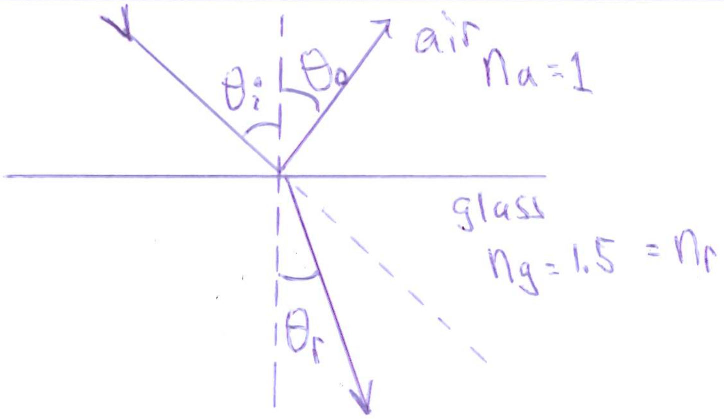
Physics 283

Quiz # 2 Solutions

1



Angle of reflection $= \theta_o = \theta_i = 35°$

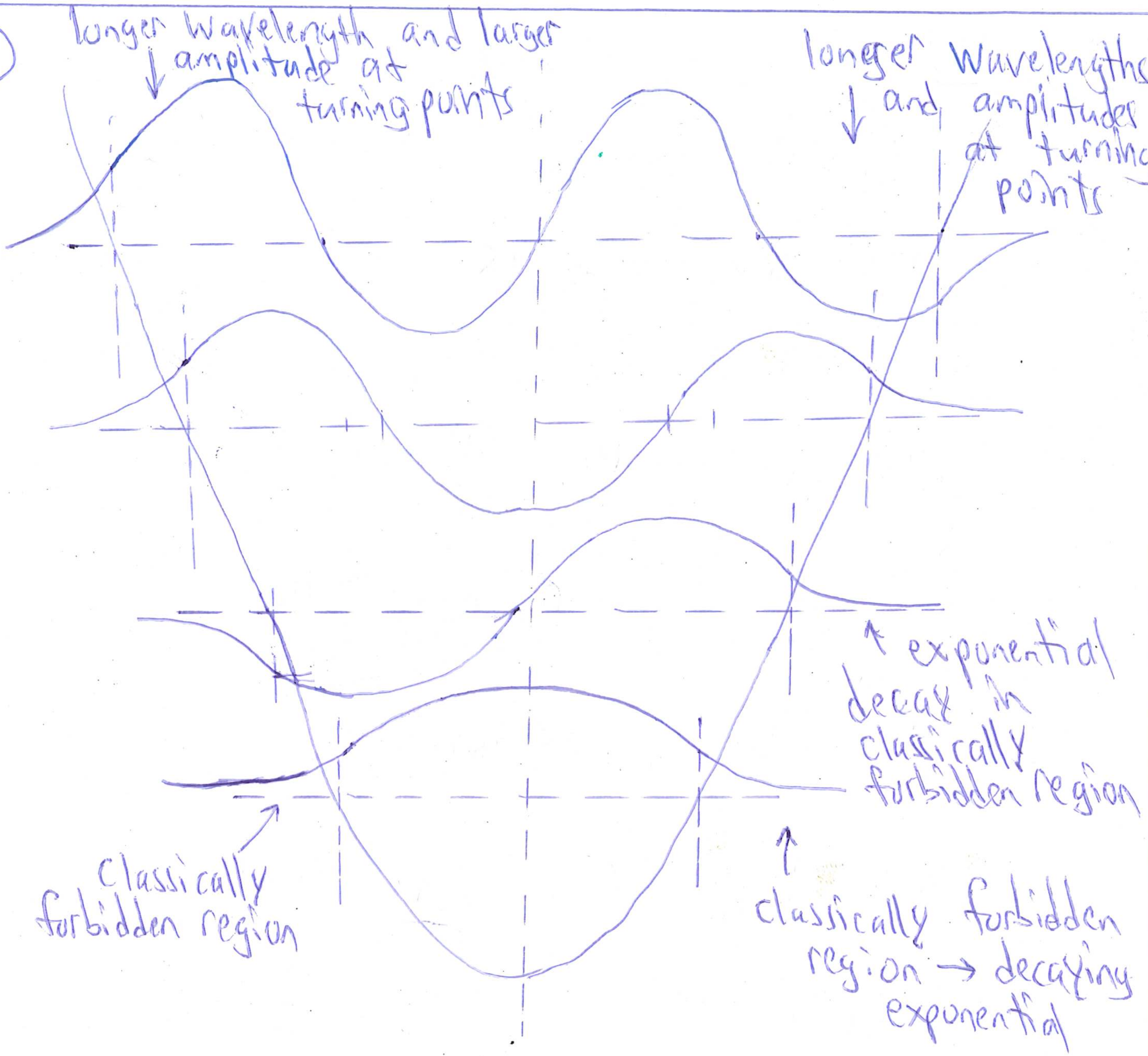
To find angle of refraction, use Snell's Law:

$$n_i \sin\theta_i = n_r \sin\theta_r$$

$$\sin\theta_r = \frac{n_i}{n_r}\sin\theta_i = \frac{1}{1.5}\sin 35°$$

$$\theta_r = \sin^{-1}\left(\frac{\sin 35°}{1.5}\right) = 22.48°$$

2


longer wavelength and larger amplitude at turning points
longer wavelengths and amplitudes at turning points
exponential decay in classically forbidden region
Classically forbidden region
classically forbidden region → decaying exponential

