Torque

Unit #6a

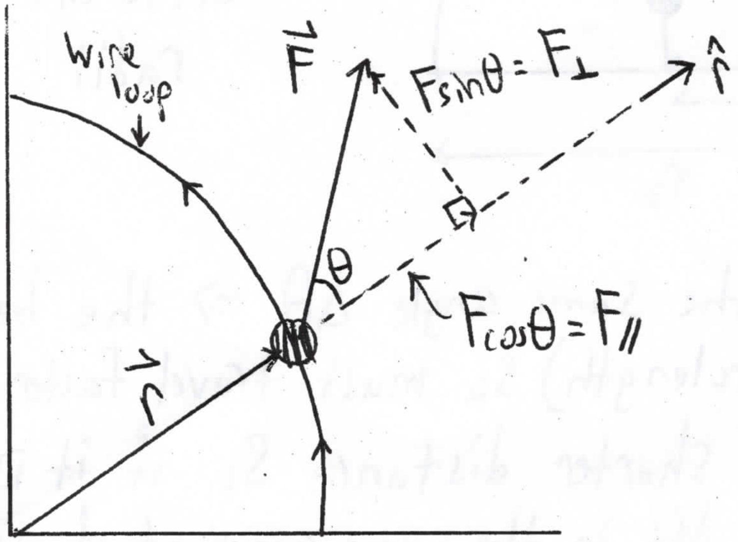
A torque is a measure of the ability to rotate an object:

Large torques: its easy to rotate an object

Small torques: its hard to rotate an object

The torque is related to the component of the force that causes rotations.

Example: A bead is attached to a wire loop and is constrained to move in a circle. A force $\vec{F}$ is applied causing motion of the bead.



$F_{\perp}$ = component of $\vec{F}$ perpendicular to $\vec{r}$

$F_{\parallel}$ = component of $\vec{F}$ parallel to $\vec{r}$

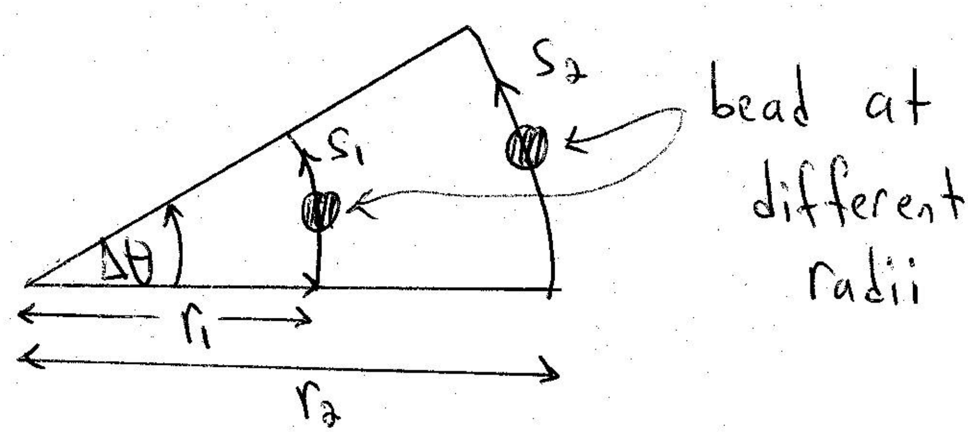
Note that $F_{\parallel}$ = a radial force $\Rightarrow$ it does not try to rotate object

Note that $F_{\perp}$ = a tangential force $\Rightarrow$ this is the force that leads to rotations

Then we can say that torque must be proportional to $F_{\perp}$:

$$\text{torque} = \boxed{\tau \sim F_{\perp}}$$

Also note that the object rotates faster as radius r increases for the same angle change $\Delta\theta$



Bead rotates ~~rotates~~ through the same angle $\Delta\theta$ $\Rightarrow$ the bead traveling a distance (arclength) S_2 must travel faster than bead traveling the shorter distance S_1 if it is to make the trip through $\Delta\theta$ in the same amount of time.

