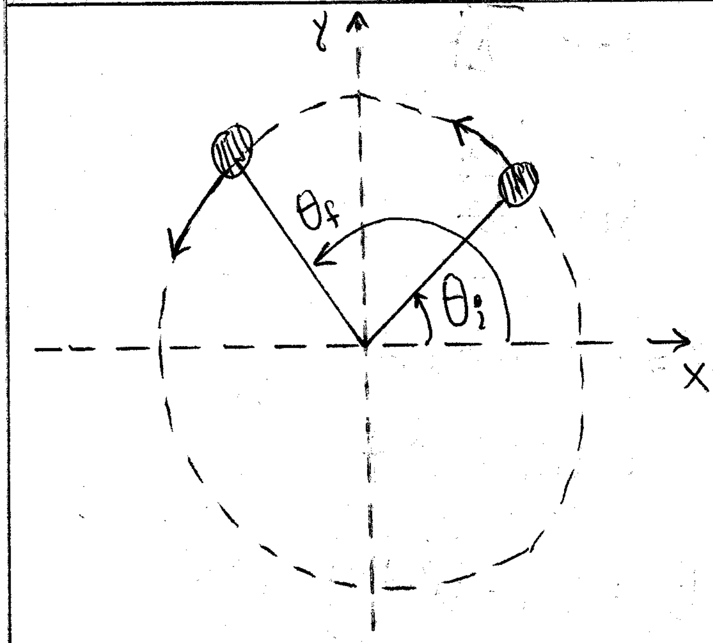
Rotational Motion

Unit #5

Kinematics for Rotational Motion:



For rotations of an object in a circle in the xy-plane:

θ = angular displacement (radians = rads)

ω = angular velocity (rads/sec)

α = angular acceleration (rads/$\sec^2$)

$$\boxed{1 \text{ revolution} = 360^\circ = 2\pi \text{ radians}} \Longrightarrow$$

useful conversion factors

There is nearly a 1-to-1 correspondence between the symbols used for translational motion and those used for rotational motion:

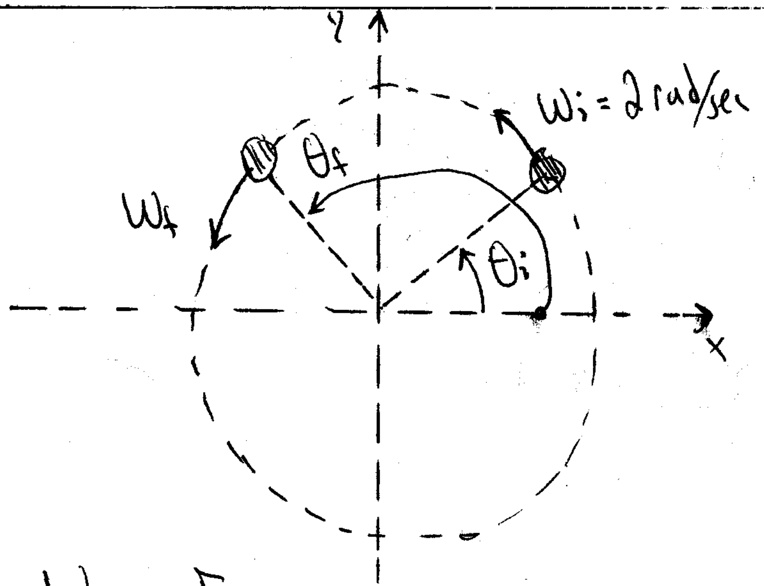
$$\boxed{\begin{array}{c} x \to \theta \\ v \to \omega \\ a \to \alpha \end{array}}$$

Translational Motion	Rotational Motion
$\Delta\vec{X} = X_f - \vec{X}_i$	$\Delta\theta = \theta_f - \theta_i$
$\boxed{\text{displacement = a vector}}$	$\boxed{\text{angular displacement however is not a vector}}$
$\vec{v} = \lim\limits_{\Delta t\to 0} \frac{\Delta X}{\Delta t}$	$\omega = \lim\limits_{\Delta t\to 0} \frac{\Delta\theta}{\Delta t}$
$\vec{a} = \lim\limits_{\Delta t\to 0} \frac{\Delta v}{\Delta t}$	$\alpha = \lim\limits_{\Delta t\to 0} \frac{\Delta\omega}{\Delta t}$