③ $\psi_{210} = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\frac{Zr}{a_0}e^{-Zr/2a_0}\cos\theta$

(a) The probability density is

$$P(r) = r^2 |\psi_{n\ell m}|^2$$

It is maximized when $\frac{dP}{dr} = 0$

$$\frac{dP(r)}{dr} = \frac{d}{dr}\left[\frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\frac{Z}{a_0}\cos\theta\right]^2 r^4 e^{-Zr/a_0}$$

$$= (\text{constants})\frac{d}{dr}\left(r^4 e^{-Zr/a_0}\right)$$

$$= (\text{const.})\left[4r^3 e^{-Zr/a_0} - \frac{Z}{a_0}r^4 e^{-Zr/a_0}\right]$$

$$= (\text{const.})\, r^3 e^{-Zr/a_0}\left[4 - \frac{Zr}{a_0}\right]$$

The maximum occurs when : $4 - \frac{Zr}{a_0} = 0$

$$\boxed{r = \frac{4a_0}{Z}}$$

(b) $\langle r \rangle = \int_V r |\Psi_{n\ell m}|^2 dV = \int_V r |\Psi_{n\ell m}|^2 r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$\langle r \rangle = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} r \left[\frac{1}{4\sqrt{2\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \frac{Z}{a_0} \right]^2 r^2 e^{-2Zr/2a_0} \cos^2\theta \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{16(2\pi)} \left(\frac{Z}{a_0}\right)^3 \left(\frac{Z}{a_0}\right)^2 \int_0^{\infty} r^5 e^{-Zr/a_0} dr \int_{\theta=0}^{\pi} \cos^2\theta \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$\int_{\phi=0}^{2\pi} d\phi = 2\pi$$

$$\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = \underbrace{-\frac{1}{3}\cos^3\theta}\Big]_0^{\pi} = -\frac{1}{3}\left[\cos^3\pi - \cos^3(0)\right] = -\frac{1}{3}(-1-1) = \frac{2}{3}$$

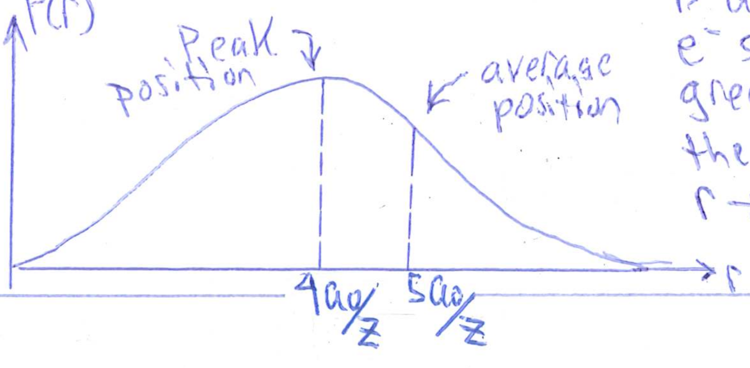
note: $\frac{d}{d\theta}\left(-\frac{1}{3}\cos^3\theta\right) = -\frac{3}{3}\cos^2\theta(-\sin\theta) = \cos^2\theta \sin\theta$

fundamental theorem of Calculus

$$\int_0^{\infty} r^5 e^{-Zr/a_0} dr = \frac{5!}{(Z/a_0)^6} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \left(\frac{a_0}{Z}\right)^6$$

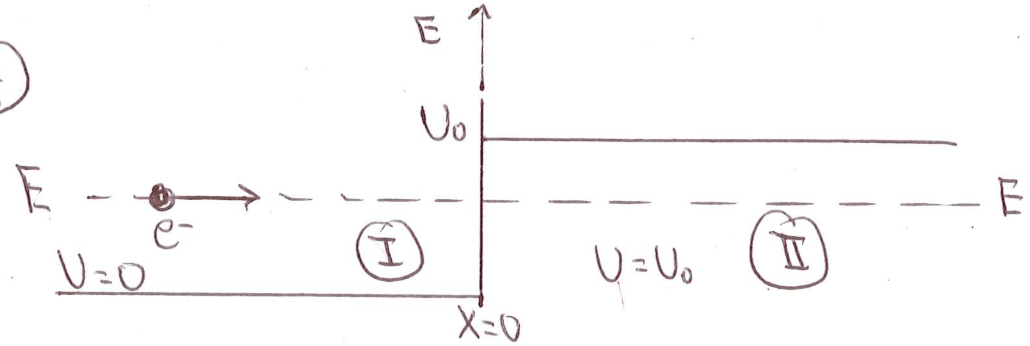
$$\langle r \rangle = \frac{1}{16\pi} \left(\frac{Z}{a_0}\right)^5 (2\pi)\left(\frac{2}{3}\right)(5 \cdot 4 \cdot 3 \cdot 2)\left(\frac{a_0}{Z}\right)^6 = 5\left(\frac{a_0}{Z}\right)$$