Since the bead on arclength S_2 rotates faster, it has a larger torque. Thus, the longer r is, the larger the torque:

$$\text{torque} = \boxed{\tau \sim r}$$

Let us the define torque as:

$$\text{torque} = \tau \equiv \underbrace{r}_{\substack{\text{the larger} \\ r \text{ is, the} \\ \text{faster it} \\ \text{rotates} \\ \text{(larger torque)}}} \underbrace{F_\perp}_{\substack{\text{the larger} \\ F_\perp \text{ is the} \\ \text{easier it} \\ \text{is to rotate} \\ \text{(there is more torque)}}} = r F \sin\theta$$

Notice that we can write torque several ways:

$$\tau = r\underbrace{(F\sin\theta)}_{r F_\perp} = \underbrace{(r\sin\theta)}_{r_\perp F} F$$

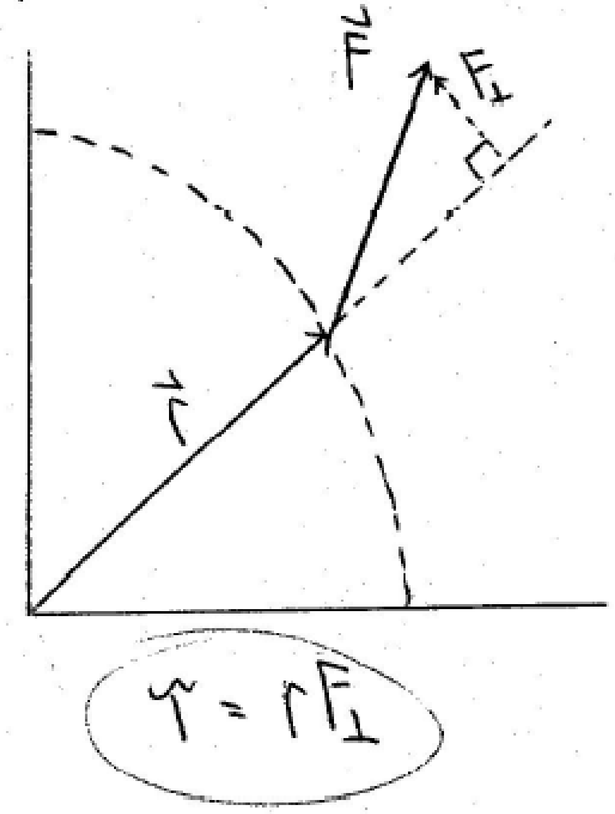
$F_\perp = F\sin\theta$ $\qquad$ $r_\perp = r\sin\theta$

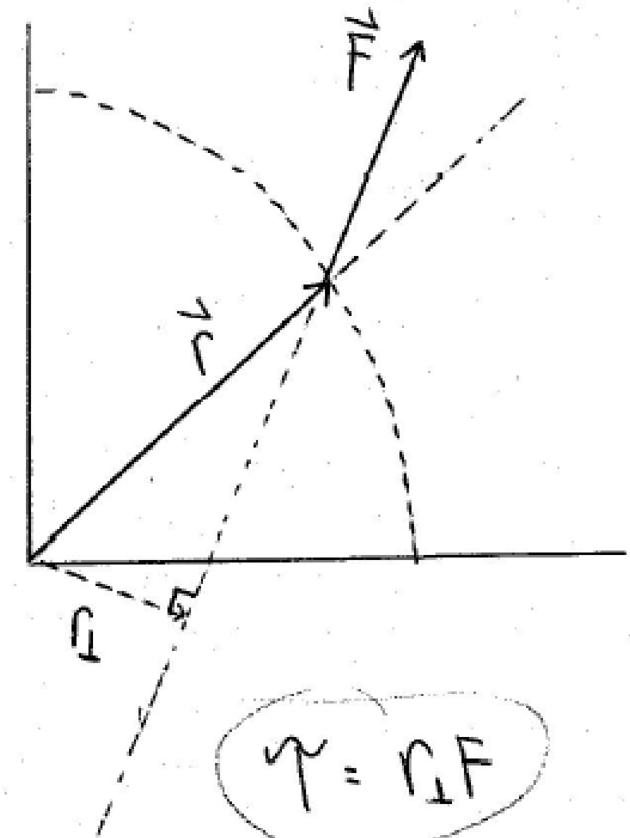
Then: $\tau = rF_\perp = (r) \times \left(\begin{array}{l}\text{the component of } \vec{F} \\ \text{perpendicular to } \vec{r}\end{array}\right)$

Or

$\tau = r_\perp F = \left(\begin{array}{l}\text{the component of } \vec{r} \\ \text{perpendicular to } \vec{F}\end{array}\right) \times F$

