Projectile Motion

Unit #2

Vectors:

Vectors $\begin{cases} \text{position} : \vec{x} & (50 \text{ miles north}) \\ \text{Velocity} : \vec{v} & (50 \text{ miles/hr north}) \\ \text{acceleration} : \vec{a} & (\text{rock falls } 9.8 \text{ m/sec}^2 \text{ downward}) \end{cases}$

Vectors have both a magnitude and direction.

Scalars have only a magnitude:

Scalars $\left\{ \begin{matrix} \text{time} = t & (5 \text{ seconds}) \\ \text{mass} = m & (5 \text{ Kg}) \\ \text{Speed} = v & (50 \text{ miles/hr}) \end{matrix} \right\}$ Scalars

Vector:

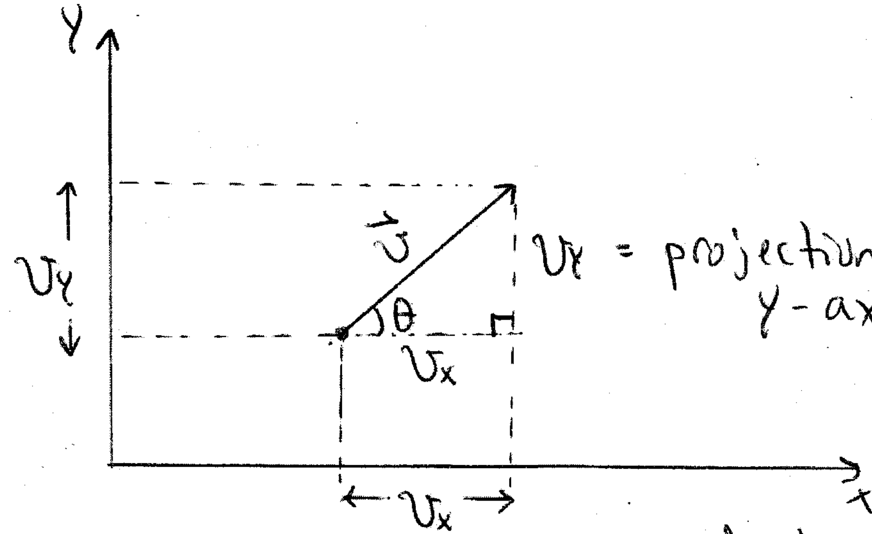
$\vec{v}$ ← half arrow on top

Scalar:

$v = |\vec{v}|$

Components of a Vector:

Let a vector $\vec{v}$ point in an arbitrary direction at an angle θ relative to the positive x-axis:



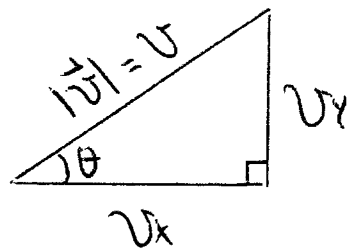
v_y = projection of $\vec{v}$ onto y-axis

v_x = projection of $\vec{v}$ onto x-axis

Notice that the:

x-component of $\vec{v}$ = v_x = projection of $\vec{v}$ onto the x-axis
= base of a right triangle

y-component of $\vec{v}$ = v_y = projection of $\vec{v}$ onto the y-axis
= height of right triangle.



From basic trigonometry, the components of $\vec{v}$ are

$$\cos\theta = \frac{v_x}{|\vec{v}|} = \frac{v_x}{v} \Rightarrow \boxed{v_x = v\cos\theta}$$

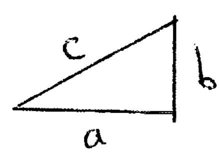
(note $|\vec{v}|$ = length or magnitude of vector $\vec{v}$)

$$\sin\theta = \frac{v_y}{|\vec{v}|} = \frac{v_y}{v} \Rightarrow \boxed{v_y = v\sin\theta}$$

The angle θ can be written in terms of the components of $\vec{v}$:

$$\tan\theta = \frac{v_y}{v_x} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)}$$

