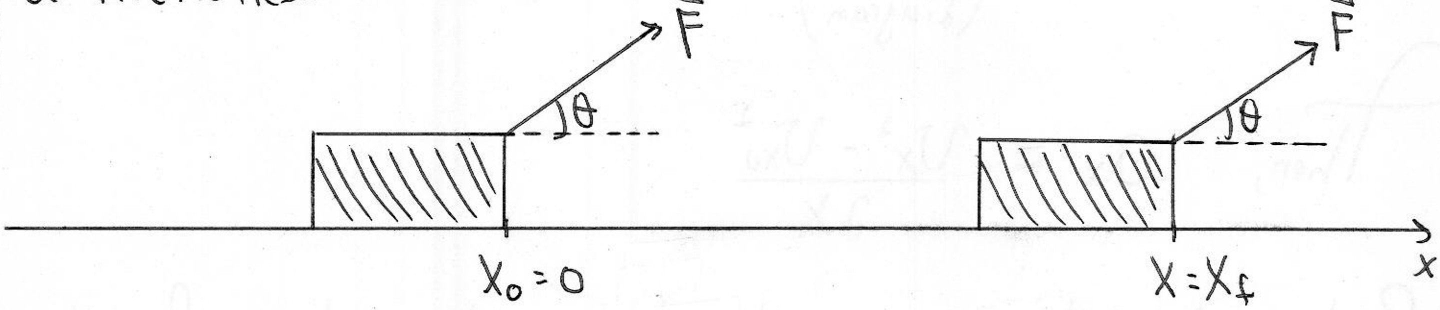
Work + Energy

Unit #7

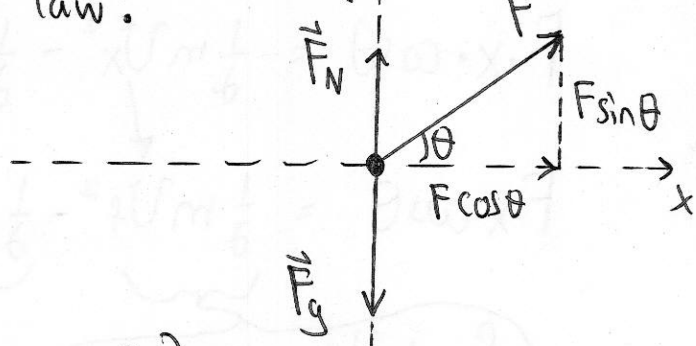
Work for Constant Forces

When we push or pull an object a certain distance, it is generally understood that we have done **work** on that object. Let's try ~~to~~ to understand this concept of "work" by examining a simple situation: A block pulled along a frictionless surface:



The block is pulled from $X_o = 0$ to $X = X_f$, and then the force is released. To examine this problem, we will draw upon what we have learned so far in this course — free body diagrams and Newton's 2^{nd} law:

(i) (Free body diagram for block):



(ii) Newton's 2^{nd} law:

$$\left\{\begin{array}{ll} F_{Total,y} = F_N - F_g + F\sin\theta = m\cancel{a_y}^{\,0} \\ F_{Total,x} = F\cos\theta = ma_x \end{array}\right\}$$

Examining just the 2nd equation ($F_{Total,x}$) gives the relation between the applied force and the acceleration of the object

$$F\cos\theta = ma_x$$

We can get an expression between the applied force an the final velocity of the object by using Eq-M4:

$$X = \underbrace{\cancel{X_0}}_{\substack{0 \\ \text{(from our diagram)}}} + \frac{V_x^2 - V_{x0}^2}{2a_x}$$

Then, $$a_x = \frac{V_x^2 - V_{x0}^2}{2X}$$

Substitution this expression in the equation above for the force gives:

$$F\cos\theta = ma_x = m\left(\frac{V_x^2 - V_{x0}^2}{2X}\right)$$

$$F \cdot X \cdot \cos\theta = \frac{1}{2}mV_x^2 - \frac{1}{2}mV_{x0}^2$$

(let $V_x = V_{final} = V_f$
$V_{x0} = V_{initial} = V_i$)

$$FX\cos\theta = \underbrace{\frac{1}{2}mV_f^2}_{\text{final Kinetic energy of the block}} - \underbrace{\frac{1}{2}mV_i^2}_{\text{Initial Kinetic energy of the block}}$$

When we express Newton's 2^{nd} law this way, we get some new, interesting terms:

$$\frac{1}{2}mV_f^2 \equiv \text{final Kinetic energy of block}$$

$$\frac{1}{2}mV_i^2 \equiv \text{initial Kinetic energy of block}$$

"defined as"

We can define expressions such as "$\frac{1}{2}mV^2$" as "Kinetic energy"

Now let's look at the term $Fx\cos\theta$ → what can this be "defined as"?

Let us define this new quantity:

$$|\vec{F}||\vec{X}|\cos\theta \equiv W = \text{Work done moving the block a distance } |\vec{X}| \text{ by force } \vec{F}$$

We can then wrap all this up as a theorem and call it the:

Work-Energy Theorem:

$$\boxed{\text{Work} = W = \frac{1}{2}mV_f^2 - \frac{1}{2}mV_i^2}$$

This expression can also be written this way:

$$\underbrace{W}_{\text{(Work done on object)}} + \underbrace{\tfrac{1}{2}mV_i^2}_{\text{(Kinetic energy before work was applied)}} = \underbrace{\tfrac{1}{2}mV_f^2}_{\text{(Kinetic energy after work was applied)}}$$

This is an example of the conservation of energy: The total energy of a closed system is a conserved quantity.

