Momentum

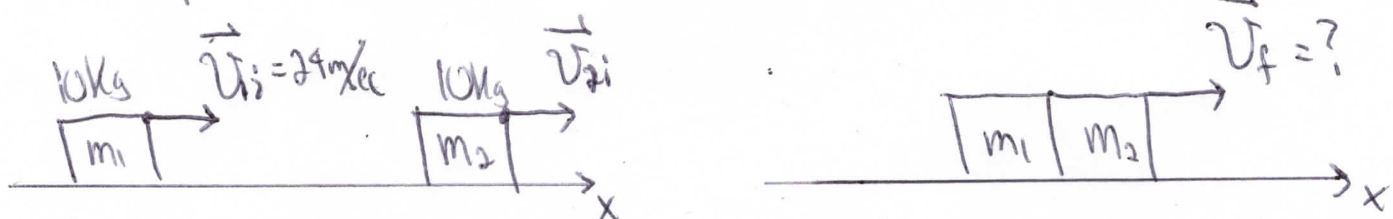
Unit #8a

$$\text{momentum} = \vec{p} = m\vec{v}$$

Conservation of Momentum

$$\vec{p}_{Total}^{\,initial} = \vec{p}_{Total}^{\,final}$$

Example: Inelastic Collision of two objects



just before collision

just after collision → they stick together

$$\left(\vec{p}_{Total}^{\,initial}\right)_x = \left(\vec{p}_{Total}^{\,final}\right)_x \qquad \left(\text{x-components of momentum}\right)$$

$$m_1 v_{1i} + m_2 \cancel{v_{2i}}^{\,0} = m_1 v_{1f} + m_2 v_{2f}$$

$v_{1i} \downarrow$ 24 m/sec, $\quad v_{2i} \downarrow$ 0, $\quad v_{1f} \downarrow$ v_f, $\quad v_{2f} \downarrow$ v_f $\quad$ (let $v_{1f} = v_{2f} = v_f$)

$$m_1 v_{1i} = m_1 v_f + m_2 v_f$$

$$m_1 v_{1i} = (m_1 + m_2) v_f$$

$$v_f = \left(\frac{m_1}{m_1 + m_2}\right) v_{1i} = \left(\frac{10\,Kg}{10\,Kg + 10\,Kg}\right)(24\,m/sec) = 12\,m/sec$$

$$\approx 10\,m/sec$$

Since friction is involved, thermal energy is generated → the objects will heat up. We can find out how much thermal energy is generated using the conservation of energy relation:

$$E_{Total}^{initial} = E_{Total}^{final}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \underset{0}{\cancel{\frac{1}{2} m_2 v_{2i}^2}} = \frac{1}{2} m_1 \underset{v_f}{v_{1f}^2} + \frac{1}{2} m_2 \underset{v_f}{v_{2f}^2} + \Delta E_{Thermal}$$

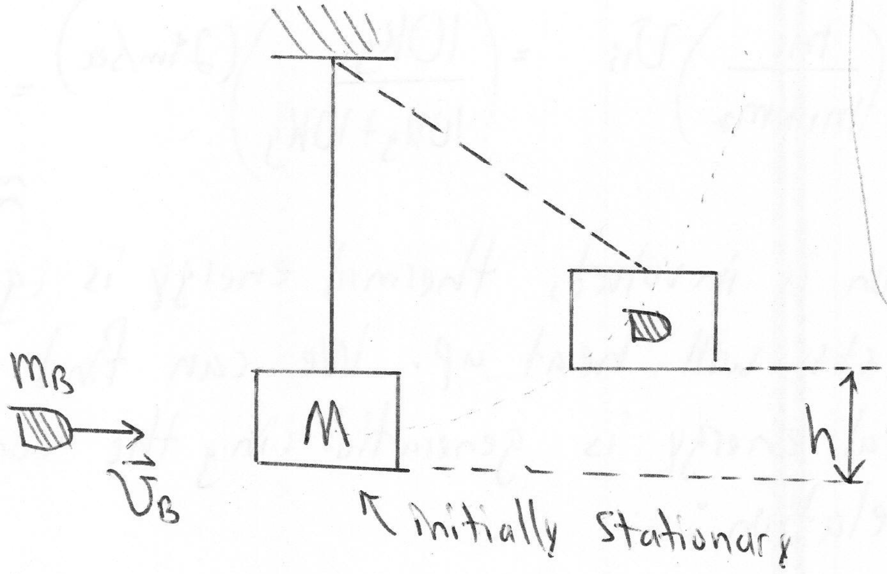
$$\Delta E_{Thermal} = \frac{1}{2} m_1 v_{1i}^2 - \frac{1}{2}(m_1 + m_2) v_f^2$$

