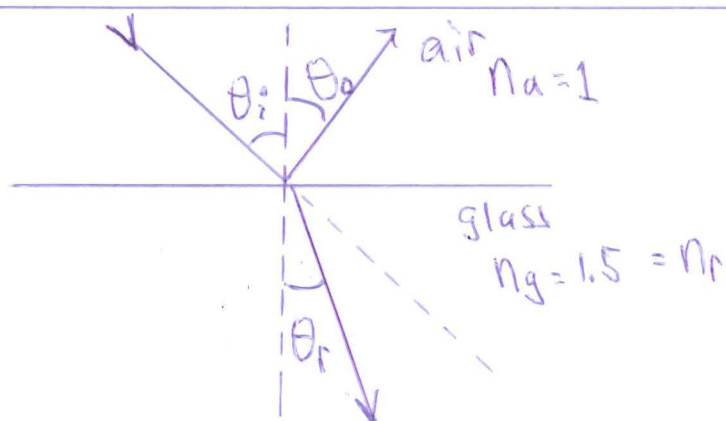


Physics 283

Quiz # 2 Solutions

①



Angle of reflection = $\theta_o = \theta_i = 35^\circ$

To find angle of refraction, use Snell's Law:

$$n_i \sin \theta_i = n_r \sin \theta_r$$

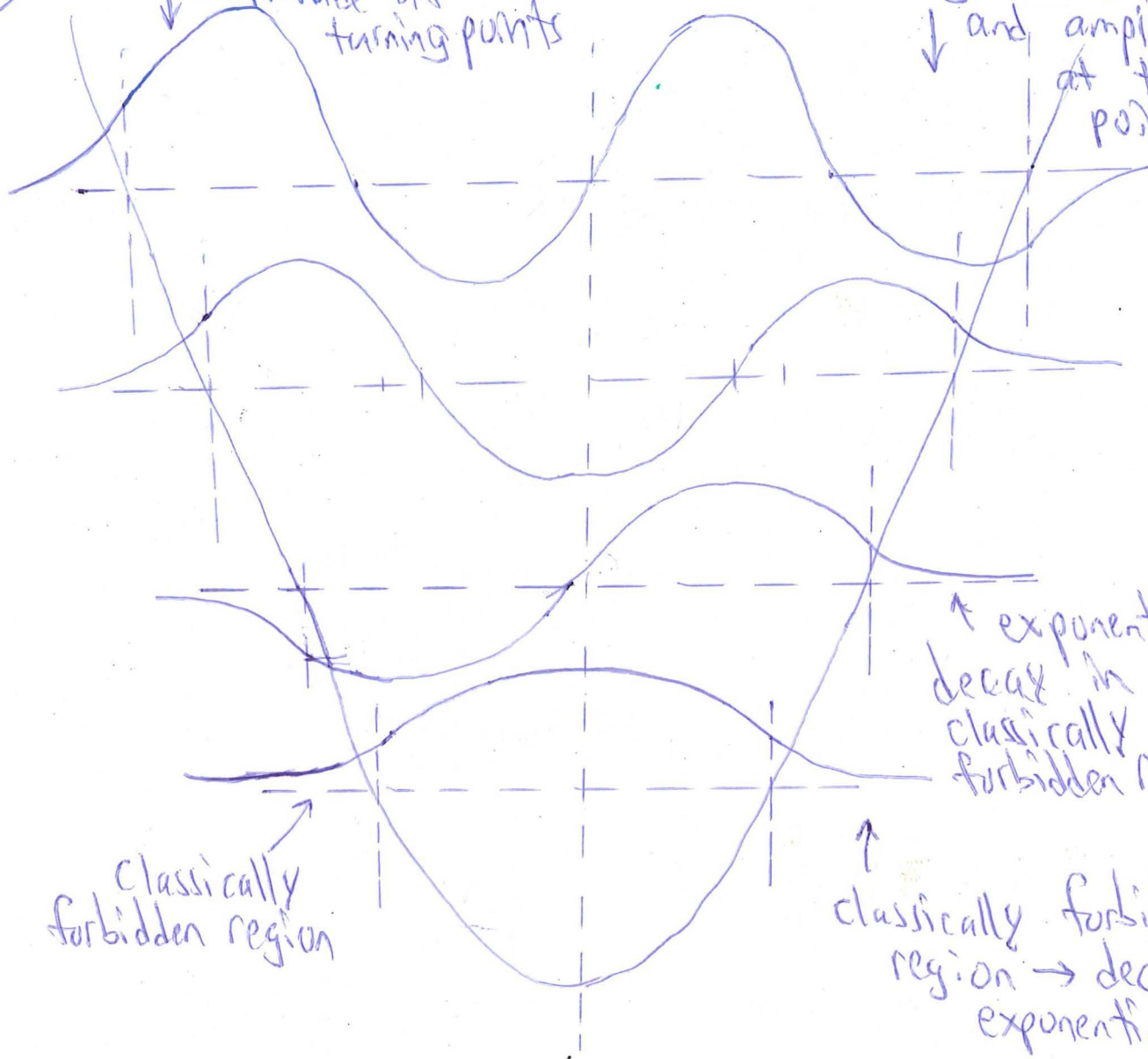
$$\sin \theta_r = \frac{n_i}{n_r} \sin \theta_i = \frac{1}{1.5} \sin 35^\circ$$