(c)



The most probable position of an e^- is at $r = 4a_0/Z$. But, on average, the e^- spends most of its time at positions greater than $4a_0/Z$ because of the long tail that extends to $r \to \infty$ of the P(r) distribution

(4)



(a) $\Psi_I(x) = Ae^{iK_I x} + Be^{-iK_I x}$, $K_I = \sqrt{\frac{2mE}{\hbar^2}}$

(b) $\Psi_{II}(x) = Ae^{-\alpha_{II} x} + De^{+\alpha_{II} x}$, $\alpha_{II} = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$

(c) $D = 0$ since probability cannot ~~equal~~ to infinity

$P(x) \sim D^2 e^{2\alpha_{II} x} \rightarrow \infty$ as $x \rightarrow \infty$

(d)

thus, must set $D = 0$ to avoid this

(d) Continuity of $\Psi(x)$:

$\Psi_I(x=0) = \Psi_{II}(x=0)$

$$\boxed{A + B = C}$$

Continuity of the slope:

$$\left.\frac{d\Psi_I}{dx}\right]_{x=0} = \left.\frac{d\Psi_{II}}{dx}\right]_{x=0}$$

$$\boxed{iK_I A - iK_I B = -\alpha_{II} C}$$

Plug C into lower eqn.

(e) $R = \frac{K_I |B|^2}{K_I |A|^2}$

$\rightarrow iK_I(A - B) = -\alpha_{II}(A + B)$

$$-ik_I B + \alpha_{II} B = -ik_I A - \alpha_{II} A$$

$$B = -\frac{(\alpha_{II} + ik_I)}{(\alpha_{II} - ik_I)} A$$

$$R = \frac{|B|^2}{|A|^2} = \left| \frac{\alpha_{II} + ik_I}{\alpha_{II} - ik_I} \right|^2$$

Complex conjugate

Note: $|a|^2 = a \cdot a^*$

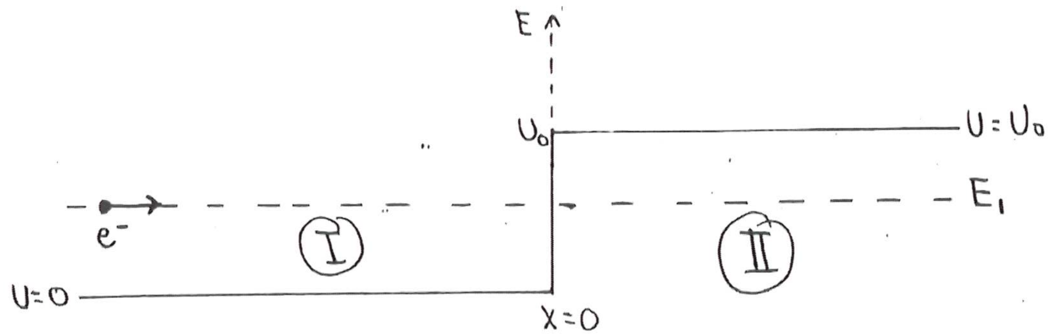
(f) $$R = \frac{(\alpha_{II} + ik_I)(\alpha_{II} - ik_I)}{(\alpha_{II} - ik_I)(\alpha_{II} + ik_I)}$$

$$= \frac{\alpha_{II}^2 + k_I^2}{\alpha_{II}^2 + k_I^2} = 1$$

$\Rightarrow$ $\boxed{R = 1}$

The particle is totally reflected at the step.

Problem 4 (25 points)



$$U(x)=\begin{cases}0 & \text{for} \quad x<0\\ U_0 & \text{for} \quad x>0\end{cases}$$

The electron has an energy less than the step height: $E < U_0$.

For this problem: ***Use only complex exponentials for wavefunctions***

(a) Write down the wave function, $\psi_I(x)$, for Region I. Let A be the amplitude of the *incident wave* and B be the amplitude of the *reflected wave*.

(b) Write down the wave function, $\psi_{II}(x)$, for region II.

(c) Which coefficient(s) in Part (a) & (b) should be set to zero? Why?

(d) Use the continuity boundary conditions for $\psi(x)$ and its slope at $x = 0$ to find two relationships among the coefficients.

(e) Evaluate the reflection coefficient:

$$R = \frac{\text{reflected probalility flux}}{\text{incident probability flux}} = \frac{k_I |B|^2}{k_I |A|^2}$$

(f) Show that the particle is totally reflected at the step by showing that $R = 1$.