That is:

$$\underbrace{\left(\text{Total initial energy of a system}\right)}_{W + \frac{1}{2}mV_i^2} = \underbrace{\left(\text{Total final energy of a system}\right)}_{\frac{1}{2}mV_f^2}$$

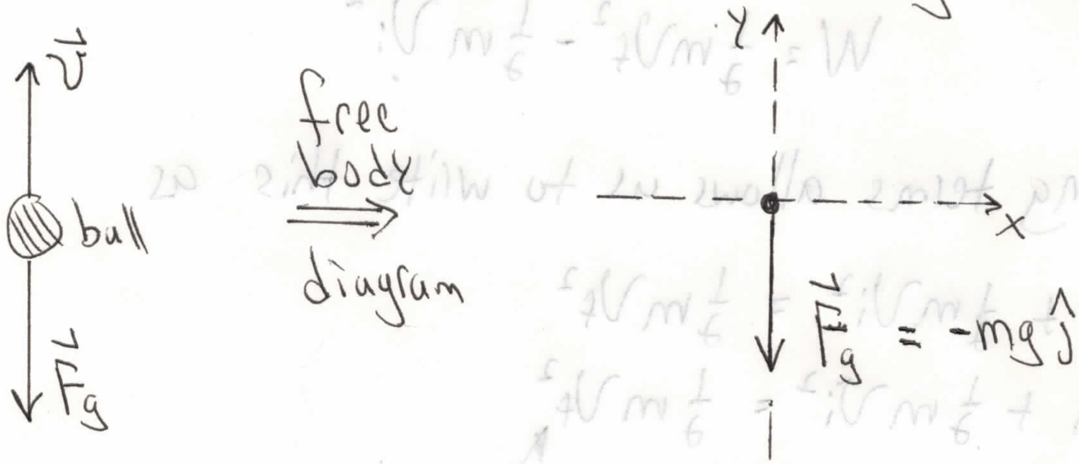
Units of Work: $W = F \cdot x \Rightarrow (\text{Newtons})(\text{meters})$

$$= N \cdot m$$
$$\equiv \text{Joule} = J$$

Work has units of Joule

Work done by gravity:

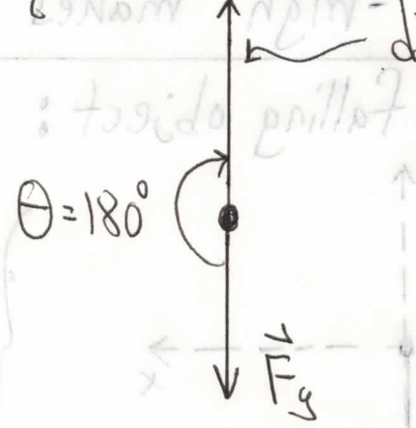
(A) Let's examine the case for a rising object



Let the ball rise a distance h. Then the work done on the ball by the gravitational force $\vec{F}_g$ is

$$W = |\vec{F}_g||\vec{y}|\cos\theta$$

where θ = angle between $\vec{F}_g$ and the displacement vector $\vec{y}$



displacement vector $\vec{y}$

where $|\vec{y}| = h$ = $\left\{ \begin{array}{l} \text{the amount} \\ \text{the ball rises} \end{array} \right.$

and $|\vec{F}_g| = mg$

Then,

$$W = mgh\underbrace{\cos(180^\circ)}_{=-1} = -mgh$$

Let's see if the minus sign makes sense by using the Work-Energy Theorem:

$$W = \frac{1}{2}mV_f^2 - \frac{1}{2}mV_i^2$$

Re-arranging terms allows us to write this as

$$W + \frac{1}{2}mV_i^2 = \frac{1}{2}mV_f^2$$

$$-mgh + \frac{1}{2}mV_i^2 = \frac{1}{2}mV_f^2$$

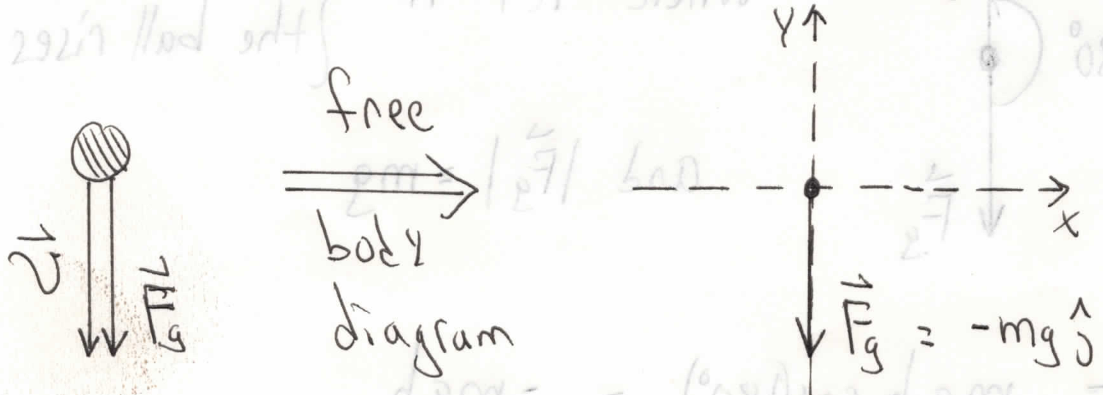
because of the minus sign, the final velocity of the ball is lower than the initial velocity as the ball rises

