Basic Mechanics

Unit #1

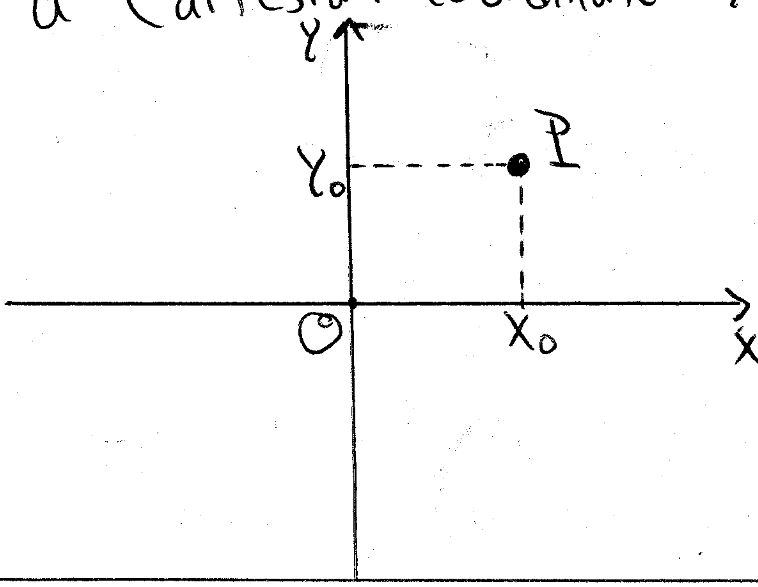
The Goal of Mechanics is to determine the <u>equations of motion</u> of a particle or a collection of particles.

The equations of motion describe the

$$\left.\begin{matrix}\text{position}\\ \text{velocity}\\ \text{acceleration}\end{matrix}\right\}\text{ of an object}$$

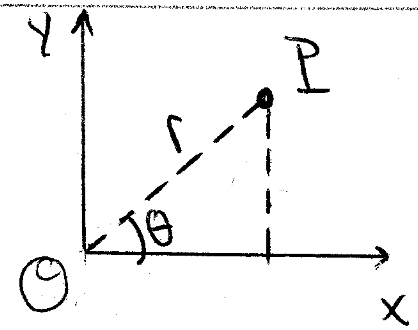
Position and Displacement:

To describe the position of a particle on a plane, a Cartesian coordinate system is convenient:



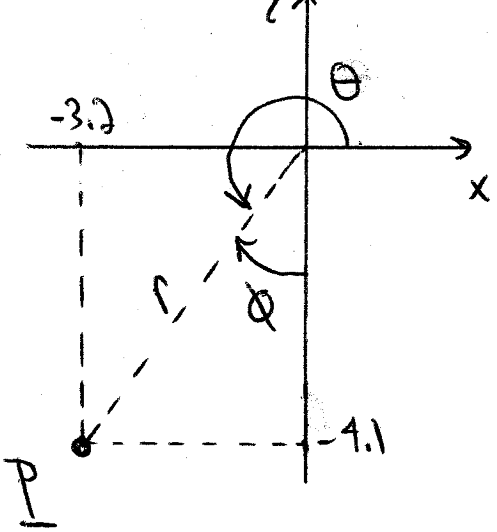
Cartesian XY-coordinate system where particle P lies at the point (X_0, Y_0).

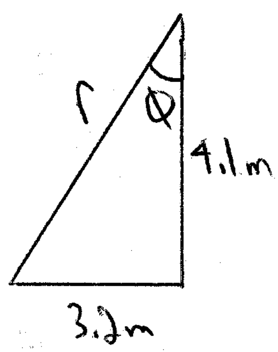
The origin, O, of the coordinate system lies at $(X=0, Y=0)$.