$$= \frac{1}{2}\left[(10\,Kg)(24\,m/sec)^2 - (10 + 10)\,Kg\,(12\,m/sec)^2\right]$$

$$= 1440\,J \approx 1000\,J$$ of heat is generated

Example: Ballistic Pendulum

What is the speed of the bullet just prior to the collision?



Let's try to solve this using conservation of energy. Since the bullet stops in the block, frictional energy was used to slow down the bullet and deform the block:

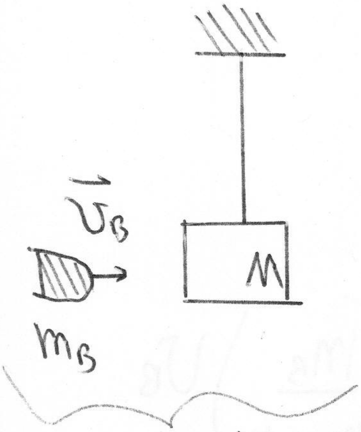
$$E_{Total}^{initial} = E_{Total}^{final}$$

$$\underbrace{\frac{1}{2} m_B v_B^2}_{\text{Kinetic energy}} = \underbrace{(m_B + M)gh}_{\text{potential energy}} + \underbrace{\Delta E_{thermal}}_{\text{heat}}$$

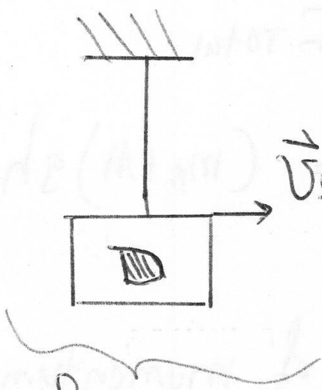
Kinetic energy → is tranformed into potential energy and heat

Since we do not know what $\Delta E_{thermal}$ is, this relationship is not very useful. Let's then apply conservation of momentum. Even though we have external forces (friction), we will take advantage of the

Circumstance that the collision is very brief:



initial
just before
collision



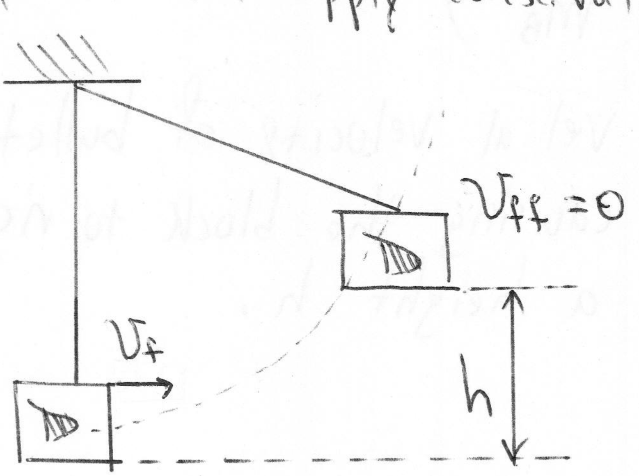
final
just after collision

$\Rightarrow$ the collision is so brief that the block has not moved upward yet

$$\left(\vec{P}^{\,initial}_{Total}\right)_x = \left(\vec{P}^{\,final}_{Total}\right)_x$$

$$m_B v_B = (m_B + M)\, v_f$$

Then we can apply conservation of energy (after the collision)



at the point of maximum height, the block stops with zero velocity

$$E^{initial}_{Total} = E^{final}_{Total}$$

$$\frac{1}{2}(m_B + M) v_f^2 = (m_B + M) g h$$

From conservation of momentum: $v_f = \left(\frac{m_B}{m_B + M}\right) v_B$

From conservation of energy: $\frac{1}{2} v_f^2 = g h$

$$v_f = \sqrt{2gh}$$

Then,

$$\left(\frac{m_B}{m_B + M}\right) v_B = \sqrt{2gh}$$

$$v_B = \left(\frac{m_B + M}{m_B}\right) \sqrt{2gh}$$

= initial velocity of bullet causing the block to rise a height h.

