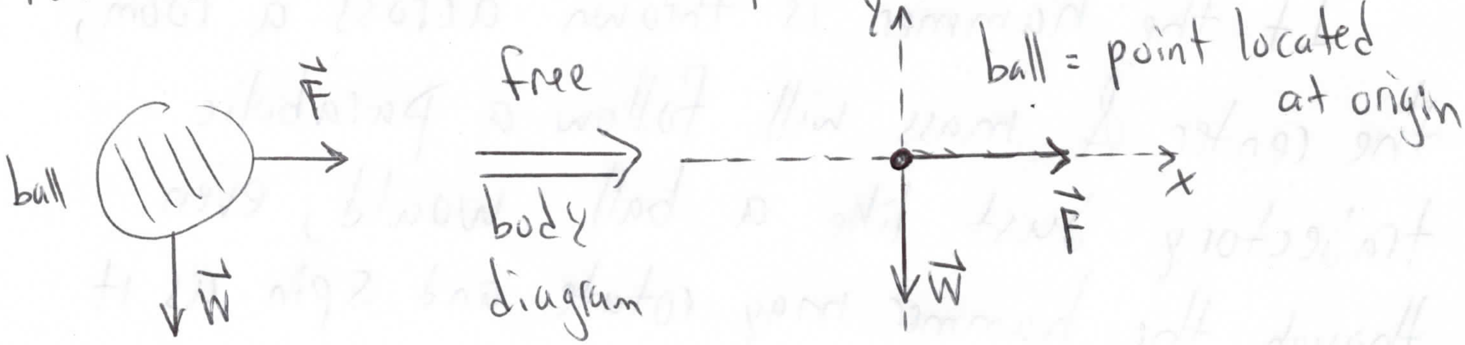
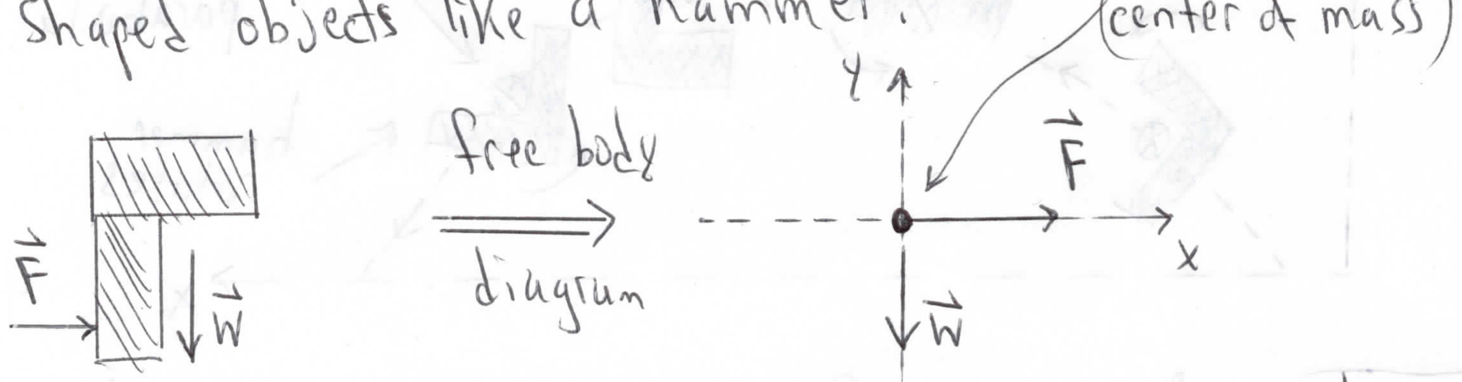
# Rigid Body Motion

Unit #6b

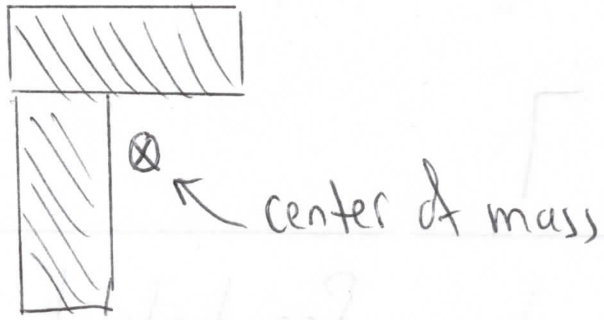
So far when we have drawn free body diagrams for an object undergoing translations, we have always idealized the object as a point particle




How do we define free body diagrams for irregularly shaped objects like a hammer?