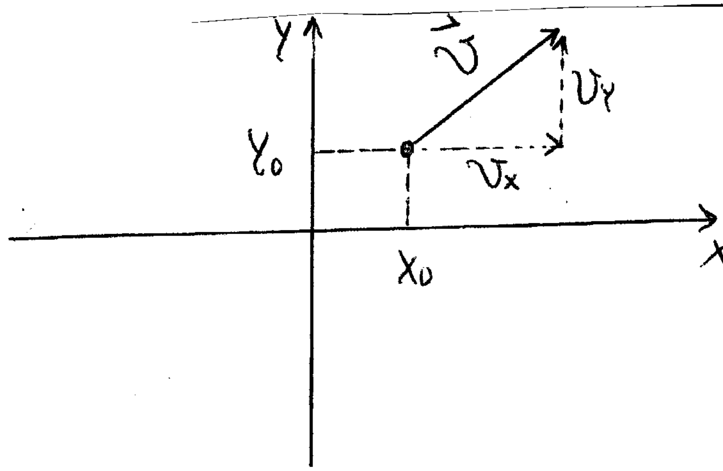
The magnitude of $\vec{v}$ can be found by applying Pythagorean's Theorem:



$$\boxed{a^2 + b^2 = c^2}$$

$$v_x^2 + v_y^2 = v^2 \Rightarrow \boxed{v = \sqrt{v_x^2 + v_y^2}}$$

Two Dimension (2-D) Motion:



For motion in two-dimensions, a particle can now have initial positions x_0 and y_0 and initial velocities v_{x0} and v_{y0}. Projectile motion means the particle is in a gravitational field. Let the y-axis be in the vertical direction (up) and the x-axis be in the horizontal direction. Then there are two sets of equations for each direction:

Equations of Motion in the y-direction (vertical):

Eq. M1: $a_y = -g$ = constant

Eq. M2: $v_y = v_{y0} - gt$

Eq. M3: $y = y_0 + v_{y0}t - \frac{1}{2}gt^2$

Eq. M4: $y = y_0 + \dfrac{v_y^2 - v_{y0}^2}{2(-g)}$

Equations of Motion in the horizontal direction:

Eq. M1: $a_x = \text{constant}$ (gravity does not pull in the horizontal direction)

Eq. M2: $v_x = v_{xo} + a_x t$

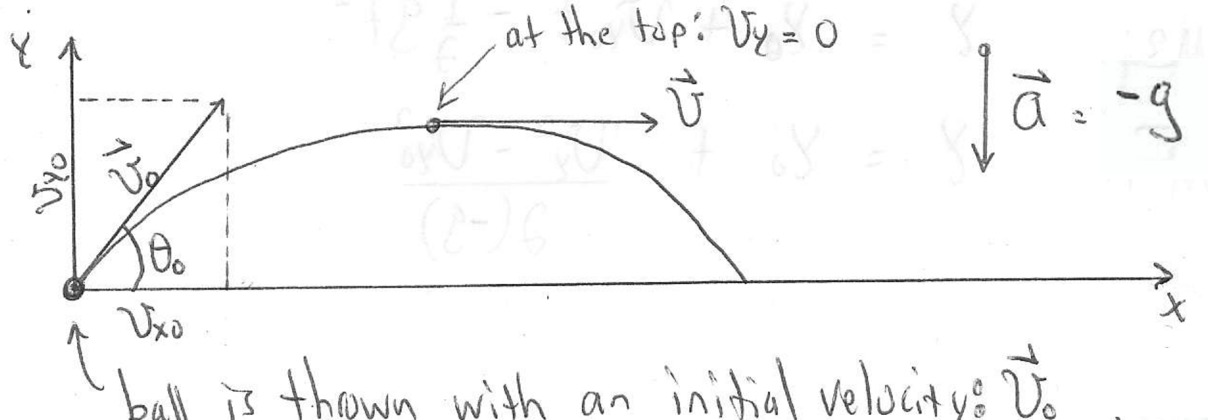
Eq. M3: $x = x_o + v_{xo} t + \frac{1}{2} a_x t^2$

Eq. M4: $x = x_o + \dfrac{v_x^2 - v_{xo}^2}{2a_x}$

Notice that in the horizontal direction, there is no pulling force from gravity. Thus, $a_x \neq g$ in the horizontal direction.

$\Rightarrow$ We are done with Projectile Motion.
To solve problems in this category, its just a matter of find out: What is x_o? What is y_o? What is v_{xo}? What is v_{yo}?

Projectile Motion:



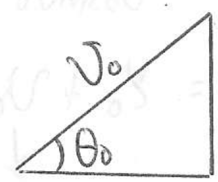
ball is thrown with an initial velocity: $\vec{v}_o$
The acceleration of the ball at all times is: $\vec{a} = -g$

There is no acceleration in the x-direction for this problem: the ball is thrown with a constant velocity $\vec{v}_0$

$$\Rightarrow a_x = 0$$

Let's write the equations of motion in the x-direction using these variables.