Eq. M1: a = constant	α = constant
Eq. M2: $v = v_o + at$	$\omega = \omega_o + \alpha t$
Eq. M3: $X = X_o + v_o t + \frac{1}{2}at^2$	$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$
Eq. M4: $X = X_o + \frac{v^2 - v_o^2}{2a}$	$\theta = \theta_o + \frac{\omega^2 - \omega_o^2}{2\alpha}$

Example: A wheel rotates with a constant angular acceleration of 3.5 rad/sec². If the angular velocity of the wheel is 2 rad/sec at $t=0$

(a) Through what angle does the wheel rotate between $t=0$ and $t = 2$ sec?



let $\theta_i = 0$

(just like we like to have $x_0 = 0$)

Use Eq. M3:

$$\underset{\substack{\downarrow\\ \text{final}\\ \text{angle}}}{\theta} = \underset{\substack{\downarrow\\ \text{initial}\\ \text{angle}\\ =0}}{\cancelto{0}{\theta_0}} + \underset{\substack{\downarrow\\ \text{initial}\\ \text{speed}\\ =2\,\text{rad/sec}}}{\omega_0} t + \frac{1}{2}\underset{\substack{\downarrow\\ 3.5\,\text{rad/sec}^2}}{\alpha} t^2$$

$$\Rightarrow \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= \left(2\frac{\text{rad}}{\text{sec}}\right)(2\text{sec}) + \frac{1}{2}\left(3.5\frac{\text{rad}}{\text{sec}^2}\right)(2\text{sec})^2$$

$$= 11\text{ rad}$$

To express this in revolutions, use conversion factor

1 revolution = 2π radians

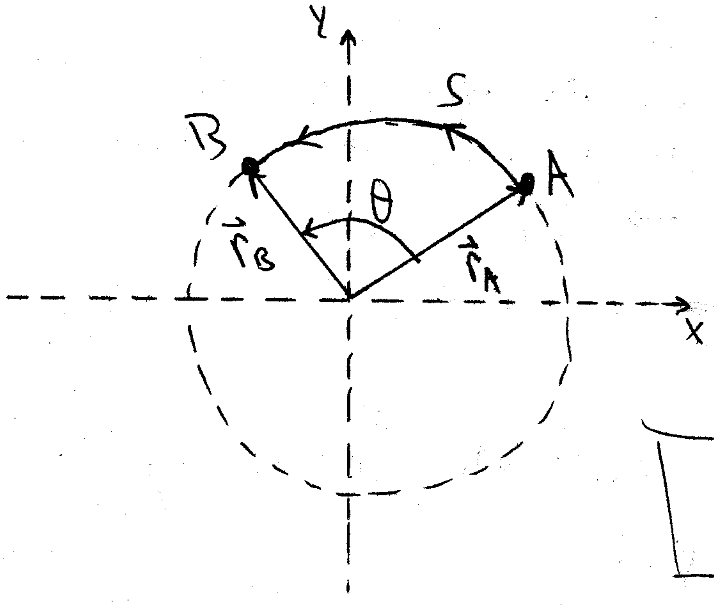
$$\theta = 11\,\cancel{\text{rad}} \cdot \frac{1\text{ rev}}{2\pi\,\cancel{\text{rad}}} = 1.75\text{ rev}$$

(b) What is the angular velocity of the wheel at $t = 2$ sec?

Eq. M2: $\omega = \omega_0 + \alpha t = \left(2\text{rad/sec}\right) + \left(3.5\text{rad/sec}^2\right)(2\text{sec})$

$$= 9\text{ rad/sec}$$

Translational and Rotational Variables:



Let an object move from point A to point B on a circle of radius r. The distance it travels is the arclength S, where

$$\boxed{S = r\theta}$$

r = radius
θ is in radians

translational variables, S
rotational variable, θ

The speed the object travels on the perimeter of the circle is

$$\boxed{v_t = \omega r}$$

ω = angular speed
v_t = tangential speed

translational
rotational

The acceleration of the object on the perimeter of the circle is

$$\boxed{a_t = \alpha r}$$

α = angular acceleration
a_t = tangential acceleration

translational
rotational

Centripetal Acceleration:

Uniform circular motion ⇒ occurs when an object moves in a circle at constant speed