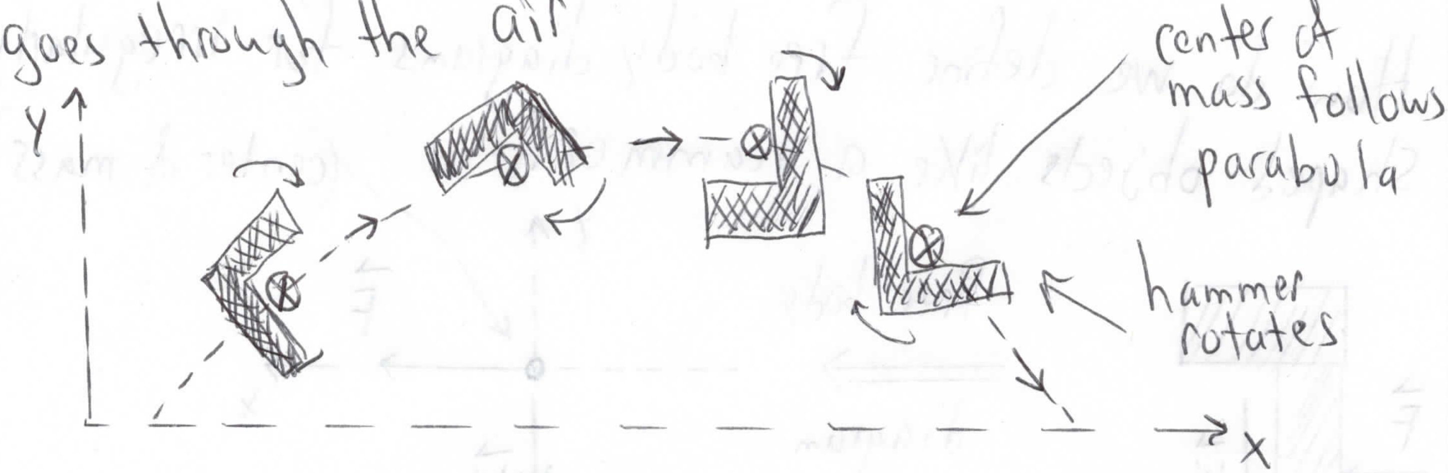

We still idealize the object above as a point that is located at the center of mass of the object




The center of mass obeys Newton's 2nd law for translations

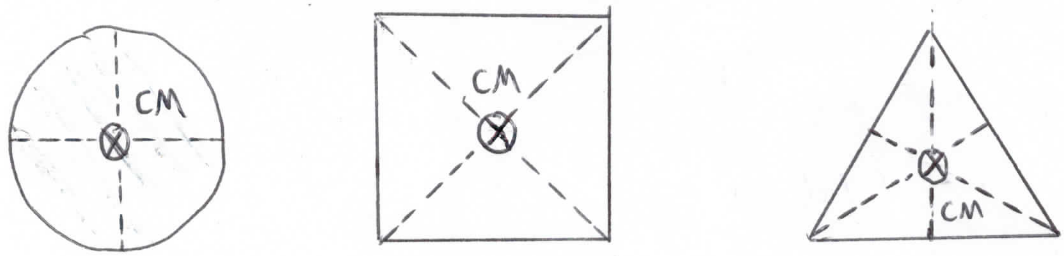
$$\begin{cases} F_{Total,y} = ma_y \\ F_{Total,x} = ma_x \end{cases}$$

If the hammer is thrown across a room, the center of mass will follow a parabolic trajectory just like a ball would, even though the hammer may rotate and spin as it goes through the air




Thus, if we can find the center of mass of an object, we can use Newton's 2nd law for translations to find its equations of motion.