Note, the components of $\vec{v}_0$:



$v_{y0} = v_0 \sin\theta_0$

$v_{x0} = v_0 \cos\theta_0$

The equations of motion are then:

Eq. M1: $a_x = 0 \longrightarrow a_x = 0$

Eq. M2: $v_x = \underbrace{v_{x0}}_{v_0\cos\theta_0} + \underbrace{\cancel{a_x}}_{0} t \longrightarrow v_x = v_{x0} \cos\theta_0$

Eq. M3: $x = x_0 + \underbrace{v_{x0}}_{v_0\cos\theta_0} t + \frac{1}{2}\underbrace{\cancel{a_x}}_{0} t^2 \longrightarrow x = x_0 + v_0\cos\theta_0 t$

$$\boxed{\begin{aligned} a_x &= 0 \\ v_x &= v_{x0}\cos\theta_0 \\ x &= x_0 + v_0\cos\theta_0 t \end{aligned}}$$

Eq. M4: There is no such equation since $a_x = 0$

The Equations of Motion in the y-direction are:

Eq. M1 : $a_y = -g$

Eq. M2 : $v_y = \underbrace{v_{y0}}_{v_0\sin\theta_0} + \underbrace{a_y}_{-g} t$

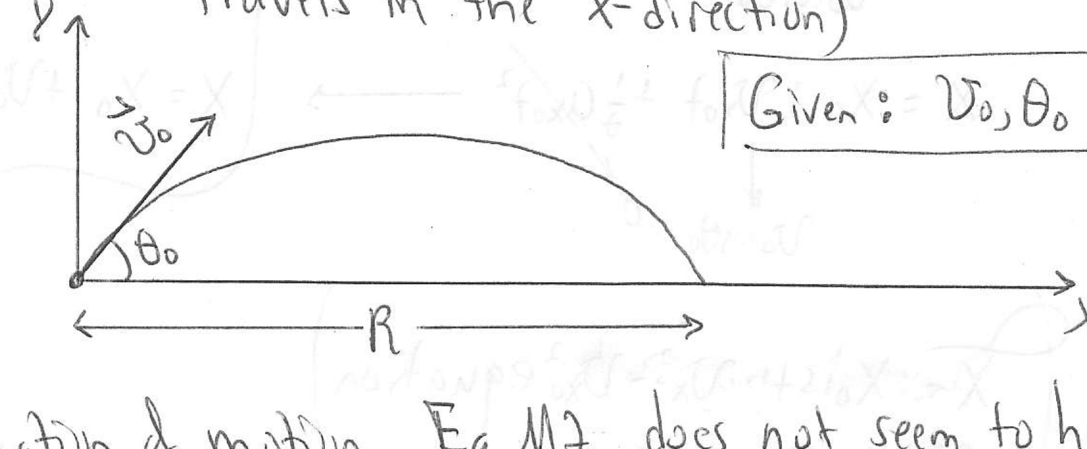
Eq. M3 : $y = y_0 + \underbrace{v_{y0}}_{v_0\sin\theta_0} t + \frac{1}{2}\underbrace{a_y}_{-g} t^2$

Eq. M4 : $y = y_0 + \dfrac{v_y^2 - v_{y0}^2}{2a_y}$

$$\boxed{\begin{aligned} a_y &= -g \\ v_y &= v_0\sin\theta_0 - gt \\ y &= y_0 + v_0\sin\theta_0 - \tfrac{1}{2}gt^2 \\ y &= y_0 + \frac{v_y^2 - v_0^2\sin^2\theta_0}{2(-g)} \end{aligned}}$$

Example: Calculate the Range (the distance the ball travels in the x-direction)

$\boxed{\text{Given: } v_0, \theta_0}$



Equation of motion Eq. M2 does not seem to help since it does not calculate distance. Then, using Eq. M3 for the x-direction:

$$\underbrace{X}_{\substack{\text{final position}\\ X = R}} = \underbrace{X_0}_{\substack{0\\ \text{initial position}}} + \underbrace{v_{x0}}_{v_0\cos\theta_0} t + \frac{1}{2}\underbrace{\cancel{a_x}}_{0} t^2$$

Then,

$$\text{Range} = R = v_0 \cos\theta_0 t$$

but, what is "t"?
we were not given this #
$\Rightarrow$ we only know v_0, θ_0

Eq. M2 in the x-direction has no "t" in it $\rightarrow$ it can't help us.