⇒ that is, the object slows down as it rises ✓

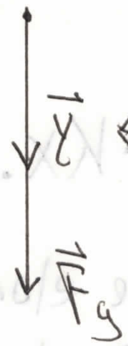
this is exactly what we expect

⇒ thus, $W = -mgh$ makes sense

(B) Let's examine the case for a falling object:



Same as before



displacement vector $\vec{y}$ is parallel to $\vec{F}_g \Rightarrow \theta = 0^\circ$ = angle between the two vectors

Then, $W = |\vec{F}_g||\vec{y}|\cos\theta = mgh\underbrace{\cos 0^\circ}_{=1} = mgh$

From the Work-Energy Theorem:

$$mgh + \frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2$$

because of the plus sign, the final velocity of the ball is larger than the initial velocity as the ball falls.

$\Rightarrow$ that is, the ball speeds up as it falls

$\rightarrow$ this is exactly what we expect

$\rightarrow W = +mgh$ makes sense.

Now, we have a general rule to determine whether work W is positive or negative:

$\boxed{W = \text{positive work if the object speeds up}}$

$\boxed{W = \text{negative work if the object slows down}}$

Work for a Nonuniform Force, $F(x)$:

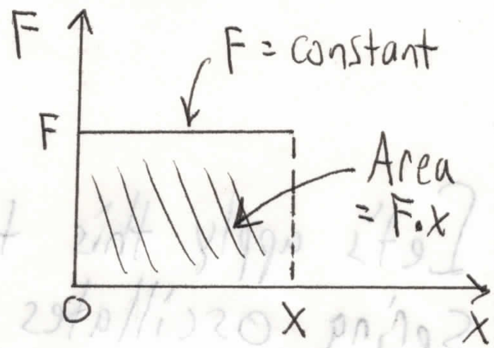
So far we have only encountered the situation in which the applied force was constant in time and space:

$$F = \text{Constant} \quad (\text{does not vary with } x \text{ or } t)$$

Lets look at the case where F = constant and $\theta = 0°$.

Then $W = F \cdot x$

Plotting "F" verses "x" gives

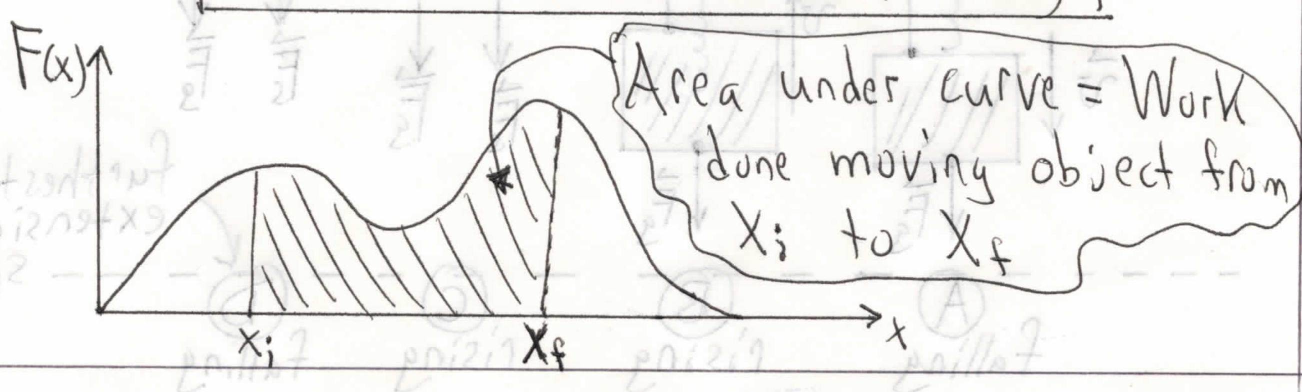


Notice that the area under F is simply the work

$$W = \text{Area under the curve "}F\text{" verses "}x\text{"}$$

Even though we have only looked at the case where F = constant, the statement is true for any function of x:

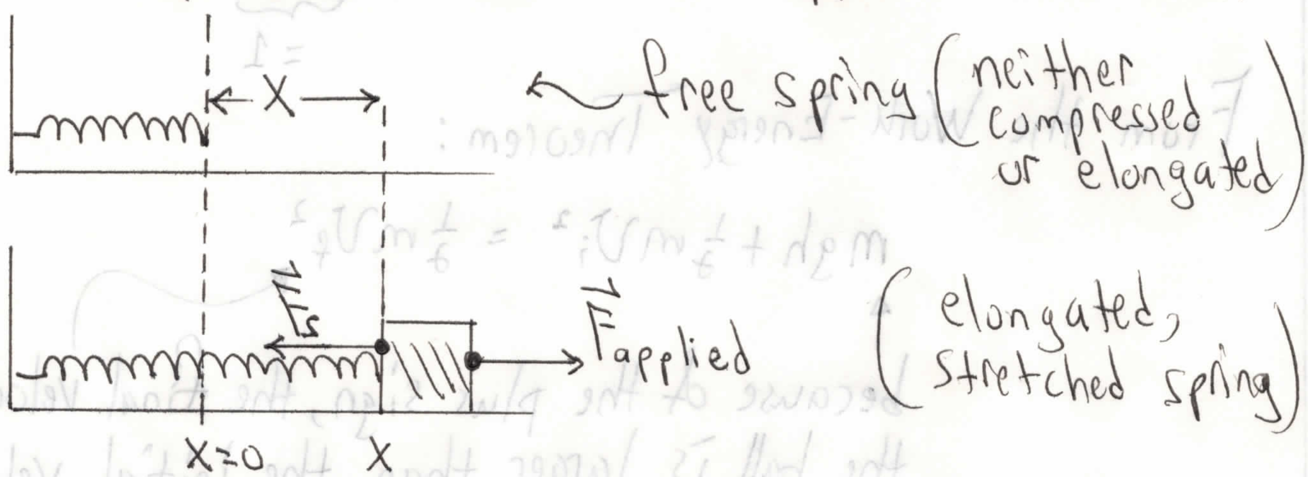
$$\boxed{W = \text{Area under the curve } F(x)}$$