For simple objects that have high symmetry and are of uniform density, the center of mass is at the geometric center:




CM = center of mass

For a system of objects, the equation for the center of mass is

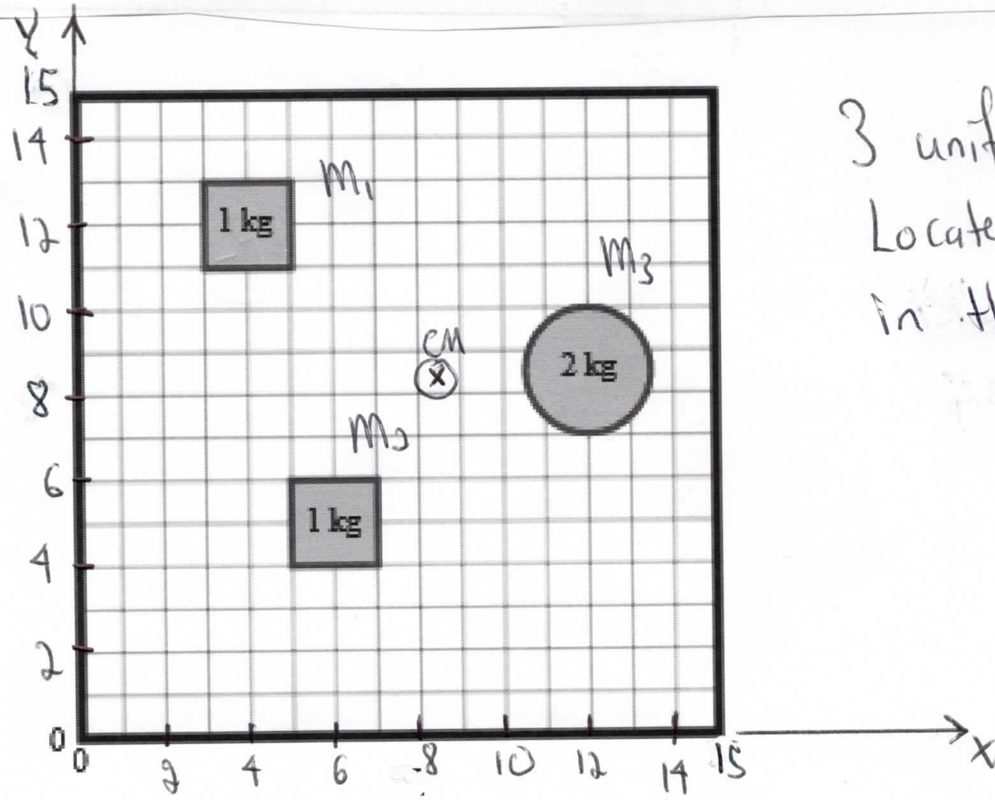
$$X_{CM} = \frac{m_1 X_1 + m_2 X_2 + m_3 X_3 + \cdots + m_N X_N}{m_1 + m_2 + m_3 + \cdots + m_N}$$

$$Y_{CM} = \frac{m_1 Y_1 + m_2 Y_2 + m_3 Y_3 + \cdots + m_N Y_N}{m_1 + m_2 + m_3 + \cdots + m_N}$$

Where $X_1, X_2, X_3, \ldots, X_N$ = x-axis coordinates of the center of mass for the N objects

$Y_1, Y_2, Y_3, \ldots, Y_N$ = y-axis coordinates of the center of mass for the N objects

$m_1, m_2, m_3, \ldots, m_N$ = masses of the N objects




3 uniform masses Located at positions in the xy-plane

$$X_{CM} = \frac{m_1 X_1 + m_2 X_2 + m_3 X_3}{m_1 + m_2 + m_3} = \frac{(1)(4) + (1)(6) + (2)(12)}{1+1+2}$$