Eq. M2 in the y-direction:

$$\underset{\substack{\text{unknown} \\ \text{(final velocity)}}}{v_y} = \underbrace{v_0 \sin\theta_0}_{\text{known}} - gt$$

$\Rightarrow$ We cannot find "t" since we do not know v_y.

Eq. M3 in the y-direction:

$$y = y_0 + v_{y0} t + \tfrac{1}{2} a_y t^2$$

final position $y = 0$ since the ball returns to the ground

initial position $= 0$ since the ball starts at the ground

$v_0 \sin\theta_0$

$-g$

Then,

$$0 = 0 + \underbrace{v_0 \sin\theta_0 t}_{\text{Known}} - \frac{1}{2} g t^2$$

we can solve for "t" since all other parameters are Known

Re-arranging the equation:

$$0 = t\left(v_0 \sin\theta_0 - \frac{1}{2} g t\right)$$

(factored out "t")

There are two possible solutions:

$$(t = 0) \longrightarrow$$

but we already Know that $y = 0$ at $t = 0$. This solution is not too useful

or

$$\left(v_0 \sin\theta_0 - \frac{1}{2} g t = 0\right)$$

$$\longrightarrow v_0 \sin\theta_0 = \frac{1}{2} g t$$

$$\Rightarrow \boxed{t = \frac{2 v_0 \sin\theta_0}{g}}$$

= time it takes for the ball to hit the ground.

Then:

$$R = v_0 \cos\theta_0 t = v_0 \cos\theta_0 \left(\frac{2 v_0 \sin\theta_0}{g}\right) = \frac{v_0^2 (2 \sin\theta_0 \cos\theta_0)}{g}$$

Using the Trigonometry Relation: $2\sin\theta_0\cos\theta_0 = \sin 2\theta_0$ allows us to simplify the expression for R:

$$\boxed{\text{Range} = R = \frac{V_0^2 \sin 2\theta_0}{g}}$$

We can also find out what angle θ_0 gives the maximum range:

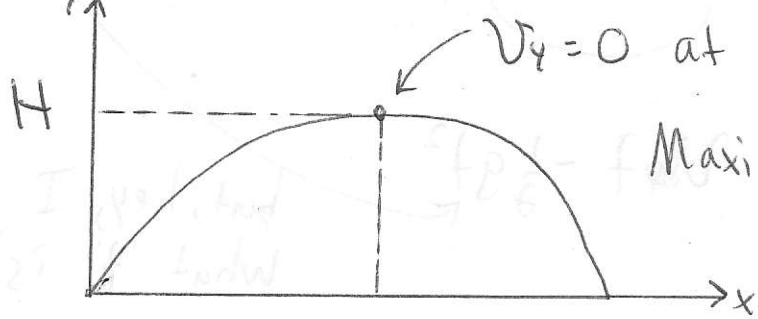
R_{max} occurs when: $\sin 2\theta_0 = 1$

$$2\theta_0 = 90^\circ$$

$$\theta_0 = \frac{90^\circ}{2} = 45^\circ$$

$\Rightarrow$ To throw a ball as far as possible, throw it at 45° relative to the ground. (good advice to the quarterback of a football te

Example: What is the maximum height of the ball?



$V_y = 0$ at the top

Maximum height $= H$

Let's use the equations of motion in the y-direction

Eq. M2: $v_y = v_{y0} - gt$

final velocity at the top = 0

I can solve for "t" since I know v_{y0}

The solution for t is:

$$0 = v_{y0} - gt$$

$$\Rightarrow \boxed{t = \frac{v_{y0}}{g}}$$

but we are looking for height, H, not "t".

Let's look at Eq. M3, maybe it will help:

Eq. M3: $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$

final position $y = H$

0 = initial position

$$\Rightarrow H = v_{y0}t - \frac{1}{2}gt^2$$

but, hey, I know what "t" is!