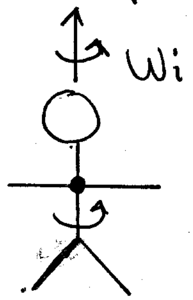
Angular Momentum

Unit #8b

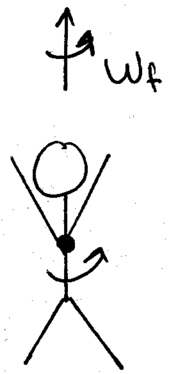
Translations	Rotations
$\vec{p} = m\vec{v}$	$\vec{L} = I\omega$
Conservation of Momentum	Conservation of Angular Momentum
$\vec{P}^{\,initial}_{Total} = \vec{P}^{\,final}_{Total}$	$\vec{L}^{\,initial}_{Total} = \vec{L}^{\,final}_{Total}$
$\downarrow$	$\downarrow$
$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$	$I_1 \omega_{1i} + I_2 \omega_{2i} = I_1 \omega_{1f} + I_2 \omega_{2f}$

②

Example:



Initial



Final

A figure skater spins with angular speed ω_i when her arms are extended outward has a moment of inertial I_i. She brings her arms inward to have moment of inertia $I_f \Rightarrow$ her new angular speed can be found using the conservation of angular momentum:

$$L_{Total}^{initial} = L_{Total}^{final}$$

$$I_i \omega_i = I_f \omega_f$$

$$\omega_f = \frac{I_i}{I_f} \omega_i$$

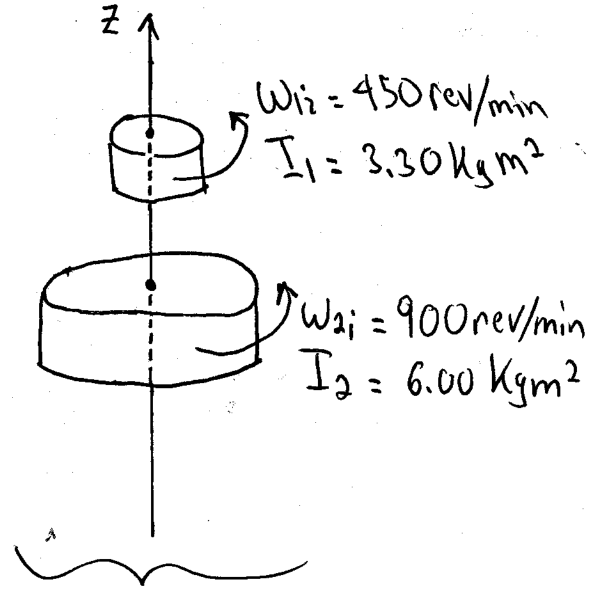
We know $I \underset{\uparrow}{\sim} mR^2 \Rightarrow \omega_f \sim \frac{R_i^2}{R_f^2} \omega_i$

"Varies as"

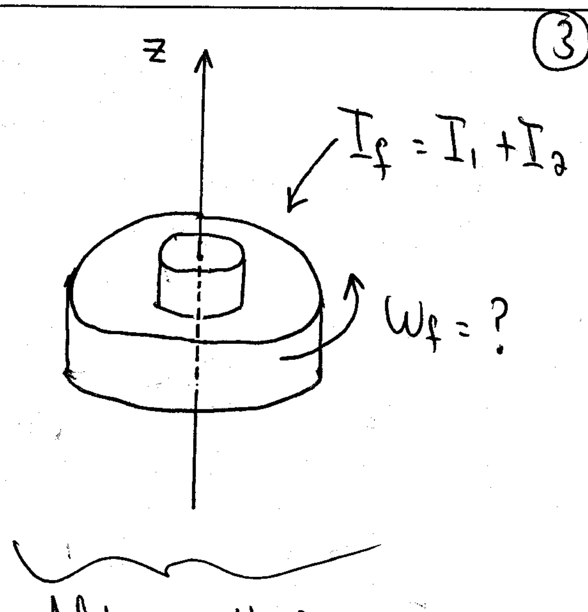
If $R_f < R_i$ then: figure skater speeds up when bringing in her arms $\rightarrow$ this is precisely what we observe.

Example:

Two disks are mounted on the same axle and can be brought together so that they can rotate as one unit. What is the angular velocity of the unit when the disks are brought together?