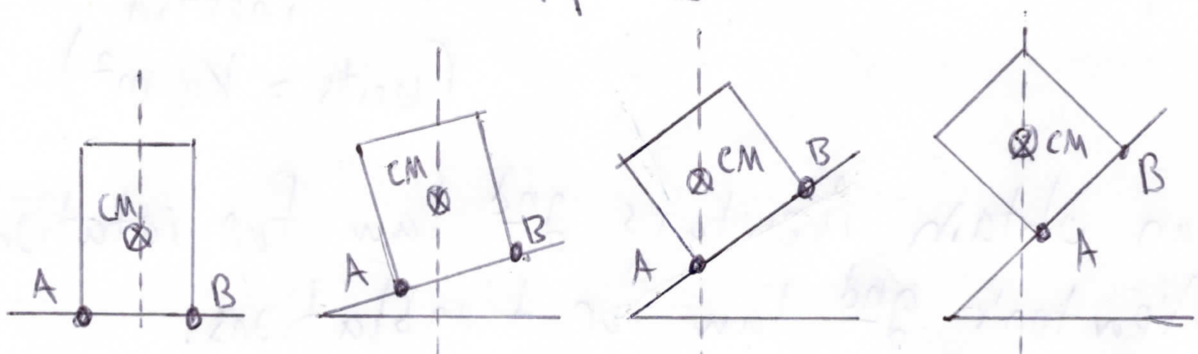
$$= 8.5$$

$$Y_{CM} = \frac{m_1 Y_1 + m_2 Y_2 + m_3 Y_3}{m_1 + m_2 + m_3} = \frac{(1)(12) + (1)(5) + (2)(8.5)}{1+1+2}$$

$$= 8.5$$

$\Rightarrow$ The center of mass is located at the coordinates: $(X_{CM}, Y_{CM}) = (8.5, 8.5)$

By examining the location of the center of mass we can determine whether an object will tip over
⇒ that is we can ascertain its
toppling stability




Stable object | Stable object | at the verge of tipping over | the object will topple over

The box above is in contact with the surface along a line between points A and B. If a vertical line draw through the center of mass (CM) intersects line $\overline{AB}$, then the object is stable and will not tip over as shown in the 1st 3 cases above. In the last case, the vertical line through the center of mass does not cross line $\overline{AB}$, and thus the object will tip over ⇒ it is unstable.

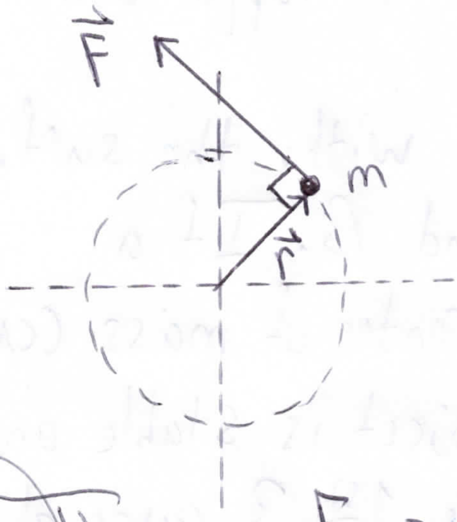
# Moment of Inertia:

Newton's $2^{nd}$ law for rotations follows

$\vec{\tau} = I\vec{\alpha}$ where $I$ = moment of inertia (units = $kg\, m^2$)

We can obtain Newton's $2^{nd}$ law for rotations from Newton's $2^{nd}$ law for translations:

$$\vec{F} = m\vec{a}$$




Example: Let there be a tangential force that is always perpendicular to $\vec{r}$ $\Rightarrow$ this causes an object, m, to move in a circle

Then,

$F = ma$ $\rightarrow$ multiply by $r$ on both sides

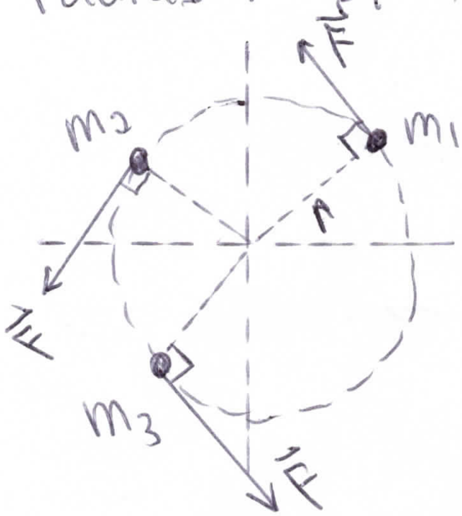
$rF = mra$ $\rightarrow$ note: $a = \alpha r$