This is shown in the diagrams below:

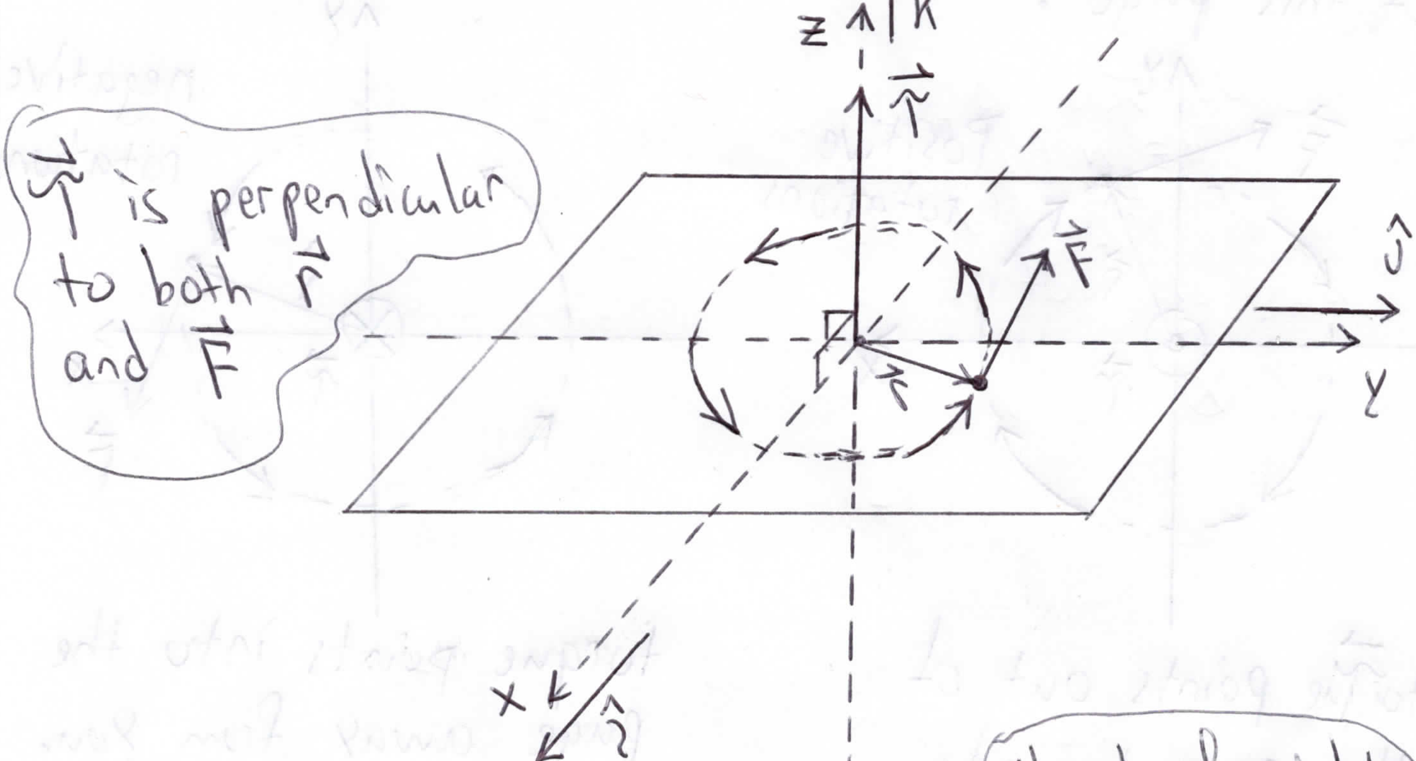




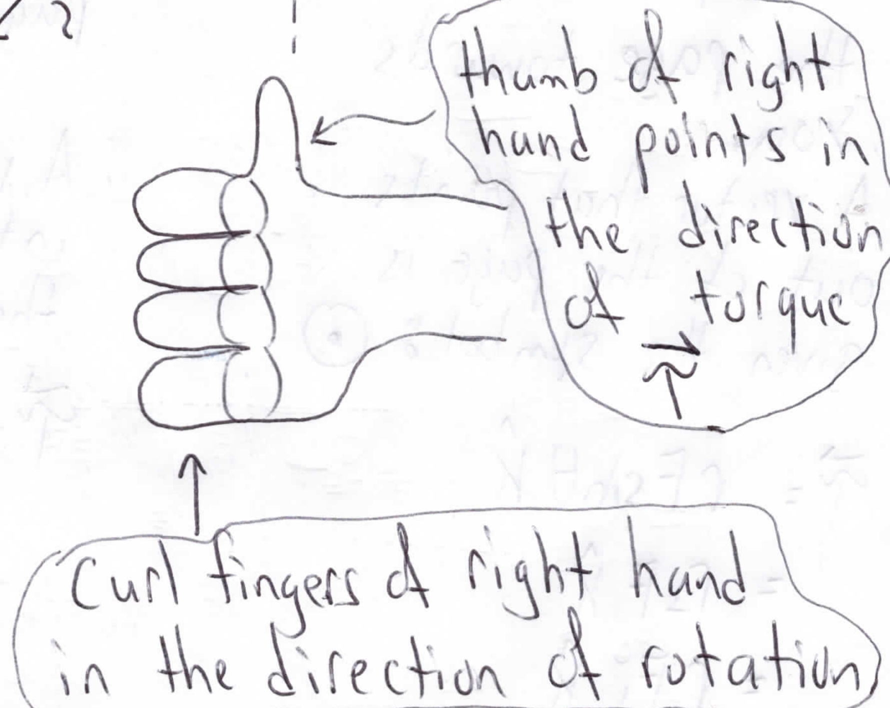
Thus, finding the perpendicular component of $\vec{r}$ or $\vec{F}$ will give the torque.