→ v_t = constant

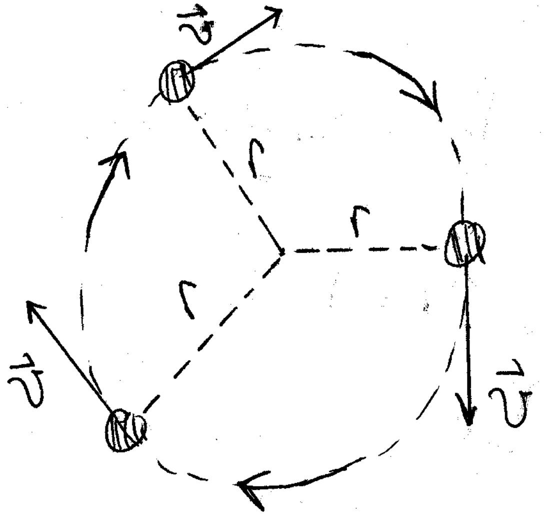
→ $\alpha_t = 0$

However, the object is still accelerating because the direction of the velocity changes:

$$\vec{a}_{ave} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{V}_f - \vec{V}_i}{\Delta t}$$

} even if $v_f = v_i$
$\vec{a}$ can be nonzero
if the direction of $\vec{v}$
changes

Example: A ball attached to a string moving in a circle at constant speed.

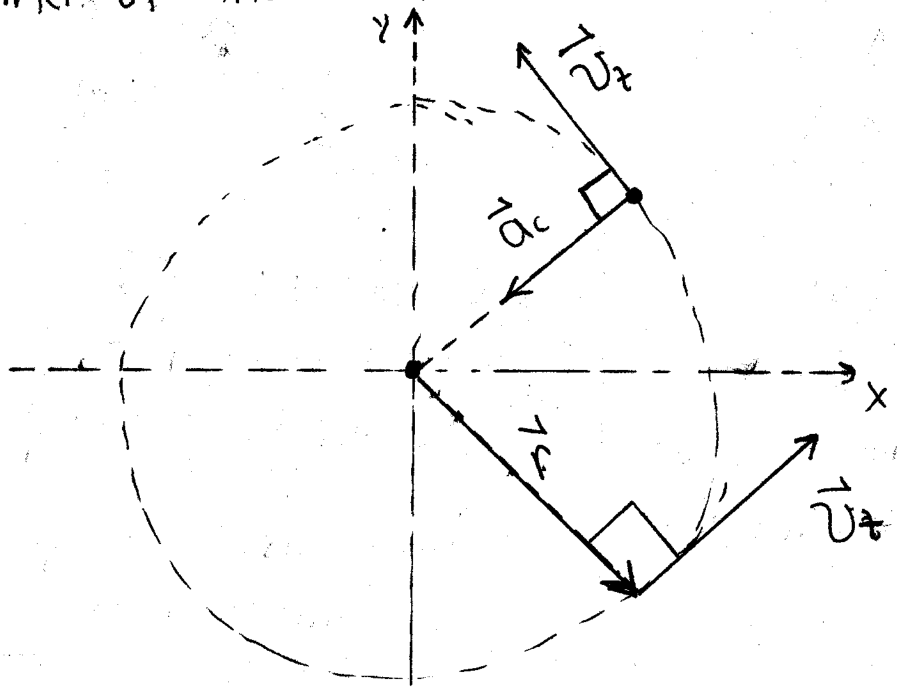


The velocity changes in direction as the ball moves in a circle

⇒ thus, the ball is accelerating even if the ball's speed is constant.

$$\vec{a}_{ave} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\text{change in velocity}}{\text{Change in time}}$$

An object in uniform circular motion undergoes centripetal acceleration, $\vec{a}_c$, that is directed towards the center of the circle:



Note: $\vec{v}_t$ is perpendicular to $\vec{r}$

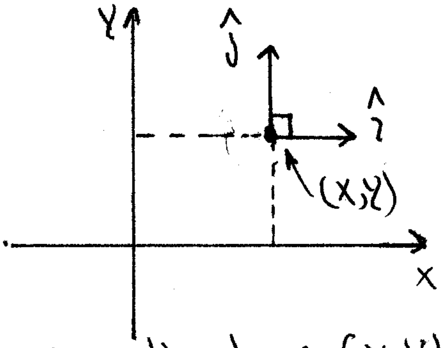
$\vec{a}_c$ points inward towards center of circle

$$|\vec{a}_c| = a_c = \frac{v_t^2}{r}$$

Note: $a_c \neq a_t = \alpha r$

Newton's 2nd law for Rotation Motion:

Cartesian Coordinates



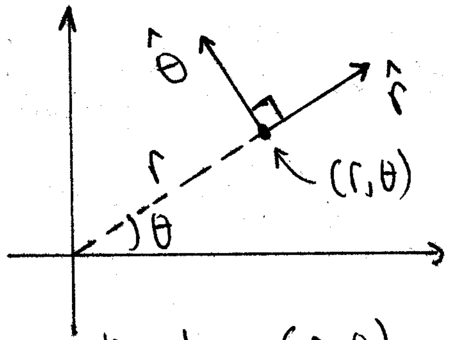
Coordinate: (x, y)

Unit vectors: $\hat{i}, \hat{j}$

$\hat{i} \rightarrow$ points in +x-direction

$\hat{j} \rightarrow$ points in +y-direction

Polar Coordinates



Coordinate: (r, θ)

Unit vectors: $\hat{r}, \hat{\theta}$

$\hat{r} \rightarrow$ points in radial outward direction

$\hat{\theta} \rightarrow$ points in tangential direction of increasing θ

Position: $\vec{r} = x\hat{i} + y\hat{j}$

$$\vec{r} = r\hat{r}$$

Velocity: v_x

v_y

$$v_r = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} \quad \left(\text{motion purely in radial direction}\right)$$

$$v_\theta = v_t = \omega r \quad \text{tangential velocity}$$

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acceleration: a_x

a_y

$$a_r = \underbrace{\left(\lim_{\Delta t \to 0} \frac{\Delta v_r}{\Delta t}\right)}_{\text{(acceleration purely in radial direction)}} + \underbrace{\left(-\frac{v_t^2}{r}\right)}_{\text{(centripetal acceleration)}}$$

$$a_\theta = \underbrace{\alpha r}_{\text{tangential acceleration}} + \underbrace{2 v_r \omega}_{\text{Coriolis acceleration}}$$

Newton's Laws in Cartesian Coordinates

$$F_{Total,y} = ma_y$$

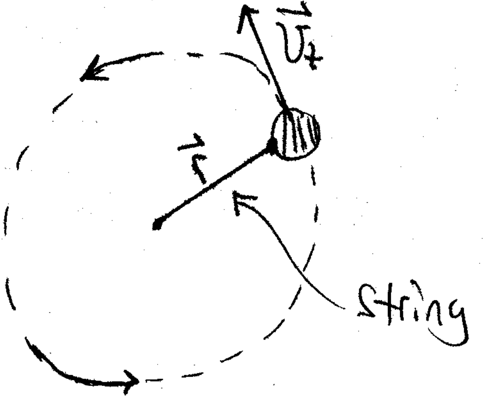
$$F_{Total,x} = ma_x$$

Newton's Laws in Polar Coordinates

$$F_{Total,r} = ma_r = -\frac{mv_t^2}{r}$$

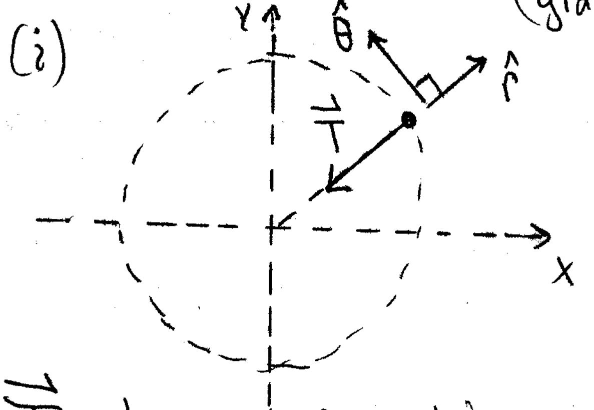
$$\underbrace{F_{Total,\theta} = ma_\theta = m\alpha r}_{\text{for objects that move in circle of constant radius}}$$

Example: Ball on a string twirled in a circle (ignore gravity)



$\xrightarrow{\text{free body}}$ diagram

(i)