Work due to a spring:

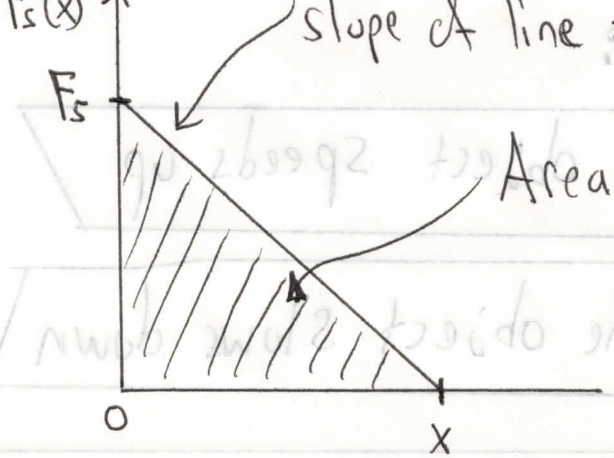
Springs obey Hooke's Law: $F_s(x) = -Kx$.

The minus sign is there because if one elongates or compresses a spring by an amount "x", there is always a spring force, F_s, in the opposite direction.



The spring force, $F_s = -Kx$, is a restoring force that tries to restore the spring to its original shape.

Let's plot $F_s(x)$ verse "x", and find the area under it to determine the work due to a spring:



$$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(x)(F) = \frac{1}{2}(x)(Kx)$$

$$= \frac{1}{2}Kx^2$$ (leaving out the sign for the moment)

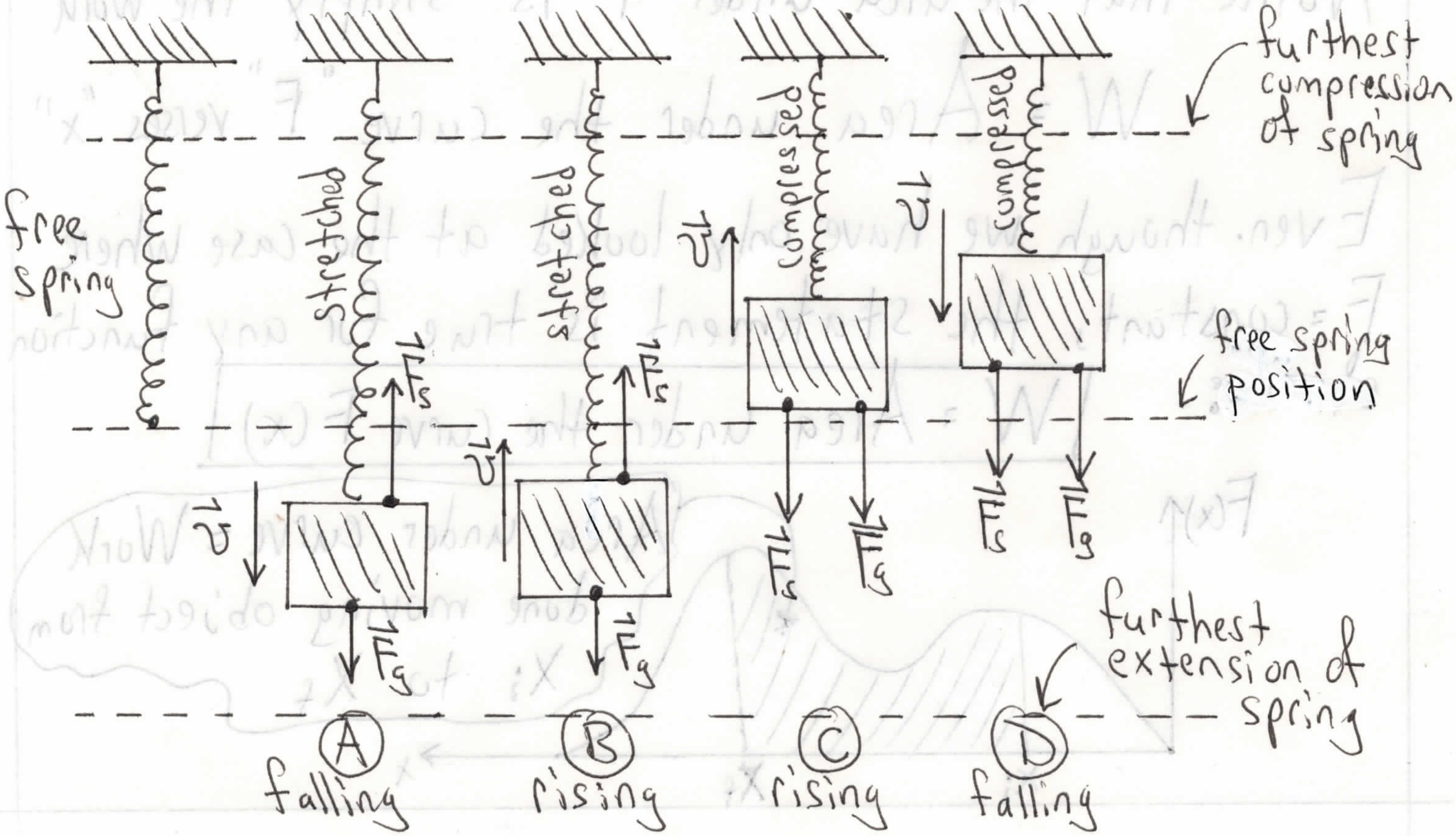
Thus, we have:

Work due to a spring: $W = \pm \frac{1}{2}kx^2$

Work due to gravity: $W = \pm mgh$

We can determine the sign by seeing if the force causes the object to speed up (plus sign) or slow down (minus sign).

Let's apply this to the following example where a spring oscillates up and down due to a block attached to it:



Using intuition or the Work-Energy Theorem, we can find the signs & work for Cases (A), (B), (C), and (D)

(A) $-\frac{1}{2}Kx^2 + mgh + \frac{1}{2}mV_i^2 = \frac{1}{2}mV_f^2$
(Slows down) (Speeds up)

(B) $+\frac{1}{2}Kx^2 - mgh + \frac{1}{2}mV_i^2 = \frac{1}{2}mV_f^2$
(Speeds up) (Slows down)

(C) $-\frac{1}{2}Kx^2 - mgh + \frac{1}{2}mV_i^2 = \frac{1}{2}mV_f^2$
(Slows down) (Slows down)

(D) $+\frac{1}{2}Kx^2 + mgh + \frac{1}{2}mV_i^2 = \frac{1}{2}mV_f^2$
(Speeds up) (Speeds up)

$$\left.\begin{aligned} &(A)\quad W_A = -\tfrac{1}{2}Kx^2 + mgh \\ &(B)\quad W_B = \tfrac{1}{2}Kx^2 - mgh \\ &(C)\quad W_C = -\tfrac{1}{2}Kx^2 - mgh \\ &(D)\quad W_D = \tfrac{1}{2}Kx^2 + mgh \end{aligned}\right\}$$

Work done by the spring force and by the gravitational force : $\vec{F}_s$ and $\vec{F}_g$

Work and Potential Energy:

Let's examine the case of a ball rising and falling

① As the ball rises, its Kinetic energy decreases as it slows down.

$$\text{Kinetic energy} = K = \frac{1}{2}mv^2$$

⇒ Kinetic energy is transfered from the ball until it stops ($K \rightarrow 0$) at the top of its path.

⇒ Where does this energy go?

⇒ It is transfered into "potential energy"

→ that is, once it reaches the top, the ball has the potential to do work as it falls.

② As the ball falls, it gains Kinetic energy (it speeds up) ⇒ energy is now transfered from the potential energy of the ball back into Kinetic energy.

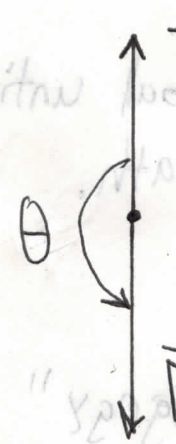
We use the symbol U = potential energy

K = Kinetic energy

The change in potential energy is defined as

$$\Delta U = -W = -|\vec{F}||\vec{x}|\cos\theta$$

Using this definition, we can find the potential energy for a rising ball:

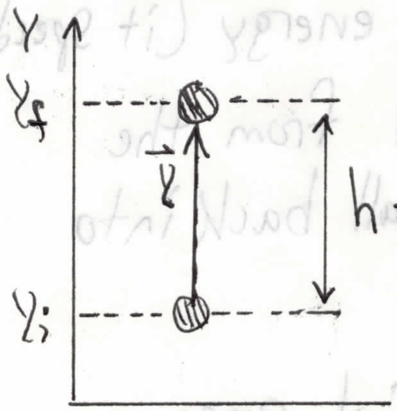


$$\vec{y} = h\hat{j} \Rightarrow |\vec{y}| = h = \text{height ball rises}$$

$$\theta = 180^\circ$$

$$\vec{F}_g = -mg\hat{j} \Rightarrow |\vec{F}_g| = mg$$

$$\Delta U = -W = -|\vec{F}||\vec{y}|\cos\theta = -mgh\underbrace{\cos(180^\circ)}_{=-1} = mgh$$



$$\Delta U = mgh = mg(y_f - y_i)$$

$$\Delta U = mgy_f - mgy_i$$

$$\Delta U = U_f - U_i = mgy_f - mgy_i$$

Now we see why there is a minus sign for:

$$\Delta U = -W$$

minus sign makes the potential energy positive for a rising ball

Conservation of Energy:

Recall the Work-Energy Theorem: $W = \underbrace{K_f}_{\frac{1}{2}mv_f^2} - \underbrace{K_i}_{\frac{1}{2}mv_i^2}$