Plugging in for "t" gives:

$$H = v_{y0}\left(\frac{v_{y0}}{g}\right) - \frac{1}{2}g\left(\frac{v_{y0}}{g}\right)^2$$

$$= \frac{v_{y0}^2}{g} - \frac{1}{2}\frac{v_{0y}^2}{g} = \frac{1}{2}\frac{v_{y0}^2}{g}$$

Using the relation $v_{y0} = v_0 \sin\theta_0$ gives

$$H = \frac{1}{2}\frac{v_0^2 \sin^2\theta_0}{g}$$

Note: we could have arrived at this using the shortcut,

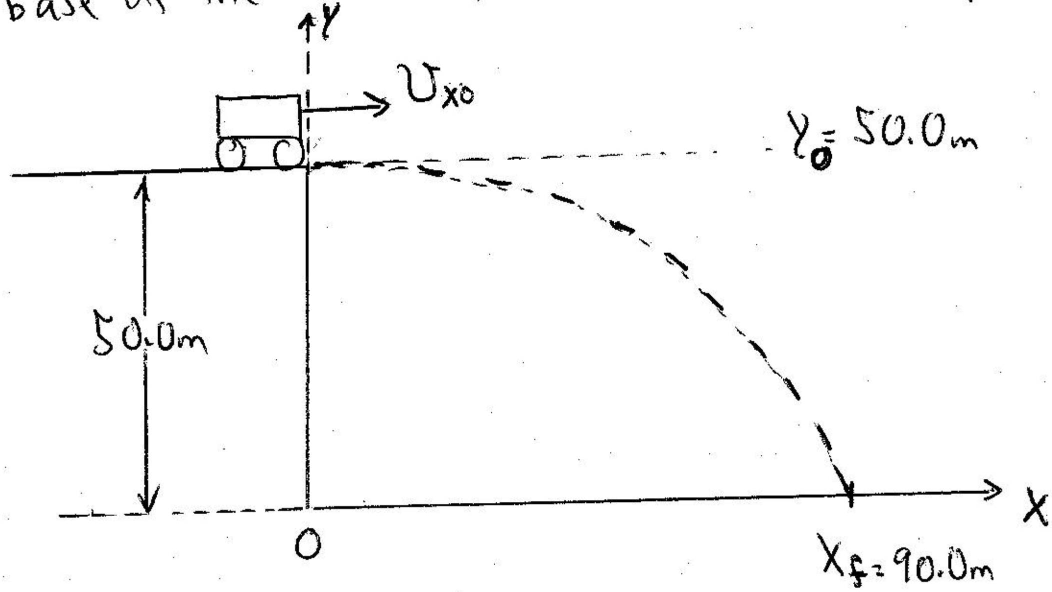
Eq. M4 $$\underset{H}{y} = \underset{0}{\cancel{y_0}} + \frac{\overset{0}{\cancel{v_y^2}} - v_{y0}^2}{2(-g)}$$

$$\Rightarrow H = \frac{1}{2}\frac{v_{0y}^2}{g} = \frac{1}{2}\frac{v_0^2\sin^2\theta_0}{g}$$

This is the reason why Eq. M4 is called a short cut approach → we arrived at the solution in two steps rather than 4 steps.

Example:

On a level road, a person drives off a 50.0m high cliff and wishes to land on level ground below 90.0m from the base of the cliff. How fast must the person drive to do so?



Let's examine the equations of motion in the x-direction:

Eq. M2-x: $v_x = v_{x0} + \underset{0}{\cancel{a_x}}\, t$ ⇒ this just says the car moves at a constant velocity in the x-direction

Eq. M3-x: $x = x_0 + v_{x0}t + \frac{1}{2}\underset{0}{a_x}\, t^2$

final position = 90.0m

initial position = 0

initial velocity (unknown)

time "t" (unknown)

We do not know v_{x0} or "t", so we cannot solve for v_{x0} yet. Let's examine the equations of motion in the y-direction:

Eq M2-y: $v_y = v_{y0} - gt$

final velocity (unknown)

initial velocity $= 0$

time "t" (unknow)

Not enough information to solve for v_y or "t".

Eq. M3-y: $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$

final position $= 0$

initial position $= 50.0\,m$

initial velocity $= 0$

time "t" (unknown)

We can solve for "t" (a quadratic eqn.) since all other parameters are known. Then we can substitute "t" into Eq. M3-x to find v_{x0}.

Eq. M3-y: $0 = y_0 - \frac{1}{2}gt^2$

$$\frac{1}{2}gt^2 = y_0$$

$$t^2 = \frac{2y_0}{g}$$

$$\Rightarrow t = \sqrt{\frac{2y_0}{g}}$$

Eq. M3-X : $x = v_{ox}t$

$$v_{xo} = \frac{x}{t} = \frac{x}{\sqrt{\frac{2y_o}{g}}} = x\sqrt{\frac{g}{2y_o}}$$

$$= (90.0\,m)\sqrt{\frac{9.8\,m/sec^2}{2(50.0\,m)}} = 28.1744\,m/sec$$

$$\approx 28.2\,m/sec$$

Example :

A punter kicks a football at an angle of 37.0° with respect to level ground and with a speed of 20.0 m/sec.