$$\theta_r = \sin^{-1} \left(\frac{\sin 35^\circ}{1.5} \right) = 22.48^\circ$$

②

longer wavelength and larger
↓ amplitude at
turning points

longer wavelengths
↓ and amplitudes
at turning
points



↑ exponential
decay in
classically
forbidden region

↑
classically
forbidden region

↑
classically forbidden
region → decaying
exponential

$$\textcircled{3} \quad \psi_{210} = \frac{1}{4\sqrt{2\pi}} \left(\frac{z}{a_0} \right)^{3/2} \frac{zr}{a_0} e^{-zr/2a_0} \cos\theta$$

(a) The probability density is

$$P(r) = r^2 |\psi_{nlm}|^2$$

It is maximized when $\frac{dP}{dr} = 0$

$$\frac{dP(r)}{dr} = \frac{d}{dr} \left[\frac{1}{4\sqrt{2\pi}} \left(\frac{z}{a_0} \right)^{3/2} \frac{z}{a_0} \cos\theta \right]^2 r^4 e^{-zr/a_0}$$

$$= (\text{constants}) \frac{d}{dr} (r^4 e^{-zr/a_0})$$

$$= (\text{const.}) \left[4r^3 e^{-zr/a_0} - \frac{z}{a_0} r^4 e^{-zr/a_0} \right]$$

$$= (\text{const.}) r^3 e^{-zr/a_0} \left[4 - \frac{zr}{a_0} \right]$$

The maximum occurs when : $4 - \frac{zr}{a_0} = 0$

$$\boxed{r = \frac{4a_0}{z}}$$

$$(b) \langle r \rangle = \int_V r |\psi_{n\ell m}|^2 dV = \int_V r |\psi_{n\ell m}|^2 r^2 \sin\theta dr d\theta d\phi$$

$$\begin{aligned} \langle r \rangle &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} r \left[\frac{1}{4\sqrt{2}\pi} \left(\frac{z}{a_0} \right)^{3/2} \frac{z}{a_0} \right]^2 r^2 e^{-2zr/a_0} \cos^2\theta r^2 \sin\theta dr d\theta d\phi \\ &= \frac{1}{16(2\pi)} \left(\frac{z}{a_0} \right)^3 \left(\frac{z}{a_0} \right)^2 \int_0^{\infty} r^5 e^{-zr/a_0} dr \int_0^{\pi} \cos^2\theta \sin\theta d\theta \int_0^{2\pi} d\phi \end{aligned}$$