Then, using $\Delta U = U_f - U_i = -W$

gives

$$U_f - U_i = -(K_f - K_i)$$

$$\underbrace{U_i + K_i}_{\text{Total initial energy of the system}} = \underbrace{U_f + K_f}_{\text{Total final energy of the system}} \Rightarrow \text{Conservation of Energy}$$

$$\boxed{E_{Total} = U_i + K_i = U_f + K_f}$$

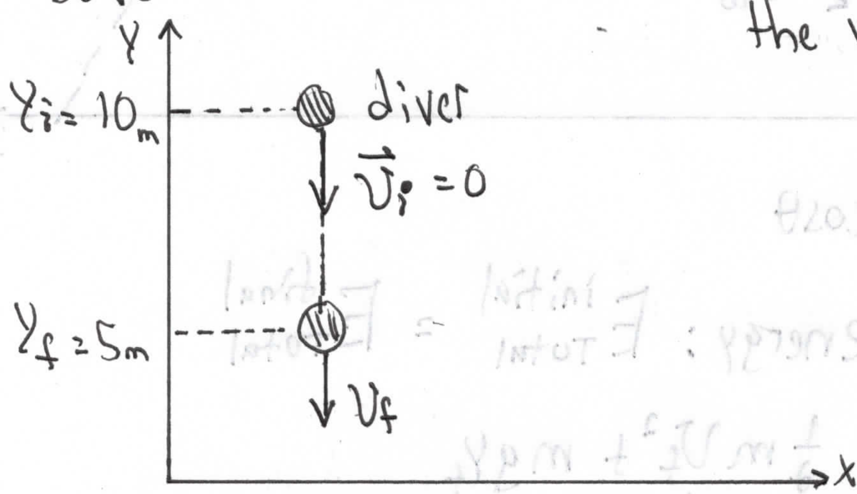
$\Rightarrow$ The total energy of a system is always a constant $\longrightarrow$ that is, the "total energy" is a conserved quantity.

If friction is involved (a nonconservative force) additional thermal energy will be generated in the final state, and the conservation of energy relation would be:

$$U_i + K_i = U_f + K_f + \Delta E_{Thermal}$$

energy generated by friction

Example: A diver of mass m drops from a board 10 m above the water's surface. What is his speed 5 m above the water's surface?



Conservation of Energy: $E_{Total}^{initial} = E_{Total}^{final}$

$$\cancel{\tfrac{1}{2} m v_i^2}^{\,0} + \cancel{m} g y_i = \tfrac{1}{2} \cancel{m} v_f^2 + \cancel{m} g y_f$$

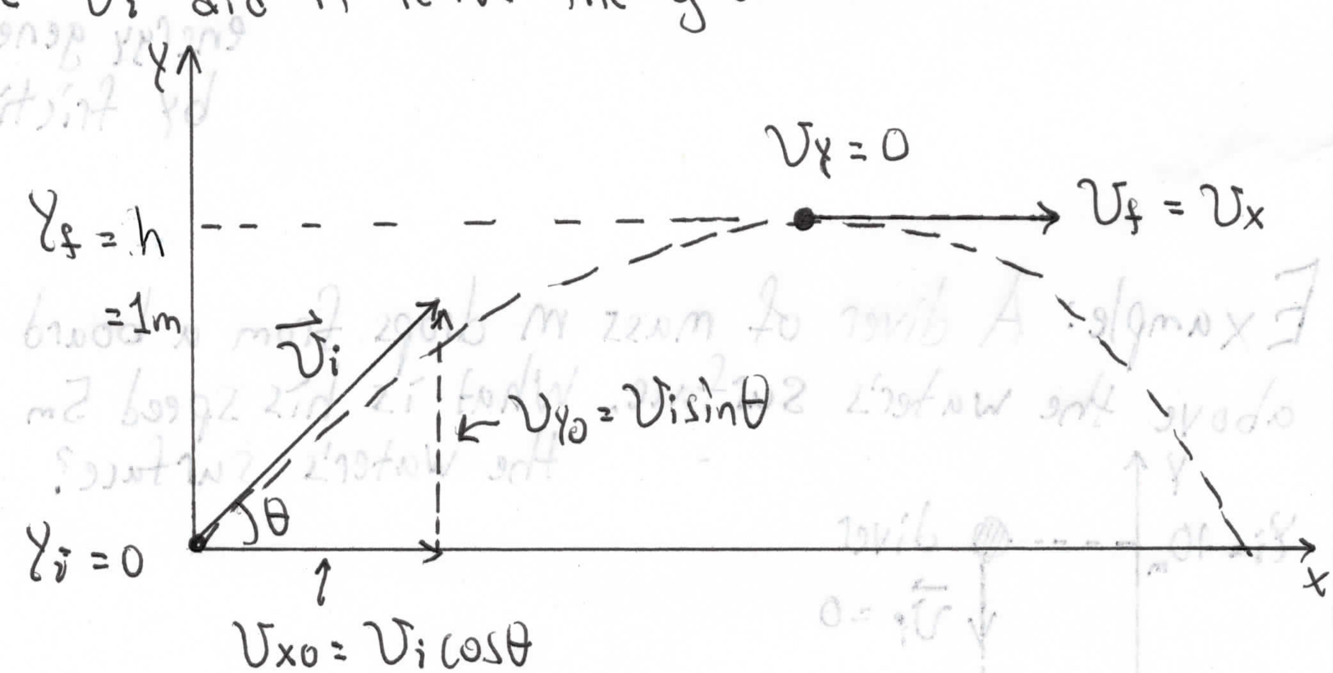
$v_i = 0$

$$g y_i = \tfrac{1}{2} v_f^2 + g y_f$$

$$v_f^2 = 2g(y_i - y_f)$$

$$v_f = \sqrt{2g(y_i - y_f)} = \sqrt{2(9.8\,m/sec^2)(10m - 5m)} = 9.90\,m/sec$$

Example: A grasshopper launches itself at an angle of 45° above the horizontal and rises to a maximum height of 1m during the leap. With what speed v_i did it leave the ground?