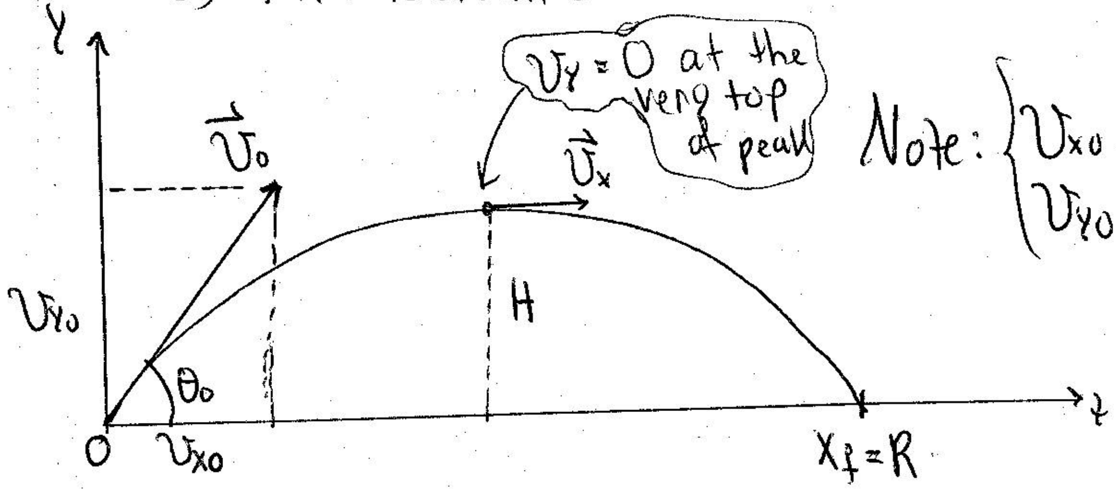
Find: (a) maximum height of the football.

(b) elapsed time the football is in the air.

(c) how far the football goes when it hits the ground

(d) the football's velocity at its maximum height.

(e) the football's acceleration at its maximum height.



Note: $\left\{ \begin{array}{l} v_{xo} = v_o \cos\theta_o \\ v_{yo} = v_o \sin\theta_o \end{array} \right\}$

(a) The maximum height is H drawn in the figure. In the y-direction, Eq. M2-y and Eq. M3-y do not give $y = H$ directly. Let's jump to the shortcut equation:

$$\text{Eq. M4-y}: \quad y = y_0 + \frac{v_y^2 - v_{y0}^2}{2(-g)}$$

final position = H

initial position = 0

initial velocity = $v_0 \sin\theta_0$

final velocity = 0 at peak

Then,

$$H = \frac{v_{y0}^2}{2g} = \frac{v_0^2 \sin^2\theta_0}{2g} = \frac{(20.0\,\text{m/sec})^2 \sin^2(37.0^\circ)}{2(9.8\,\text{m/sec}^2)}$$

$$= 7.3914\,\text{m} \approx 7.39\,\text{m}$$

(b) Notice the time it takes to get to the peak can be found from

$$\text{Eq. M2-y}: \quad v_y = v_{y0} - gt$$

time "t" (unknown)

final velocity = 0

initial velocity = $v_0 \sin\theta_0$

$$0 = v_0 \sin\theta_0 - gt$$

$$\Rightarrow t = \frac{v_0 \sin\theta_0}{g}$$

The time it takes to get to the peak is equal to the time it takes to fall from the peak to the ground:

$$\Rightarrow \begin{pmatrix}\text{total time it}\\ \text{is in the air}\end{pmatrix} = 2t = \frac{2v_0 \sin\theta_0}{g} =$$

$$= \frac{2(20.0\,m/sec)\sin(37.0^\circ)}{9.8\,m/sec^2}$$

$$= 2.45638\,sec \approx 2.46\,sec$$

(c) Using Eq. M3-X gives the distance the ball travels:

Eq. M3-X :
$$\underbrace{x}_{\substack{\text{final position}\\ =\text{ range}\\ = R}} = \underbrace{x_0}_{\substack{\text{initial position}\\ = 0}} + \underbrace{v_{x0}}_{\substack{\text{initial velocity}\\ = v_0\cos\theta_0}}\underbrace{t}_{\substack{\text{time ``}t\text{''}\\ \text{solved above}\\ t \approx 2.46\,sec}} + \underbrace{\cancel{\tfrac{1}{2}a_x t^2}}_{0}$$

$$\Rightarrow \quad R = v_0\cos\theta_0 t$$

$$= v_0\cos\theta_0\left(\frac{2v_0\sin\theta_0}{g}\right) = \frac{2v_0^2\sin\theta_0\cos\theta_0}{g}$$

$$= \frac{v_0^2\sin(2\theta_0)}{g} = \frac{(20.0\,m/sec)^2\sin[2(37.0^\circ)]}{9.8\,m/sec^2}$$

$$= 39.23517\,m \approx 39.2\,m$$

(d) The footballs velocity at maximum height can be found by solving for v_x :

Eq. M2-X : $v_x = v_{x0} + \cancel{a_x}t$

0

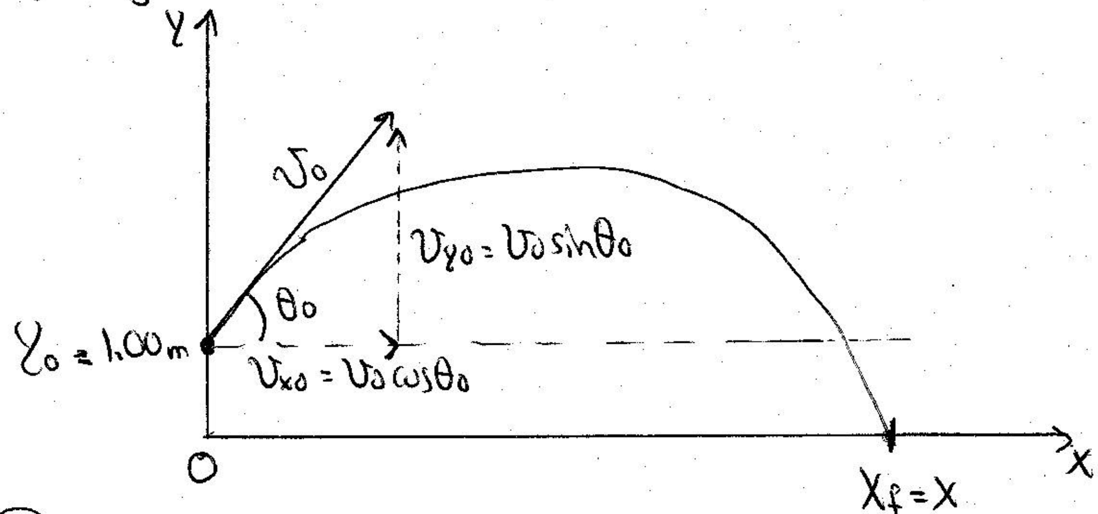
$v_x = v_0 \cos\theta_0 = (20.0 m/sec) \cos(37°)$
$= 15.9727 m/sec \approx 16.0 m/sec$

(e) $a_y = -g = -9.8\ m/sec^2$
everywhere along its path

(

Example:

A football is punted 1.00m above the ground with an initial velocity of 20.0 m/sec at an angle of $\theta_0 = 37.0°$ relative to the horizontal. How far does the football travel before hitting the ground?



The equation of motion, Eq. M3-X, gives information about how far ball travels:

Eq. M2-X : $X = X_0 + v_{x0}t + \cancel{\frac{1}{2}a_x t^2}$

X: final position (unknown)

X_0: initial position $= 0$

$a_x = 0$ → 0

$v_{x0} = v_0 \cos\theta_0$

t = unknown

Since "x" and "t" are unknown, we must find "t" to get "x"

$$X = (v_0 \cos\theta_0)t$$

Let's use Eq. M3-y to find "t":