$rF = (mr)(r\alpha) = mr^2\alpha$ $\rightarrow$ note: $\tau = rF$

$\tau = (mr^2)\alpha$

$\tau = I\alpha$ where $I = mr^2$

If one has 3 objects moving in a circle of radius $r$ by tangential force $\vec{F}$,




Then,

$$\tau = (m_1 r^2 + m_2 r^2 + m_3 r^2)\alpha$$

$$= I\alpha$$

where $I = m_1 r^2 + m_2 r^2 + m_3 r^2$

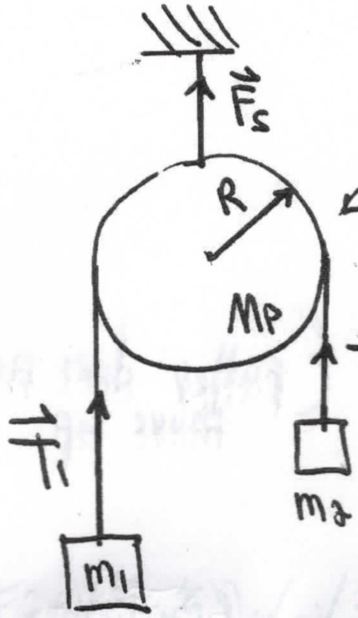
In general, the moment of inertia is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_N r_N^2$$

for $N$ objects at different distances $r_N$ from the axis of rotation.

For discrete objects like the example above, the moment of inertia can be easily calculated. For continuous objects, where $N \to \infty$, it requires calculus. Thus, your textbook has a table for the moment of inertia for solid and hollow spheres and cylinders, and for thin rods. You must use the table when trying to determine the equations of motion for these types of solid objects (or Rigid Bodies).

# Atwood Machine



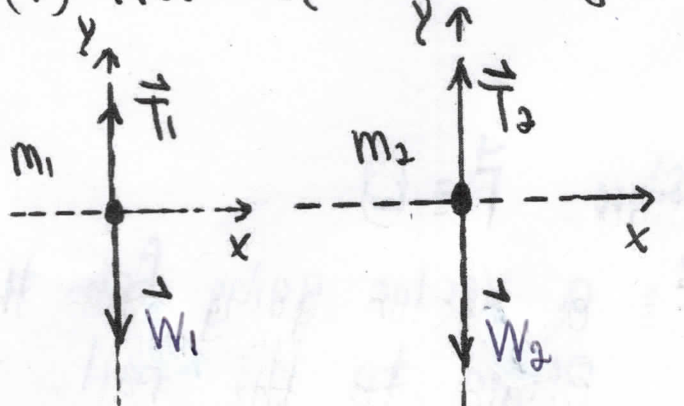

Since the pulley has mass, $M_P$

$T_1 \neq T_2$ anymore

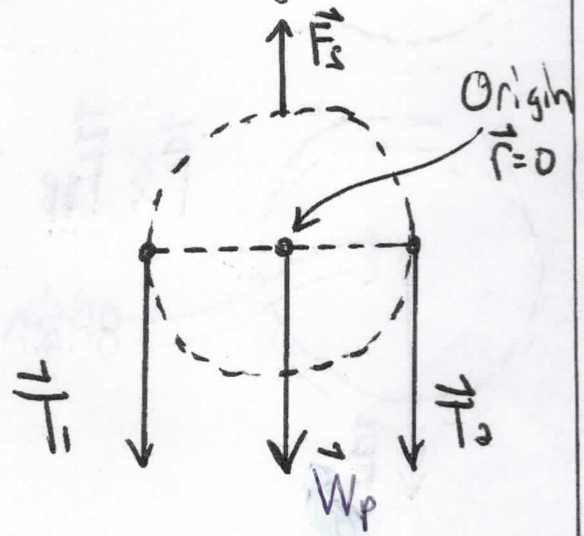
(this will be shown below)