Conservation of energy: $E^{initial}_{Total} = E^{final}_{Total}$

$$\frac{1}{2} m v_i^2 + \cancel{m g \underset{\substack{\downarrow \\ y_i = 0}}{y_i}}^{\,0} = \frac{1}{2} m \underset{\substack{\downarrow \\ v_x}}{v_f}^2 + m g \underset{\substack{\downarrow \\ h = 1m}}{y_f}$$

Note that: $v_f = v_x$ has only a horizontal velocity at the top of the parabola.

Also, the horizontal velocity is constant everywhere along its path $\longrightarrow$ then $v_x = v_{x0} = v_i \cos\theta$

$$\frac{1}{2} m v_i^2 = \frac{1}{2} m (v_i \cos\theta)^2 + mgh$$

$$\frac{1}{2} \cancel{m} v_i^2 = \frac{1}{2} \cancel{m} v_i^2 \cos^2\theta + \cancel{m}gh$$

$$v_i^2 = v_i^2 \cos^2\theta + 2gh$$ (multiplied by 2 on both sides)

$$v_i^2 (1 - \cos^2\theta) = 2gh$$

$$v_i^2 \sin^2\theta = 2gh$$ $\left(\begin{array}{l} \sin^2\theta + \cos^2\theta = 1 \\ \sin^2\theta = 1 - \cos^2\theta \end{array}\right)$

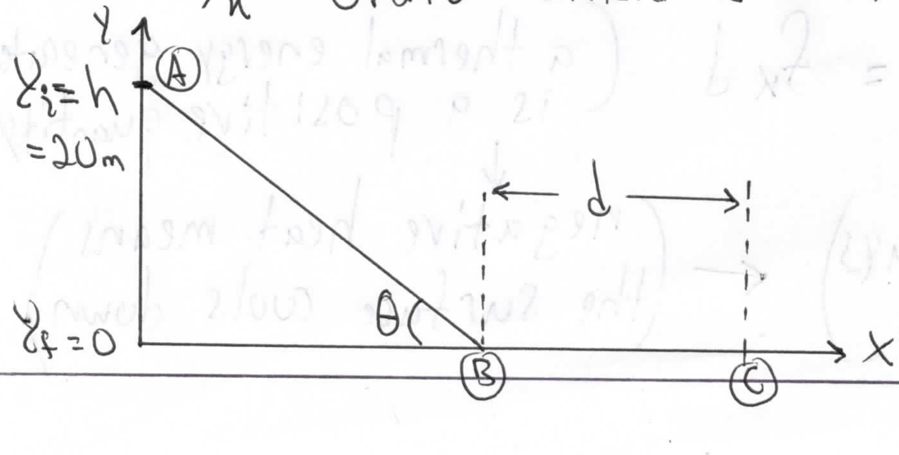
$$v_i = \frac{\sqrt{2gh}}{\sin\theta}$$ $\left(\sin 45^\circ = \frac{1}{\sqrt{2}}\right)$

$$v_i = \frac{\sqrt{2gh}}{\frac{1}{\sqrt{2}}} = \sqrt{2}\sqrt{2gh} = \sqrt{2}\cdot\sqrt{2}\sqrt{gh}$$

$$v_i = 2\sqrt{gh} = 2\sqrt{(9.8\,m/sec^2)(1\,m)} =$$

$$= 6.26\,m/sec$$

Example: A skier starts from rest at the top of a frictionless incline of height 20m. At the bottom of the incline, the skier encounters a horizontal surface where $\mu_k = 0.210$ (there is friction along horizontal surface)



(a) what is the skier's speed at the bottom (at point B)?

$$E^{initial}_{Total} = E^{final}_{Total}$$

$$\underbrace{\tfrac{1}{2}\cancel{m}\cancel{V_i^2}}_{V_i=0} + \cancel{m}g\underbrace{y_i}_{h} = \tfrac{1}{2}\cancel{m}V_f^2 + \cancel{mg\underbrace{y_f}_{y_f=0}}$$

$$gh = \tfrac{1}{2}V_f^2$$

$$V_f = \sqrt{2gh} = \sqrt{2(9.8\,m/sec^2)(20m)}$$

$$= 19.8\ m/sec^2$$

(b) How far does the skier travel along the horizontal surface before coming to rest?