Using a Polar Coordinate system, particle P lies at the point (r, θ).
It lies a distance r from the origin at an angle θ relative to the positive x-axis

Example: Point P is located at $X = -3.2m$, $Y = -4.1m$





Its distance from the origin can be found using Pythagorean's Theorem;

$$r^2 = a^2 + b^2$$
$$= (-3.2m)^2 + (-4.1m)^2$$

$$\Rightarrow r = \sqrt{(-3.2m)^2 + (-4.1m)^2} = 5.2\,m$$

The angle ϕ can be found using basic trigonometry:

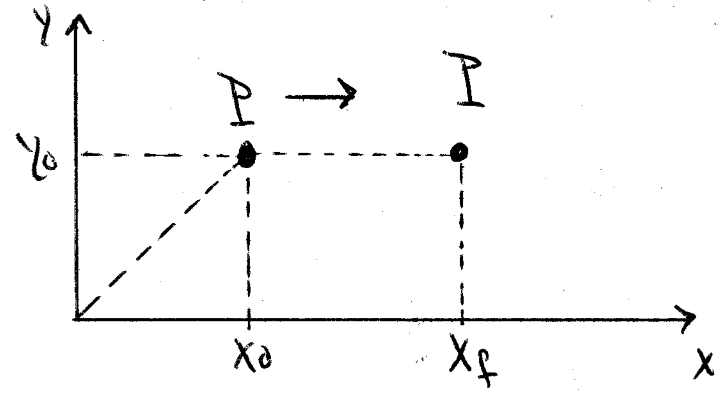
$$\tan\phi = \frac{\text{opposite}}{\text{adjacent}} = \frac{3.2m}{4.1m} \Rightarrow \phi = \tan^{-1}\left(\frac{3.2}{4.1}\right) = 37.97^\circ$$

Then, $\theta = 270^\circ - \phi = 270^\circ - 37.97^\circ = 232.03^\circ$

$\left\{ \begin{array}{l} \text{Position of} \\ \text{Point P} \end{array} \right\} = \left\{ \begin{array}{l} \text{5.2 m from the origin at an angle of} \\ \theta = 232^\circ \text{ relative to the positive x-axis} \end{array} \right\}$

Now, let particle P move in only the x-direction

(this is motion in 1-dimension)



Point P moves from point (x_0, y_0) to point (x_f, y_0) in time Δt

We no longer need to draw the y-axis since there is no motion in the y-direction. So a simpler picture is:

takes time Δt

x_0 → (initial position) → $x_{initial}$ → $\underbrace{x_i \equiv x_0}$

We usually set the initial position to be x_0

x_f → (final position) → x_{final} → $\underbrace{x_f \equiv x}$

We usually set the final position to be x

(we drop the "f" subscript)

Object I undergoes a displacement (it moves a distance)

$$\Delta X = X_f - X_i = X - X_0$$

Note Δ = Greek symbol delta

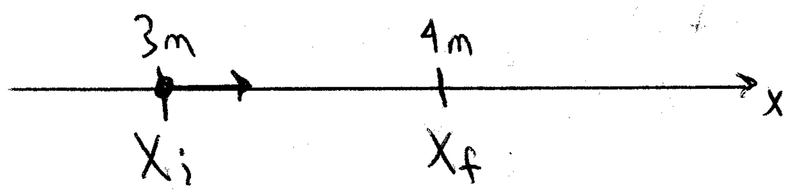
and in physics: Δ = "the change in"

ΔX = "the change in X"

The SI units of position, X, and displacement, ΔX, are meters (abbreviation: m).

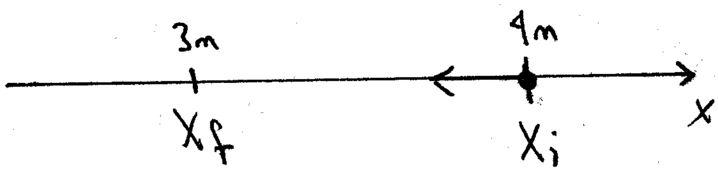
Notice: displacement ΔX can be positive or negative

Example: Object I moves from $X = 3m$ to $X = 4m$:



$\Delta X = X_f - X_i = 4m - 3m = 1m \Rightarrow$ a positive #

If object I moves from $X = 4m$ to $X = 3m$:



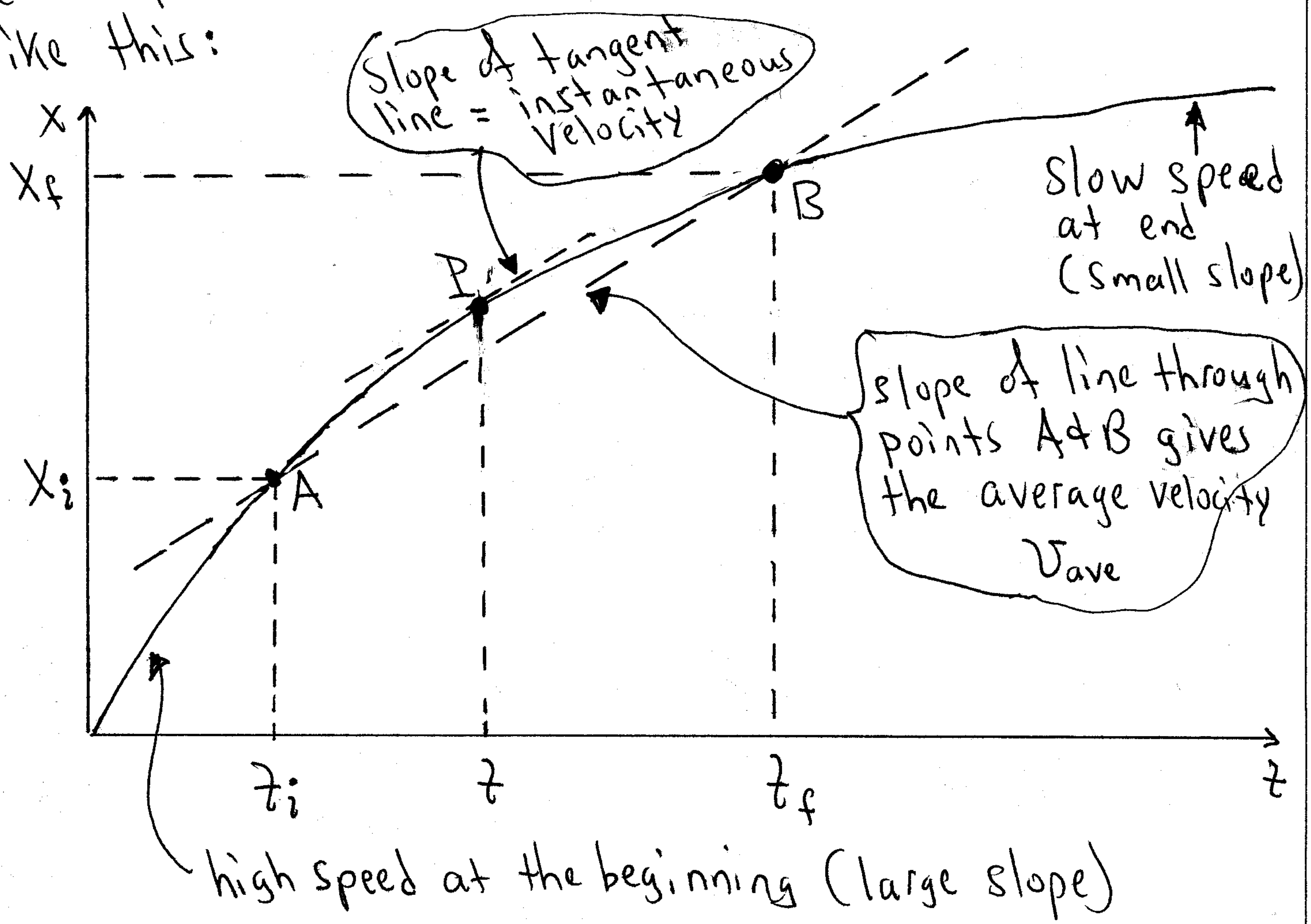
$\Delta X = X_f - X_i = 3m - 4m = -1m \Rightarrow$ a negative #

Object I moves in the negative x-directi

Since Δx can be negative or positive, the displacement Δx is a vector. We will discus more about vectors in the next chapter, Unit 2.