$\vec{T}$ = tension in string = the only force on the ball

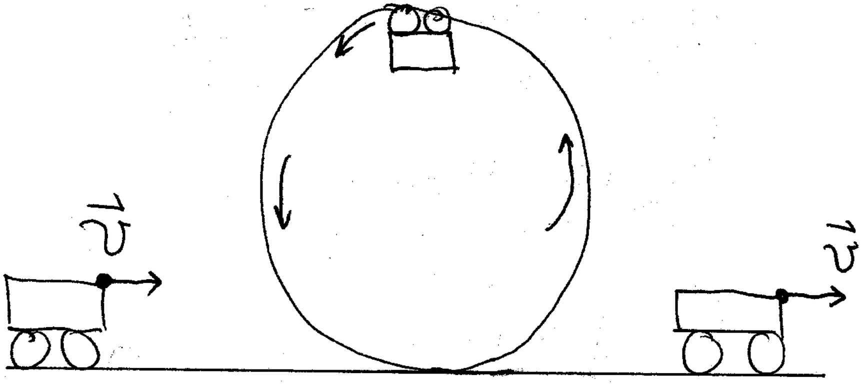
(ii) Newton's 2nd law

$$F_{Total,r} = -T = -\frac{mv_t^2}{r}$$

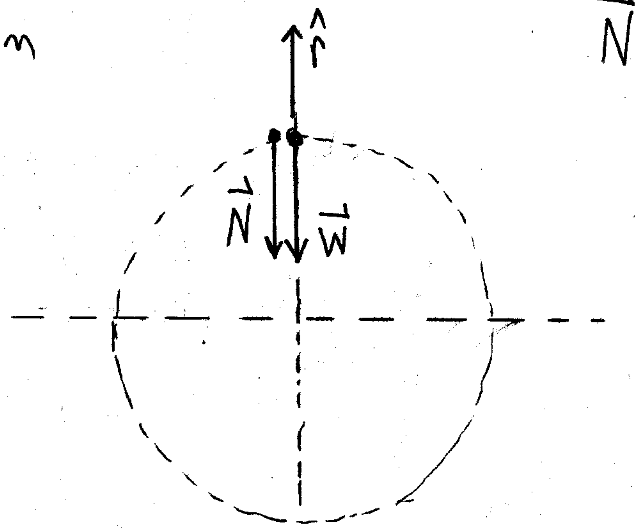
↑ points in the negative $\hat{r}$-direction

$$\Rightarrow \boxed{T = \frac{mv_t^2}{r}} = \text{the tension in the string}$$

Example: A stunt car drives in a vertical loop at constant speed. What is the lowest speed he can have and still remain in contact with the top of the loop?



(i) Free body diagram



$\vec{N}$ = normal force of loop on the car

(ii) Newton's 2nd law:

$$F_{Total,r} = -N - W = \frac{-m v_T^2}{r}$$

$$\Rightarrow \quad N + mg = \frac{m v_T^2}{r}$$

} In order for car to remain in contact with loop: $N > 0$

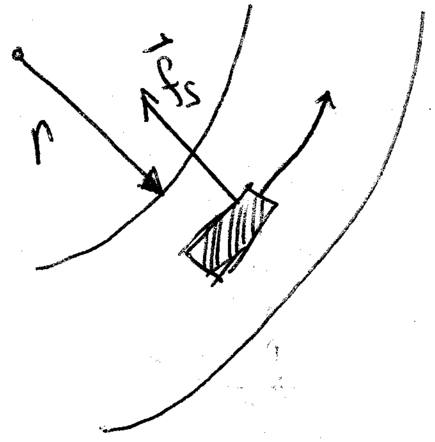
Then, $N = m v_T^2/r - mg > 0$

$$v_T^2 > gr \Rightarrow v_T > \sqrt{gr}$$

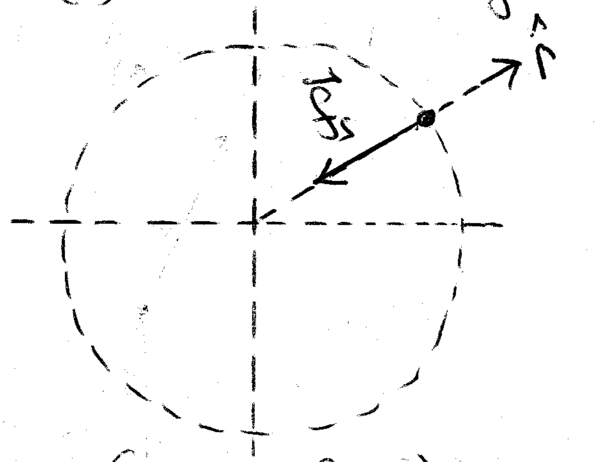
} The car's speed must be <u>greater</u> than $\sqrt{gr}$

Example: Serway Example 7.6

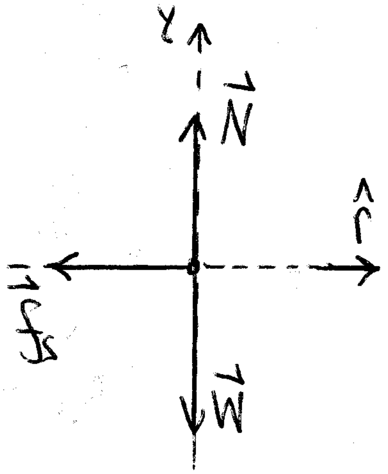
A car travels at a constant speed of 30mi/hr on a level circular turn of radius 50m. What is the minimum μ_s that will allow the car to navigate the curve without sliding?