$$\int_0^{2\pi} d\phi = 2\pi$$

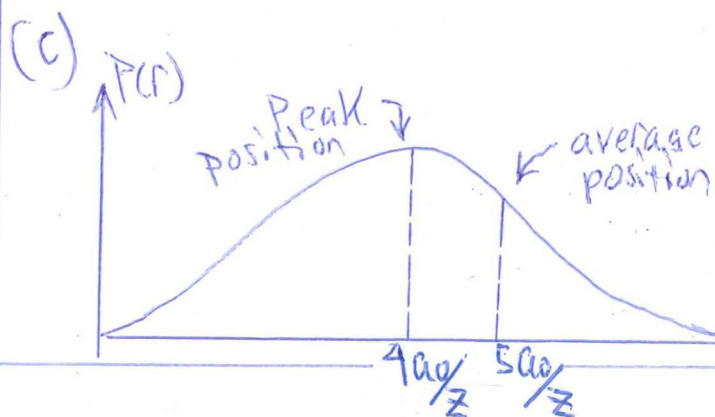
$$\int_0^{\pi} \cos^2\theta \sin\theta d\theta = \left[-\frac{1}{3} \cos^3\theta \right]_0^{\pi} = -\frac{1}{3} [\cos^3\pi - \cos^3(0)] = -\frac{1}{3} (-1 - 1) = \frac{2}{3}$$

note: $\frac{d}{d\theta} \left(-\frac{1}{3} \cos^3\theta \right) = -\frac{2}{3} \cos^2\theta (-\sin\theta) = \cos^2\theta \sin\theta$

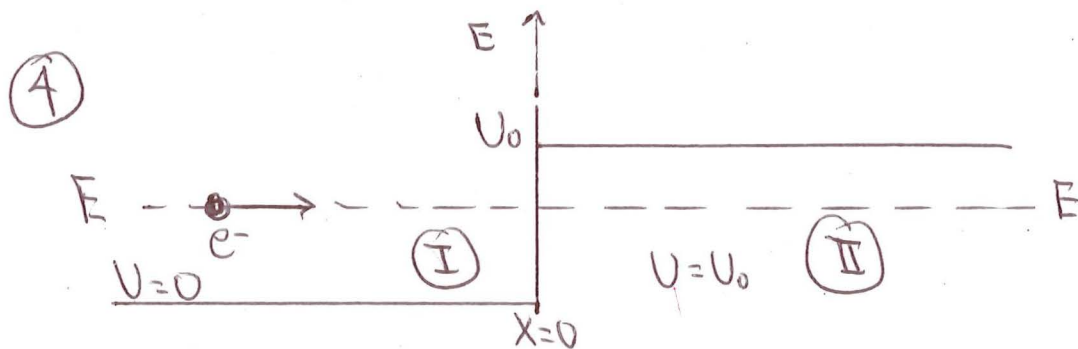
Fundamental theorem of Calculus

$$\int_0^{\infty} r^5 e^{-zr/a_0} dr = \frac{5!}{(z/a_0)^6} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \left(\frac{a_0}{z} \right)^6$$

$$\langle r \rangle = \frac{1}{16\pi} \left(\frac{z}{a_0} \right)^5 (2\pi) \left(\frac{2}{3} \right) (5 \cdot 4 \cdot 3 \cdot 2) \left(\frac{a_0}{z} \right)^6 = 5 \left(\frac{a_0}{z} \right)$$



The most probable position of an e^- is at $r = 4a_0/z$. But, on average, the e^- spends most of its time at positions greater than $4a_0/z$ because of the long tail that extends to $r \rightarrow \infty$ of the $P(r)$ distribution



(a) $\Psi_I(x) = Ae^{ik_I x} + Be^{-ik_I x}$, $k_I = \sqrt{\frac{2mE}{\hbar^2}}$

(b) $\Psi_{II}(x) = \cancel{A}e^{-\alpha_{II} x} + De^{+\alpha_{II} x}$, $\alpha_{II} = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$