Speed and Velocity:

Let's assume that on one courageous day you decided to run the Chicago marathon. Your friend has a stop watch and measures your position as a function of time. Initially you run very fast and then slow down significantly towards the end. Your position verses time plot may look something like this:



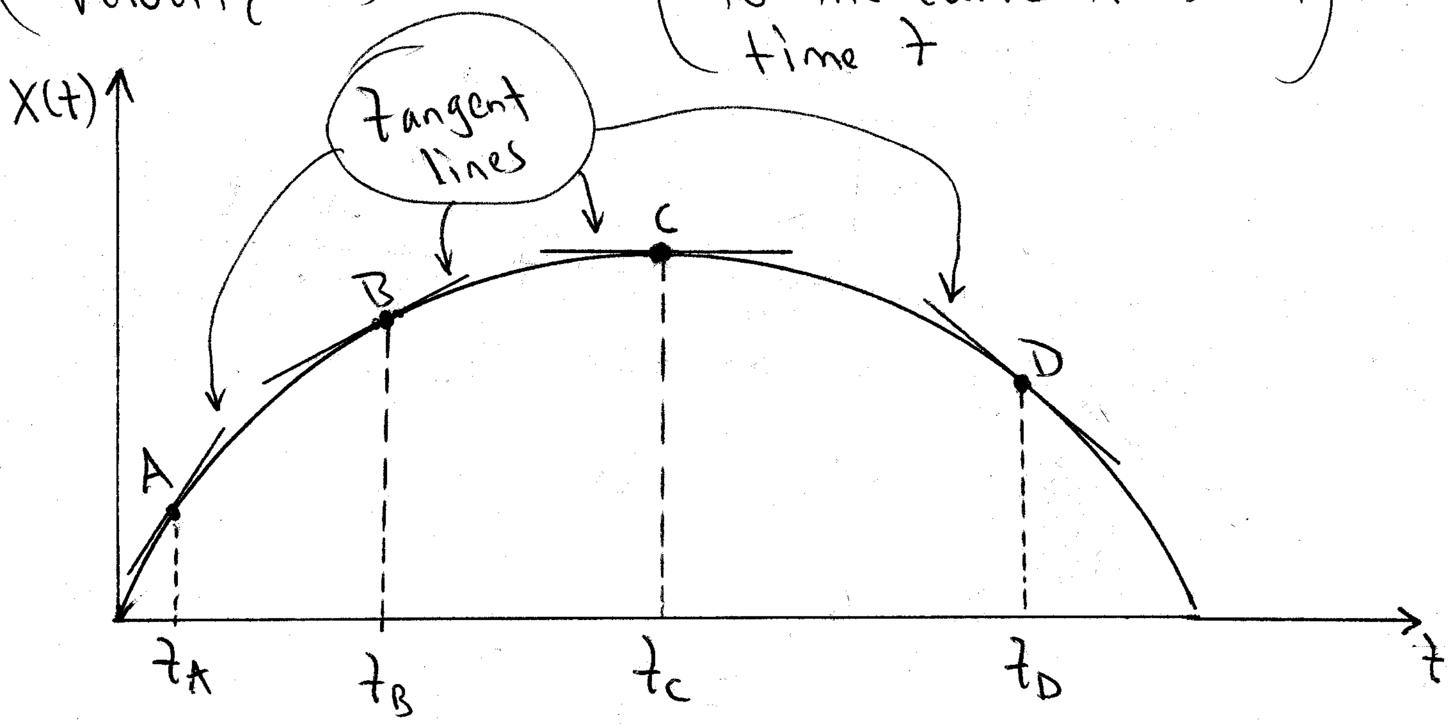
Your average velocity between points A & B is

$$v_{ave} = \frac{\text{change in distance}}{\text{change in time}} = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t}$$

$$= \left(\begin{array}{l}\text{the slope of the line}\\ \text{through points A \& B}\end{array}\right)$$

What if you wanted to know your speed at an instant in time (such as at time t at point P in the figure above). If the distance between points A & B becomes very small, one can see that the slope of the line becomes essentially the slope of a line tangent to the curve at point P

$$\left(\begin{array}{l}\text{Instantaneous}\\ \text{velocity}\end{array}\right) = v = \left(\begin{array}{l}\text{slope of tangent line}\\ \text{to the curve } x(t) \text{ at}\\ \text{time } t\end{array}\right)$$



Notice: Slope at point A > slope at point B

then → velocity at point A > speed at point B

$$v(t_A) > v(t_B)$$

$v(t_A)$: (Velocity at time t_A) $v(t_B)$: (Velocity at time t_B)