Let's solve for the tensions, $T_1, T_2$, as well as the acceleration of mass $m_1$. The $1^{st}$ thing we should do is draw free body diagrams for forces and torques:

(i) Free body force diagram




Free body torque diagram



(ii) Newton's 2nd law for translations:

$$F_{Total,y_1} = T_1 - W_1 = m_1 a_{y_1}$$

$$F_{Total,y_2} = T_2 - W_2 = m_2 a_{y_2}$$

$$F_{Total,y_p} = F_s - W_p - T_1 - T_2 = \cancel{M_p a_{y_p}}^{\,0}$$ (pulley does not move up)

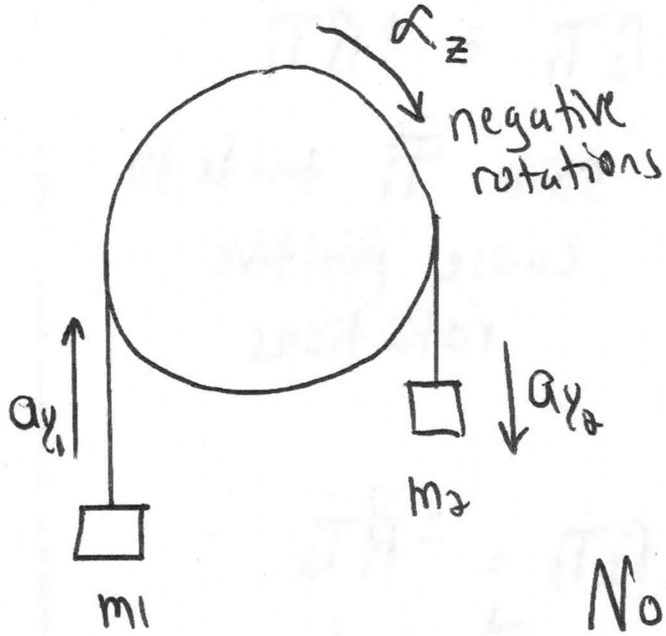
Newton's 2nd law for torques:

$$\tau_{Total,z} = (\pm r_1 F_s) + (\pm r_1 W_p) + (\pm r_1 T_1) + (\pm r_1 T_2) = I_z \alpha_z$$

Next, we need a relationship among the accelerations

$$a_{y_1},\ a_{y_2},\ \alpha_z$$

This follows on the next page




Thus, if block $m_2$ is falling, $m_1$ is rising and

$$\begin{cases} \text{if: } a_{y_1} = +a \\ \text{then: } a_{y_2} = -a \end{cases}$$

Note that $M_p$ rotates in the negative direction, then $\alpha_z$ is negative:

From the relation: $a = \alpha R \Rightarrow \alpha = a/R$

$$\Rightarrow \boxed{\alpha_z = \frac{-a}{R}}$$ for negative rotations

Summarizing Again:

$$\begin{cases} T_1 - m_1 g = m_1 a & ① \\ T_2 - m_2 g = -m_2 a & ② \\ RT_1 - RT_2 = -\frac{I a}{R} & ③ \end{cases}$$

3 equations, 3 unknowns $(a, T_1, T_2)$

$\Rightarrow$ We can solve this set of simultaneous eqns. uniquely

Note: we do not need the 4th eqn. to find $a, T_1, T_2$.

Eq. 3 gives: $T_1 - T_2 = -Ia/R^2$

Eq. 1 gives: $T_1 = m_1 a + m_1 g$

Eq. 2 gives: $T_2 = m_2 g - m_2 a$

Substituting into Eq. 3 gives

$$(m_1 a + m_1 g) - (m_2 g - m_2 a) = -I a/R^2$$

$$(m_1 + m_2 + I/R^2)a = (m_2 - m_1)g$$

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2 + I/R^2}$$

If the moment of inertia is ignored ($I = 0$), then

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$$

And from the torque eqn: $RT_1 - RT_2 = I\alpha$
we also get

$$T_1 = T_2 \quad (\text{when } I = 0)$$

This is what we expect for a massless pulley.