(c) $D = 0$ since probability cannot ~~grow~~ to infinity
 $P(x) \sim D^2 e^{2\alpha_{II} x} \rightarrow \infty$ as $x \rightarrow \infty$

(d) thus, must set $D = 0$ to avoid this

⑤ Continuity of $\Psi(x)$:

$$\Psi_I(x=0) = \Psi_{II}(x=0)$$

$$A + B = C$$

Continuity of the slope:

$$\left. \frac{d\Psi_I}{dx} \right|_{x=0} = \left. \frac{d\Psi_{II}}{dx} \right|_{x=0}$$

$$ik_I A - ik_I B = -\alpha_{II} C$$

Plug C into lower eqn.

(e) $R = \frac{k_I |B|^2}{k_I |A|^2}$

$$\rightarrow ik_I (A - B) = -\alpha_{II} (A + B)$$

$$-ik_I B + \alpha_{II} B = -ik_I A - \alpha_{II} A$$

$$B = \frac{-(\alpha_{II} + ik_I)}{(\alpha_{II} - ik_I)} A$$

$$R = \frac{|B|^2}{|A|^2} = \left| \frac{\alpha_{II} + ik_I}{\alpha_{II} - ik_I} \right|^2$$

Complex
conjugate
↓

$$(f) R = \frac{(\alpha_{II} + ik_I)(\alpha_{II} - ik_I)}{(\alpha_{II} - ik_I)(\alpha_{II} + ik_I)}$$

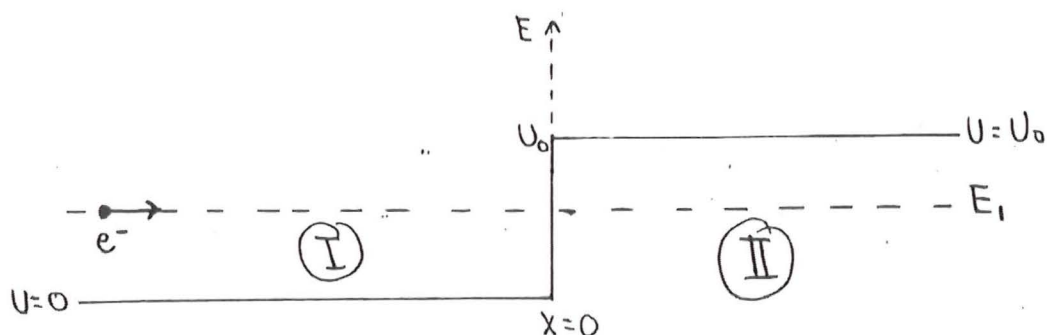
$$\text{Note: } |a|^2 = a \cdot a^*$$

$$= \frac{\alpha_{II}^2 + k_I^2}{\alpha_{II}^2 + k_I^2} = 1$$

$$\Rightarrow \boxed{R = 1}$$

The particle is totally reflected at the step.

Problem 4 (25 points)



$$U(x) = \begin{cases} 0 & \text{for } x < 0 \\ U_0 & \text{for } x > 0 \end{cases}$$

The electron has an energy less than the step height: $E < U_0$.

For this problem: *Use only complex exponentials for wavefunctions*

- Write down the wave function, $\psi_I(x)$, for Region I. Let A be the amplitude of the *incident wave* and B be the amplitude of the *reflected wave*.
- Write down the wave function, $\psi_{II}(x)$, for region II.
- Which coefficient(s) in Part (a) & (b) should be set to zero? Why?
- Use the continuity boundary conditions for $\psi(x)$ and its slope at $x = 0$ to find two relationships among the coefficients.
- Evaluate the reflection coefficient:

$$R = \frac{\text{reflected probability flux}}{\text{incident probability flux}} = \frac{k_I |B|^2}{k_I |A|^2}$$

- Show that the particle is totally reflected at the step by showing that $R = 1$.