Notice: Slope at point C = 0

then → velocity at point C = $v(t_c) = 0$

Notice: slope at point D is negative

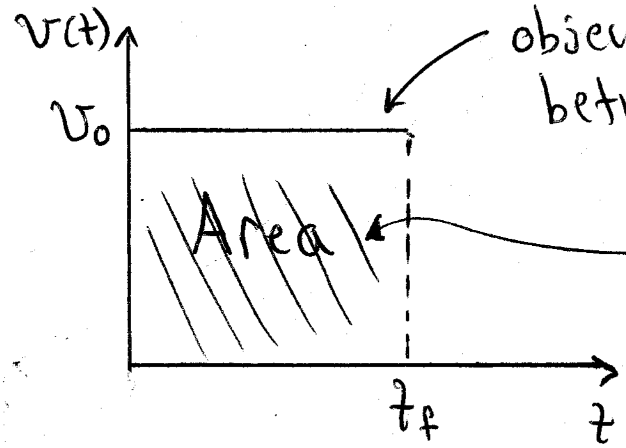
then → Velocity at point D is negative

→ object is moving in the negative x-direction

Note: speed is always a positive number

Velocity and Displacement:

Let's plot the simplest curve for $v(t)$: A straight line



object moves with constant velocity v_0 between times t_i and t_f

Notice: Area = (base)(height)

$= (t_f - t_i)\, v_0$

$= v_0 \Delta t$

Since: $v_{ave} = \dfrac{v_i + v_f}{2} = \dfrac{v_0 + v_0}{2} = \dfrac{2v_0}{2} = v_0$

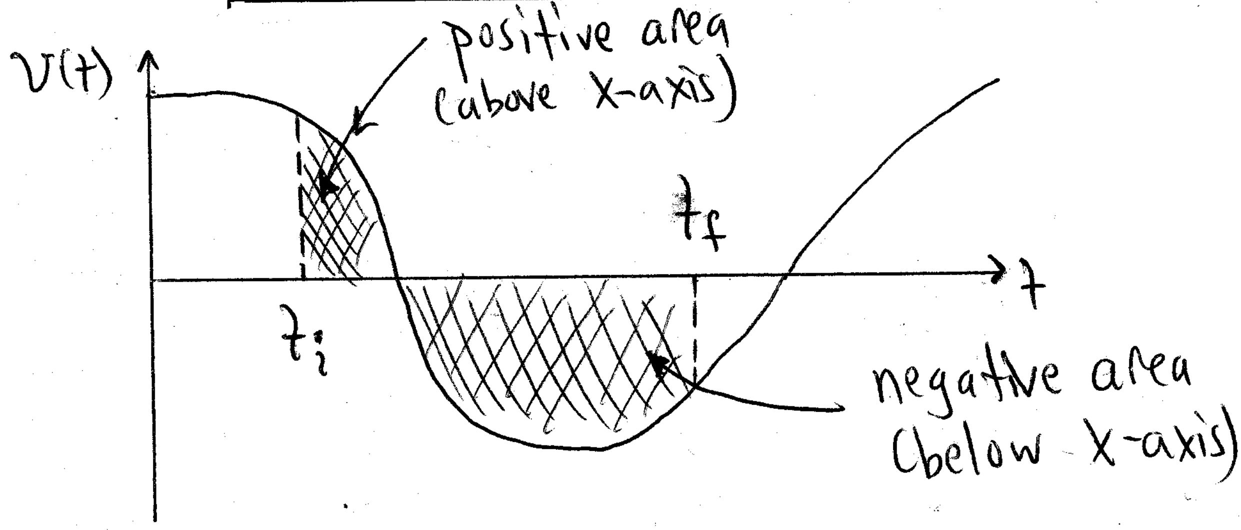
for object moving at constant velocity

and since:

$$v_{ave} = \frac{\Delta x}{\Delta t} \Rightarrow v_0 = \frac{\Delta x}{\Delta t}.$$

Then, $\underset{\substack{\uparrow \\ \text{displacement}}}{\Delta x} = v_0 \Delta t = \left(\begin{array}{c}\text{Area under curve} \\ \text{when } v = \text{constant}\end{array}\right)$

In General: $\boxed{\text{Displacement} = \Delta x = \text{Area under } v(t)}$



Δx = total area under the curve.