Before collision



After collision

$$L^{\text{initial}}_{\text{Total}} = L^{\text{final}}_{\text{Total}}$$

$$I_1 \omega_{1i} + I_2 \omega_{2i} = I_1 \omega_{f1} + I_2 \omega_{f2} = (I_1 + I_2)\omega_f$$

$\omega_{f1} = \omega_{f2} \equiv \omega_f$ since the disks are now in contact

$$\Rightarrow \omega_f = \frac{I_1 \omega_{1i} + I_2 \omega_{2i}}{I_1 + I_2} = \frac{(3.30\,\text{Kgm}^2)(450\,\text{rev/min}) + (6.00\,\text{Kgm}^2)(900\,\text{rev/min})}{3.30\,\text{Kgm}^2 + 6.00\,\text{Kgm}^2}$$

$$= 740.32258\ \text{rev/min} \approx 700\ \text{rev/min}$$

Question: what if ω_{2i} were initially spinning in the opposite direction?

$\Rightarrow \omega_{2i} = -900$ rev/min since $\vec{\omega}_{2i}$ now points downward (curl fingers in direction of rotation → thumb of right hand points downward)

Then, $I_1 \omega_{1i} - I_2 \omega_{2i} = (I_1 + I_2)\omega_f$

$$\Rightarrow \omega_f = \frac{I_1 \omega_{1i} - I_2 \omega_{2i}}{I_1 + I_2} \approx -400\ \text{rev/min}$$

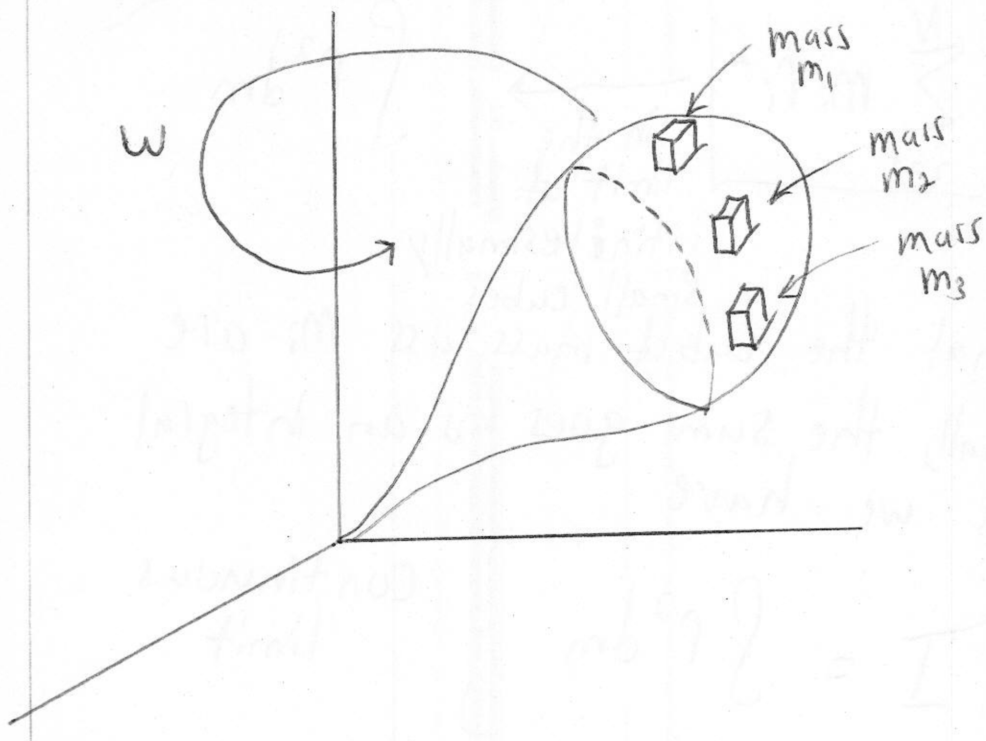
(the two disks now rotate in the opposite direction)

Section 12.3 Rigid Body Motion Unit #8c

Newton's law's for translations and rotations are very similar, as is the form for Kinetic energy

Translations	Rotations
$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$
$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$

For rotations, we replace mass, m, with the moment of Inertia, I. Let's see why by try to derive the Kinetic energy of a rigid body.