Torque is a vector that is perpendicular to both $\vec{r}$ and $\vec{F}$.

For example, let $\vec{r}$ and $\vec{F}$ lie in the xy-plane so that rotations are purely in the xy-plane:



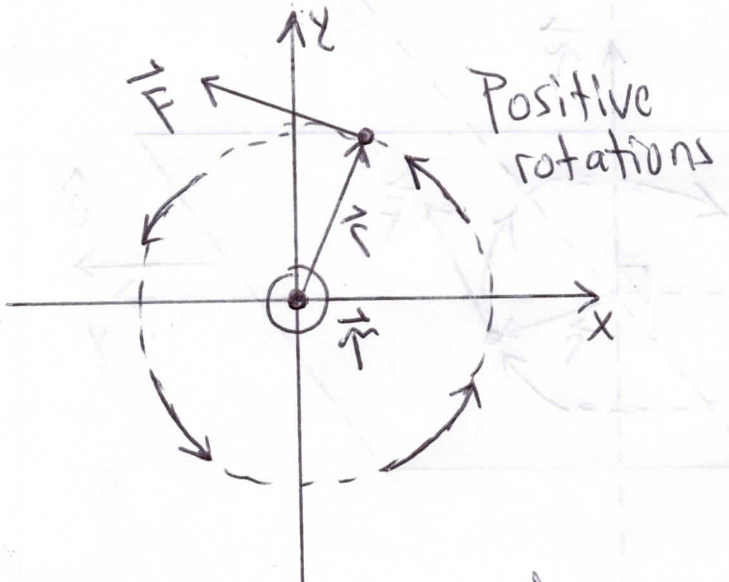
Right-hand Rule:

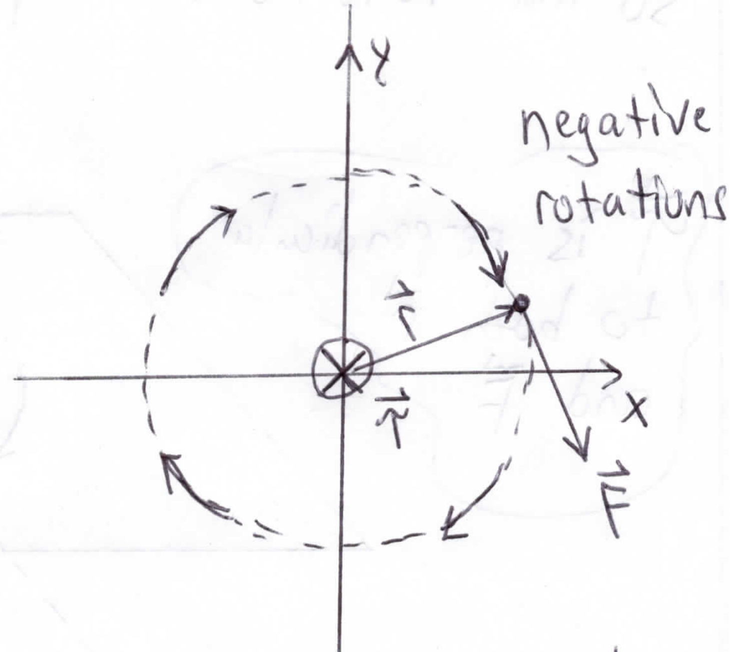


For the example above, $\vec{\tau}$ points in the z-direction which is the $\hat{k}$-direction:

$$\boxed{\vec{\tau} = rF\sin\theta\,\hat{k} = rF_{\perp}\hat{k}}$$

Thus, when the xy-plane lies in the sheet of this page → torque points either into or out of this page:





torque points out of the page towards you.

A vector that points out of the page is given the symbol: ⊙

$$\vec{\tau} = rF\sin\theta\,\hat{k}$$
$$= r_{\perp}F\,\hat{k}$$
$$= rF_{\perp}\,\hat{k}$$

The rotations above are positive rotations

ω = positive, α = positive

torque points into the page away from you.

A vector that points into the page is given the symbol: ⊗

$$\vec{\tau} = -rF\sin\theta\,\hat{k}$$
$$= -r_{\perp}F\,\hat{k}$$
$$= -rF_{\perp}\,\hat{k}$$

The rotations above are negative rotations

ω = negative, α = negative

Equilibrium

An object in equilibrium is a stationary object. Thus, the total forces and torques on an object are zero.

$$\left.\begin{array}{l} \vec{F}_{Total} = 0 \\ \vec{\tau}_{Total} = 0 \end{array}\right\} \text{Equilibrium Condition}$$

This directly comes from Newton's 2nd law:

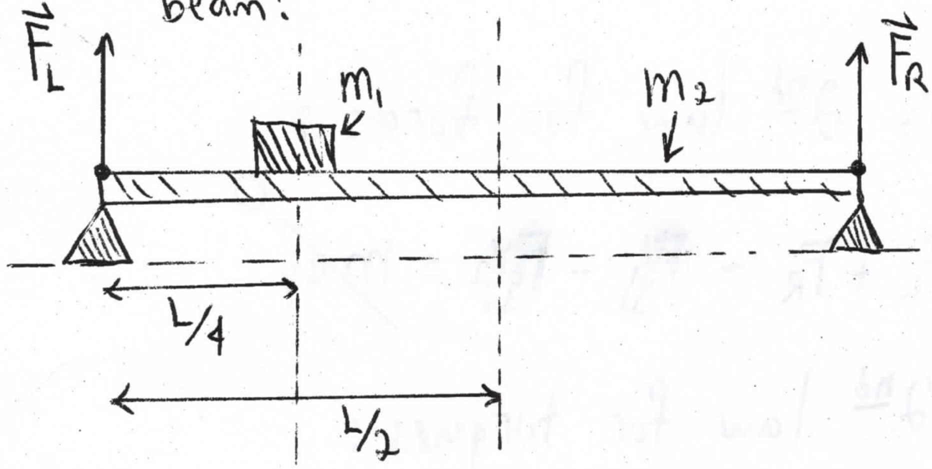
$$\left(\begin{array}{c}\text{Newton's 2nd law} \\ \text{for translations}\end{array}\right) = \vec{F}_{Total} = m\vec{a} = 0 \text{ for no movement}$$

$$\left(\begin{array}{c}\text{Newton's 2nd law} \\ \text{for rotations}\end{array}\right) = \vec{\tau}_{Total} = \underbrace{I}_{\text{moment of inertia}}\ \underbrace{\vec{\alpha}}_{\text{angular acceleration}} = 0 \text{ for no rotations}$$