(i) Free body diagram



(Viewed from the top)



(Viewed from the side)

(ii) Newton's 2nd law:

$$\begin{cases} F_{Total,r} = -f_s = -mv_t^2/r \\ F_{Total,y} = N - W = \cancel{ma_y}^{0} \end{cases} \Rightarrow \begin{cases} f_s = mv_t^2/r & ① \\ N - mg = 0 & ② \end{cases}$$

Note that $\underset{\underset{mv_t^2/r}{\downarrow}}{f_s} \le \mu_s \underset{\underset{mg}{\downarrow}}{N}$, but Eq.② gives: $N = mg$

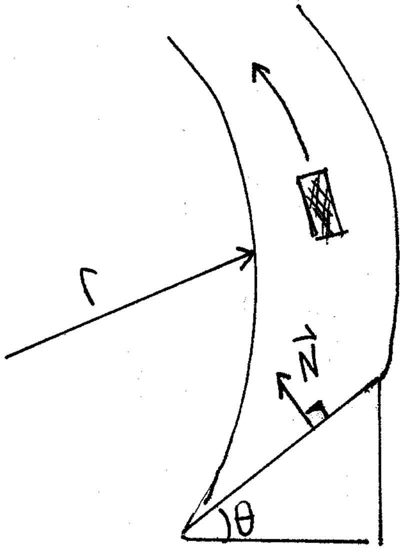
Then,

$$\Rightarrow \quad \frac{mv_t^2}{r} \le \mu_s mg \Rightarrow \mu_s \ge \frac{v_t^2}{gr} = \frac{(13.4m/sec)^2}{(9.8m/sec^2)(50m)}$$

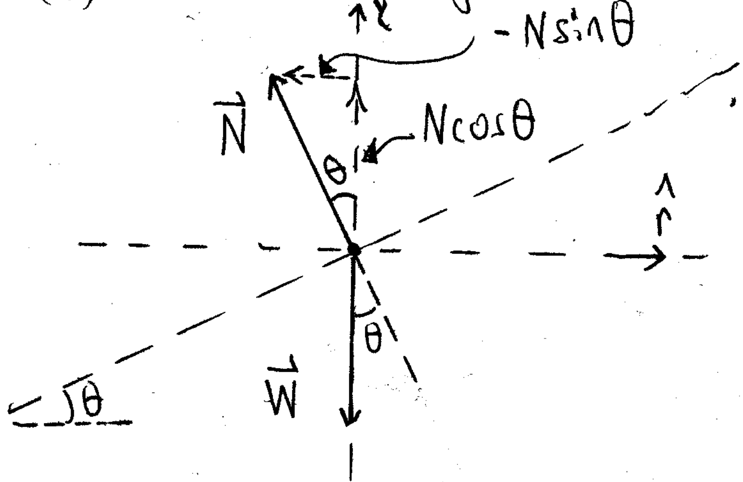
$$\mu_s \ge 0.366$$

The minimum μ_s is: $\mu_s = 0.366$. Anything small and car slides.

Example: A car is negotiating a curve in a road having an inclination angle of θ. If the road is wet (no friction) what is the maximum speed it can have before sliding off the road?



(i) Free body diagram



(ii) Newton's 2nd law:

$$\begin{cases} F_{Total,y} = N\cos\theta - W = \cancel{ma_y}^{\,0} \\ F_{Total,r} = -N\sin\theta = -\frac{mv_t^2}{r} \end{cases} \Rightarrow \begin{cases} N\cos\theta - mg = 0 \\ N\sin\theta = \frac{mv_t^2}{r} \end{cases}$$

To find minimum speed, re-arrange eqns. so that N is eliminated

$$N\sin\theta = mv_t^2/r$$
$$N\cos\theta = mg$$

divide: $\frac{N\sin\theta}{N\cos\theta} = \frac{mv_t^2/r}{mg} \Rightarrow \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{v_t^2}{gr}$

$$\Rightarrow \boxed{v_t = \sqrt{gr\tan\theta}}$$