divide up the solid body into little cubes of mass m_i

The total Kinetic energy is the sum of the Kinetic energy of each point particle:

$$K = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 + \tfrac{1}{2} m_3 v_3^2 + \cdots + \tfrac{1}{2} m_N v_N^2$$

$$= \sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^2 = \underbrace{\sum_{i=1}^{N} \tfrac{1}{2} m_i (\omega r_i)^2}_{\text{note: } v = \omega r} = \tfrac{1}{2} \underbrace{\left(\sum_{i=1}^{N} (m_i r_i^2) \right)}_{\substack{\text{define this term as} \\ I = \text{moment of inertia}}} \omega^2$$

Then,

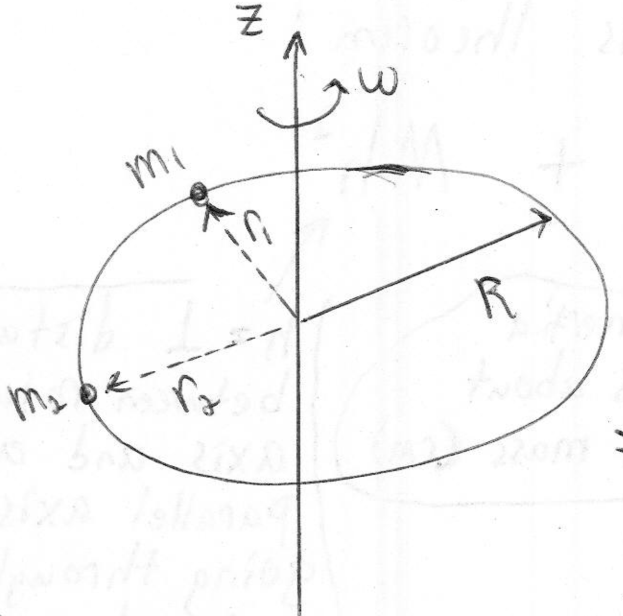
$$\boxed{K = \tfrac{1}{2} I \omega^2}$$

where: $\boxed{I = \sum_{i=1}^{N} m_i r_i^2}$

In the limit that the cubes of mass m_i are infinitesimally small, the sum goes to an integral $\Sigma \to \int$ and we have

$$I = \int r^2 dm \quad \Big\} \text{ continuous limit}$$

Example: Thin hoop of radius R



$$I = \sum_{i=1}^{N} m_i r_i^2$$

all radii are equal:

$$r_1 = r_2 = r_3 = \cdots = r_i = R$$

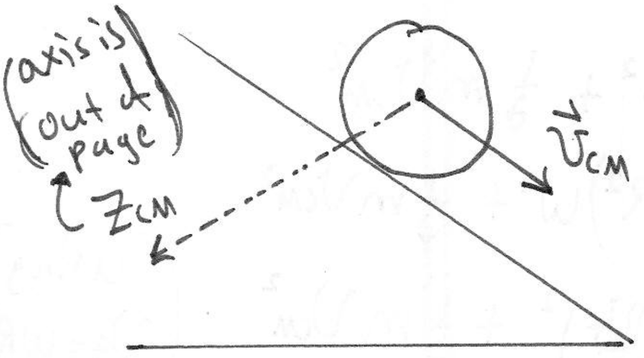
$$\Rightarrow I = \sum_{i=1}^{N} m_i R^2 = R^2 \underbrace{\left(\sum_{i=1}^{N} m_i\right)}_{\text{total mass of hoop} = M}$$

Then, $\boxed{I = MR^2}$, M = mass of hoop

for a thin hoop

$$\text{Kinetic energy} = K = \frac{1}{2} I \omega^2 = \frac{1}{2} M R^2 \omega^2$$