$$E^{initial}_{Total} = E^{final}_{Total}$$

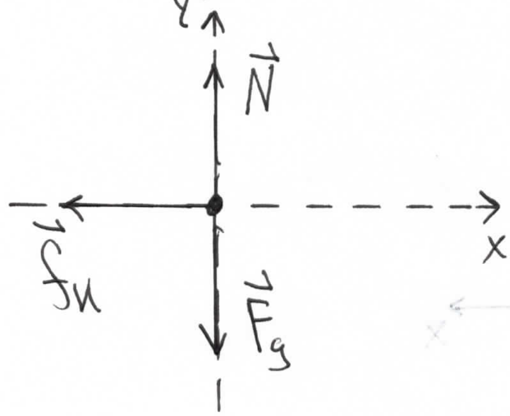
$$\underbrace{\tfrac{1}{2}mV_i^2}_{\substack{\sqrt{2gh} \\ \text{from Part (a)}}} + \cancel{mg\underbrace{y_i}_{y_i=0}} = \cancel{\tfrac{1}{2}m\underbrace{V_f^2}_{\substack{V_f=0 \\ \text{Skier comes} \\ \text{to rest}}}} + \cancel{mg\underbrace{y_f}_{y_f=0}} + \underbrace{\Delta E_{Thermal}}_{\substack{\text{due to Work} \\ \text{due to friction}}}$$

Work due to friction $= W = -f_k d$ (negative since skier slows down)

Then, $\Delta E_{Thermal} = f_k d$ (thermal energy generated is a positive quantity)

(friction always heats up surfaces) ← (negative heat means the surface cools down)

(i) free body diagram



(ii) Newton's 2nd law

$$F_{Total,y} = N - F_g = \cancel{ma_y}^{\,0}$$

$$\downarrow$$

$$N - mg = 0$$

$$N = mg$$

But $f_k = \mu_k N \Rightarrow \boxed{f_k = \mu_k mg}$

Then,

$$\frac{1}{2} m v_i^2 = \Delta E_{Thermal}$$

$$\downarrow \qquad\qquad \downarrow$$

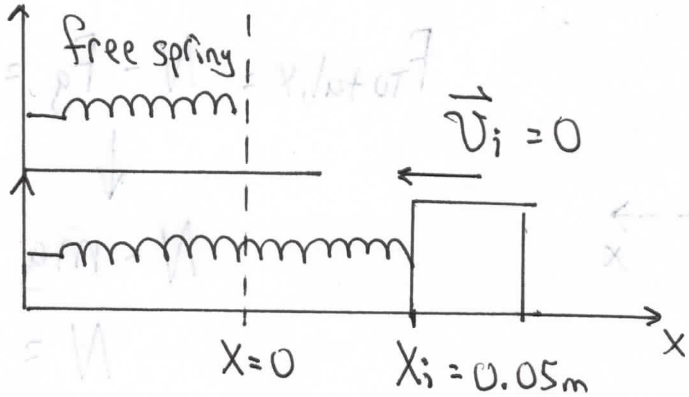
$$\sqrt{2gh} \qquad f_k d = \mu_k mgd$$

$$\Rightarrow \frac{1}{2}\cancel{m}(\sqrt{2gh})^2 = \mu_k \cancel{m} g d$$

$$\cancel{\frac{1}{2}}(\cancel{2}\cancel{g}h) = \mu_k \cancel{g} d$$

$$d = \frac{h}{\mu_k} = \frac{20m}{0.210} = 95.2m$$

Example: A block with mass 5 Kg is attached to a spring with $K = 4 \times 10^2$ N/m. The surface is frictionless, and the block is pulled out to $X_i = 0.05$ m and released.



(a) find the speed of the block when it reaches the equilibrium point (at $X_f = 0$).

$$E_{Total}^{initial} = E_{Total}^{final}$$

$$\frac{1}{2}m\cancel{v_i^2}^{\,0} + \frac{1}{2}KX_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}\cancel{KX_f^2}^{\,0}$$

$$X_f = 0$$

$$KX_i^2 = mv_f^2$$

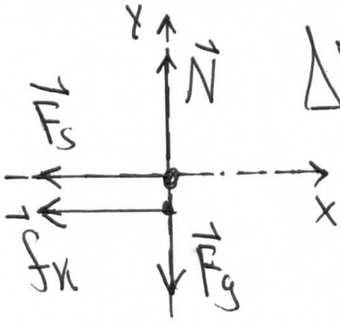
$$v_f = X_i\sqrt{\frac{K}{m}} = (0.05\text{m})\sqrt{\frac{4\times10^2 \text{ N/m}}{5 \text{ Kg}}}$$

$$= 0.447 \text{ m/sec}$$

(b) find the speed if friction acts on the block with $\mu_k = 0.15$.

$$E_{Total}^{initial} = E_{Total}^{final}$$

$$\frac{1}{2}m\cancel{v_i^2}^{\,0} + \frac{1}{2}KX_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}\cancel{KX_f^2}^{\,0} + \Delta E_{Thermal}$$



$$\Delta E_{Thermal} = f_k d = \mu_k N d = \mu_k m g d$$

Since $N = F_g = mg$
from free body diagrum

$$\frac{1}{2} K X_i^2 = \frac{1}{2} m V_f^2 + \mu_k m g X_i \quad (\text{note: } d = X_i)$$

$$V_f = \sqrt{\left(\frac{K X_i^2 - 2 \mu_k m g X_i}{m}\right)}$$

$$= \sqrt{\frac{(4 \times 10^2 \, N/m)(0.05 m)^2 - 2(0.15)(5 Kg)(0.05 m)}{5 Kg}}$$

$$= 0.23 \, m/sec$$