Eq. M3-Y : $y = y_0 + v_{y0}t - \frac{1}{2}gt^2$

y: final position $= 0$

y_0: initial position $= 1m$

$v_{y0} = v_0 \sin\theta_0$

t = unknown

$$0 = y_0 + (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

Then,

$$t = \frac{-v_0\sin\theta_0 \pm \sqrt{(v_0\sin\theta_0)^2 - 4(-\frac{1}{2}g)(y_0)}}{2(-\frac{1}{2}g)}$$

$$= \frac{-v_0\sin\theta_0 \pm \sqrt{v_0^2\sin^2\theta_0 + 2gy_0}}{-g}$$

$$= \frac{-(20.0\,\text{m/sec})(\sin 37.0^\circ) \pm \sqrt{(20.0\,\text{m/sec})^2(\sin 37.0^\circ)^2 + 2(9.8\,\frac{\text{m}}{\text{sec}^2})(1.00\text{m})}}{-9.8\,\text{m/sec}^2}$$

$$= \frac{-12.0363\,\text{m/sec} \pm 12.82468\,\text{m/sec}}{-9.8\,\text{m/sec}^2}$$

$$= 2.536835\,\text{sec} \quad \text{or} \quad -0.0804473\,\text{sec}$$

We can safely ignore the negative time solution (it tries says something about the past prior to the ball being kicked), thus

$$t = 2.536835\,\text{sec}$$

Substituting this into the expression for "X" gives:

$$X = (v_0\cos\theta_0)t$$

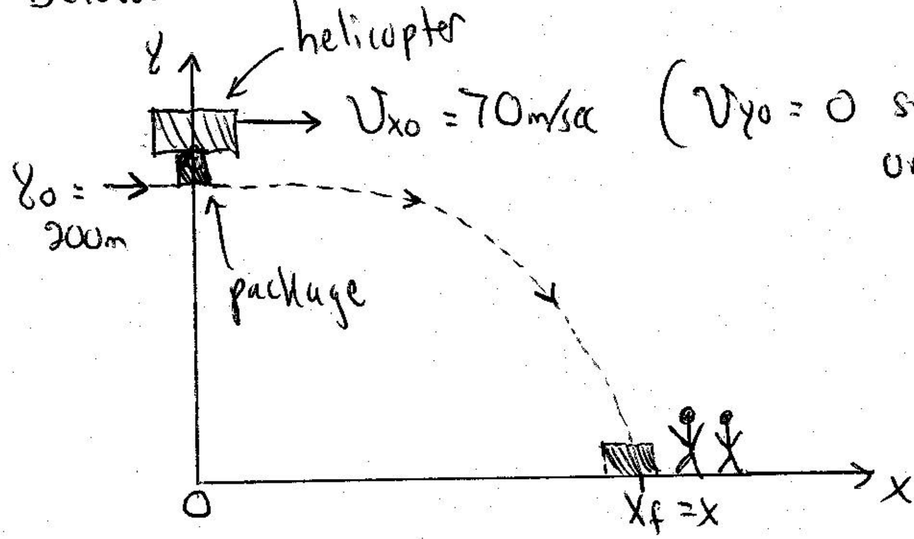
$$= (20.0\,\text{m/sec})(\cos 37.0^\circ)(2.536835\,\text{sec})$$

$$= 40.520133\,\text{m}$$

$X \approx 40.5\text{m}$ is the distance the football travels.

Example:

A helicopter traveling at a constant speed in the horizontal direction of 70 m/sec drops a package to people 200 m below.



($v_{y0} = 0$ since helicopter moves only in the horizontal direction)

(a) At what distance from the people should the helicopter release the package so that it gets to them? (to far away → the package never makes it, to close to them → the package goes to far past them)

Equation of motion, Eq. M3-x, gives this distance:

Eq. M3-x: $x = x_0 + v_{x0}t + \cancel{\frac{1}{2}a_x t^2}$

$a_x = 0$ helicopter moves at constant speed.

x: final distance (unknown)

x_0: initial distance $= 0$

v_{x0}: initial velocity $= 70$ m/sec

$t =$ unknown

Then

(distance from people before package is dropped) $= x = v_{x0}t$

we need to find "t"

Use the equation of motion in the y-direction to find "t"

Eq. M3-4: $y = y_0 + v_{y0} t - \frac{1}{2} g t^2$

final position $= 0$

initial position $= 200\text{m}$

initial velocity $= 0$

$t =$ unknown

$$0 = y_0 - \frac{1}{2} g t^2$$

$$\Rightarrow t = \sqrt{\frac{2y_0}{g}}$$

Then,

$$X = v_{x0} t = v_{x0}\sqrt{\frac{2y_0}{g}} = \left(70 \frac{\text{m}}{\text{sec}}\right)\sqrt{\frac{2(200\text{m})}{9.8\text{m}/\text{sec}^2}}$$

$$= 447.2136\text{ m}$$

$$\approx 400\text{m}$$