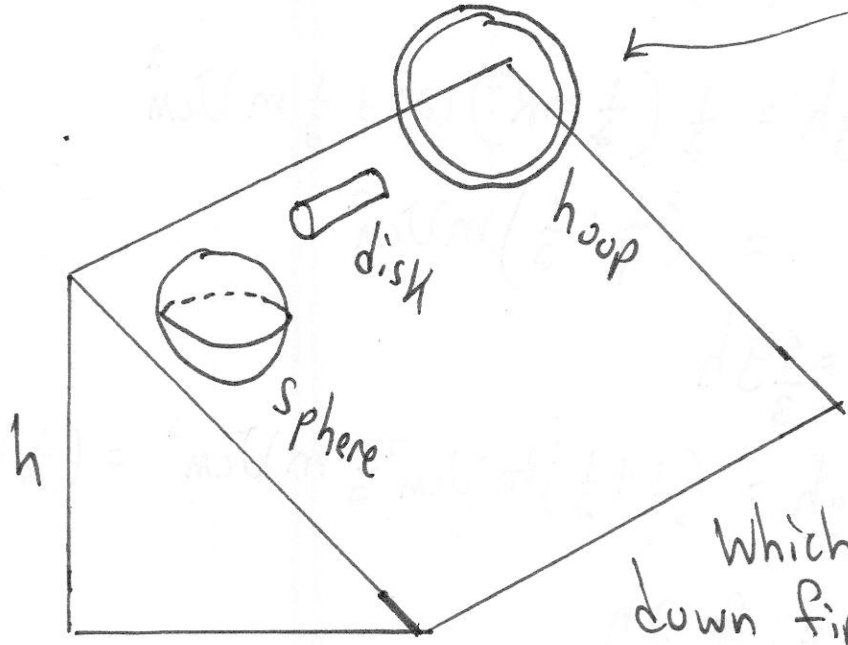
Section 12.8: Rolling Down an Incline: Conservation of Energy



We know the kinetic energy for an object rolling down an incline is

$$K = \underbrace{\tfrac{1}{2} I_{CM} \omega^2}_{\text{rotation}} + \underbrace{\tfrac{1}{2} m v_{CM}^2}_{\text{translation}}$$

Let's examine a sphere, disk, and hoop all rolling down an incline:



all have the same mass an radius

$I_{sphere} = \frac{2}{5} m R^2$

$I_{disk} = \frac{1}{2} m R^2$

$I_{hoop} = m R^2$

Which one makes it down first from height h?

Using conservation of energy for each object:

$$\text{Potential energy} \xrightarrow[\text{converted to}]{} \left(\text{Kinetic energy of rotation}\right) + \left(\text{Kinetic energy of translation}\right)$$

For the sphere:

$$E^{initial}_{Total} = E^{final}_{Total}$$

$$mgh = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} m V_{cm}^2$$

$$= \frac{1}{2}\left(\frac{2}{5} mR^2\right)\omega^2 + \frac{1}{2} m V_{cm}^2$$

$$= \frac{1}{5} m R^2 \left(\frac{V_T}{R}\right)^2 + \frac{1}{2} m V_{cm}^2 \quad \left(\text{using } V_T = \omega R\right)$$

$$= \left(\frac{1}{5} + \frac{1}{2}\right) m V_{cm}^2 \quad \left(\text{using } V_T = V_{cm}\right)$$

$$= \frac{7}{10} m V_{cm}^2$$

$$\Rightarrow \left(V_{cm}^2\right)_{sphere} = \left(\frac{10}{7}\right) gh$$

For disk: $mgh = \frac{1}{2}\left(\frac{1}{2} mR^2\right)\omega^2 + \frac{1}{2} m V_{cm}^2$

$$= \left(\frac{1}{4} + \frac{1}{2}\right) m V_{cm}^2$$

$$\Rightarrow \left(V_{cm}^2\right)_{disk} = \frac{4}{3} gh$$

For hoop: $mgh = \frac{1}{2}\left(mR^2\right)\omega^2 + \frac{1}{2} m V_{cm}^2 = \left(\frac{1}{2} + \frac{1}{2}\right) m V_{cm}^2$

$$\Rightarrow \left(V_{cm}^2\right)_{hoop} = gh$$

Since $(V_{cm})_{sphere} > (V_{cm})_{disk} > (V_{cm})_{hoop}$

The Sphere arrives at the bottom 1st

For the general case:

$$mgh = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} m v_{cm}^2$$

$$= \frac{1}{2} I_{cm} \left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2} m v_{cm}^2$$

$$= \left(\frac{1}{2} \frac{I_{cm}}{R^2} + \frac{1}{2} m\right) v_{cm}^2$$

$$\Rightarrow v_{cm}^2 = \frac{2mgh}{m + \frac{I_{cm}}{R^2}}$$

the larger I_{cm} is, the smaller v_{cm} is

Or conversely → the smaller I_{cm} is, the larger v_{cm} is

→ the sphere has the smallest moment of inertia

→ the sphere has the fastest velocity

→ the sphere arrives at the bottom first.