For rotations, I = moment of inertia $(Kg \cdot m^2)$

$\vec{\alpha}$ = angular acceleration $(rads/sec^2)$

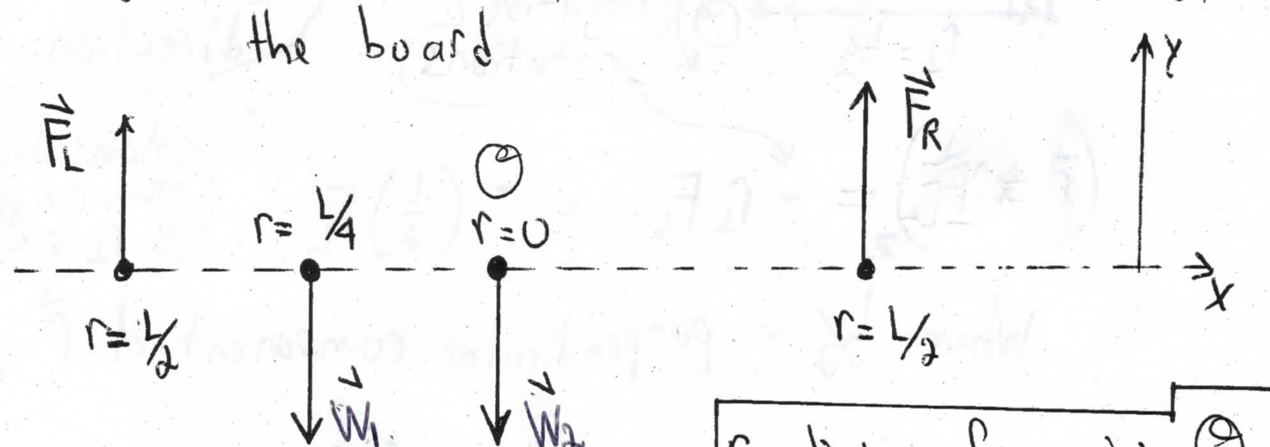
Example: A block of mass $m_1 = 2.7\,Kg$ rests on a wooden beam of mass $m_2 = 1.8\,Kg$ a distance $L/4$ from its end (the length of the beam is L). The wooden beam is supported at two ends, and the system is in equilibrium. What are the forces F_L and F_R on the ends of the beam?



(A) First, we draw a free body diagram. For each mass, we must determine where the center of mass is:

For $m_1 \rightarrow$ the center of mass lies $L/4$ from the center of the board

For $m_2 \rightarrow$ the center of mass is at the center of the board



$r =$ distance from origin Θ

Note: $\vec{W}_1$ = force due to gravity on mass m_1 placed at its center of mass

$\vec{W}_2$ = force due to gravity on the board m_2 placed at the center of mass of the board.

I have defined the center of mass of the board to be at the origin $\mathcal{O}$ where $\vec{r} = 0$.

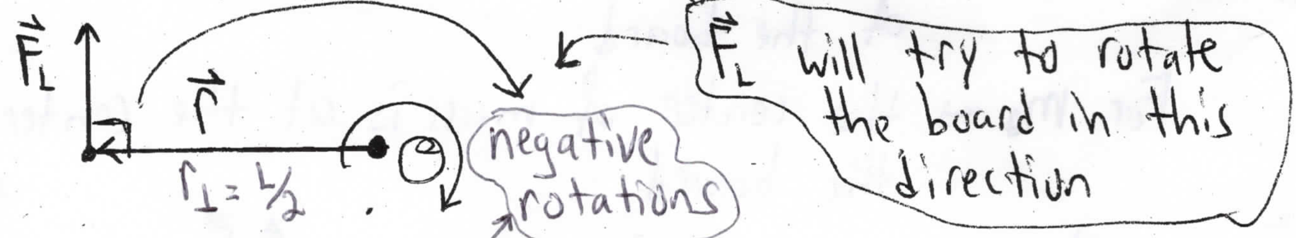
(B) Newton's 2nd law for forces:

$$F_{Total,y} = F_L + F_R - W_1 - W_2 = \cancel{m a_y}^{\,0}$$

Newton's 2nd law for torques:

$$\tau_{Total,z} = (\pm r_\perp F_L) + (\pm r_\perp F_R) + (\pm r_\perp W_1) + (\pm r_\perp W_2) = \cancel{I \alpha_z}^{\,0}$$

Let's see, step by step, how to get each of the terms:



$$\pm r_\perp F_L = - r_\perp F_L = -\left(\frac{L}{2}\right) F_L$$

(note: distance from origin to $\vec{F}_L$ is $r_\perp = L/2$)

Where $L/2$ = perpendicular component of $\vec{r}$ onto $\vec{F}_L$

To get the minus sign, note that if you curl the fingers of your right hand in the direction shown in the diagram, your thumb points into the page. Let us define

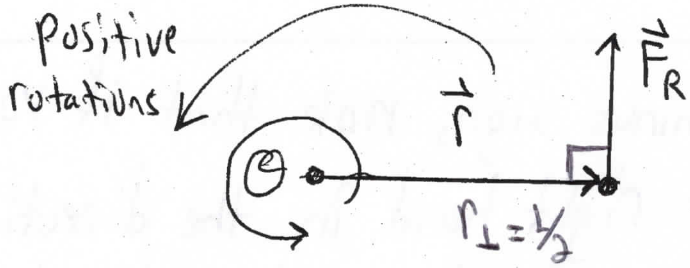
(Positive rotations and torques) = (Curl fingers of right hand in direction of rotation: thumb points out of page.)

(Negative rotations and torques) = (Curl fingers of right hand in direction of rotation: thumb points into the page)

Then, $\vec{F}_L$ will try to induce a negative torque on the board.

So, we get the 1st term in Newton's 2nd law for torques:

$$\pm r_\perp F_L = -\left(\frac{L}{2}\right) F_L$$



$$\pm r_\perp F_R = + r_\perp F_R = \underbrace{+\left(\frac{L}{2}\right) F_R}_{\text{2nd term in the torque equations}}.$$

note: distance from origin to $\vec{F}_R$ is $r_\perp = \frac{L}{2}$

Note that the force at the center of the board will not try try to rotate the board at all ;

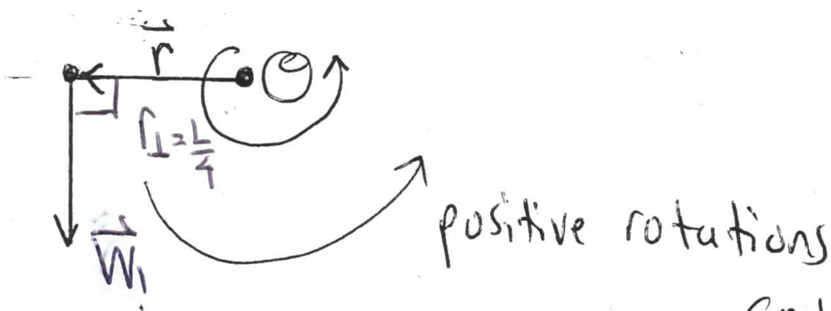
torque due to $\vec{W}_2 = 0$ intuitively

$$\pm r_\perp W_2 = 0 \quad \text{since } \underbrace{\vec{r} = 0}_{\text{it is at the origin}}$$

θ

$(r_\perp = 0)$

$\Rightarrow$ There is no torque due to $\vec{W}_2$



$$\pm r_\perp W_1 = + \underbrace{r_\perp W_1}_{\text{3rd term in the torque eqns.}} = \left(\frac{L}{4}\right) m_1 g$$

(note: distance from origin to $\vec{F}_{g1}$ is $r_\perp = \frac{L}{4}$)

Summarizing:

$$\begin{cases} F_L + F_R - m_1 g - m_2 g = 0 & \rightarrow \text{Newton's 2nd law for forces} \\ -F_L\left(\frac{L}{2}\right) + F_R\left(\frac{L}{2}\right) + m_1 g\left(\frac{L}{4}\right) = 0 & \rightarrow \text{Newton's 2nd law for torques} \end{cases}$$

Simplifying:

$$\begin{cases} F_L + F_R = m_1 g + m_2 g & (1) \\ -F_L + F_R = -\dfrac{m_1 g}{2} & (2) \end{cases}$$

add: $0 + 2F_R = \dfrac{m_1 g}{2} + m_2 g$

$$\Rightarrow \boxed{F_R = \frac{1}{2}\left(\frac{m_1}{2} + m_2\right) g}$$

We can substitute this into Eq.(2) to get F_L:

$$F_L = F_R + \frac{m_1 g}{2} = \frac{1}{2}\left(\frac{m_1}{2} + m_2\right)g + \frac{m_1 g}{2}$$

$$= \left(\frac{3}{4}m_1 + \frac{1}{2}m_2\right)g$$

$$\boxed{F_L = \frac{1}{2}\left(\frac{3m_1}{2} + m_2\right)g}$$