For the curve above, Δx is negative (there is more negative area than positive area from t_i to t_f). Thus, the object has moved in the negative x-direction.

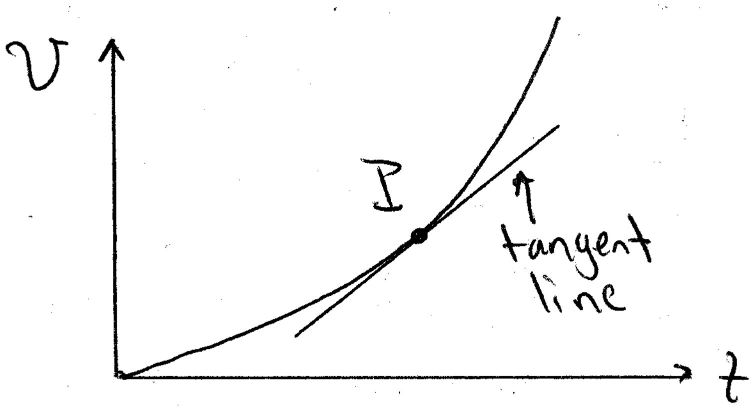
Acceleration:

In this course, acceleration will always be constant. Then, the average acceleration is

$$a_{ave} = \underbrace{\frac{a_i + a_f}{2} = \frac{a_i + a_i}{2}}_{\left(\substack{a_i = a_f \text{ if} \\ \text{acceleration is constant}}\right)} = \frac{2a_i}{2} = \underbrace{a_i \equiv a}_{\text{we will just define } \boxed{a_{ave} \equiv a}} = \text{const.}$$

$$a_{ave} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

Thus we can write this as: $a = \dfrac{\Delta v}{\Delta t}$



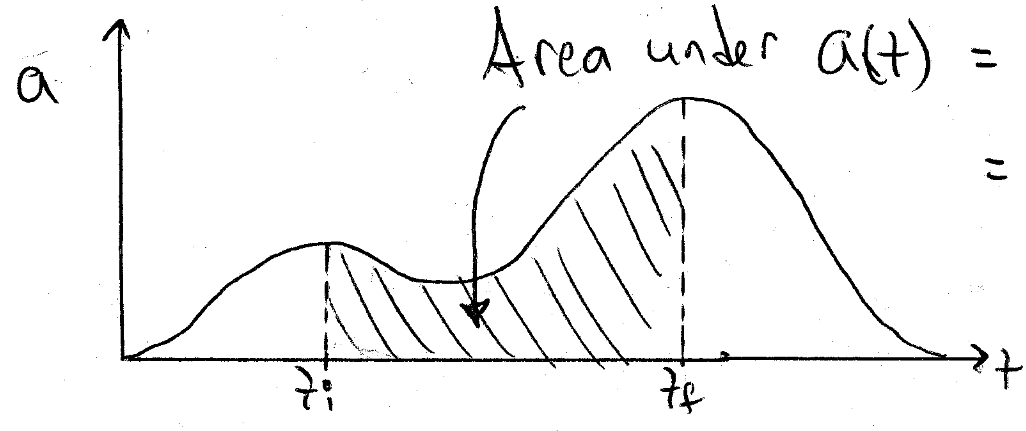
$\Rightarrow$ Slope of the tangent line at point I

= acceleration

= $a(t)$

Area under $a(t) = \Delta v$ (the change in velocity)

$= v_f - v_i$



Terminology of symbols:

Position: x Position at time t: $x(t)$

Displacement: Δx

Velocity (average): $v_{ave} = \frac{v_f + v_i}{2} = \frac{\Delta x}{\Delta t}$

(Instantaneous Velocity): $v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$ = Slope of tangent line to curve $x(t)$

Velocity at time t: $v(t)$

$$\boxed{\text{Note: } v \neq v_{ave}}$$

Speed $= |v|$ = always a positive #

(absolute value symbol) Example: $|-4| = 4$
$|4| = 4$

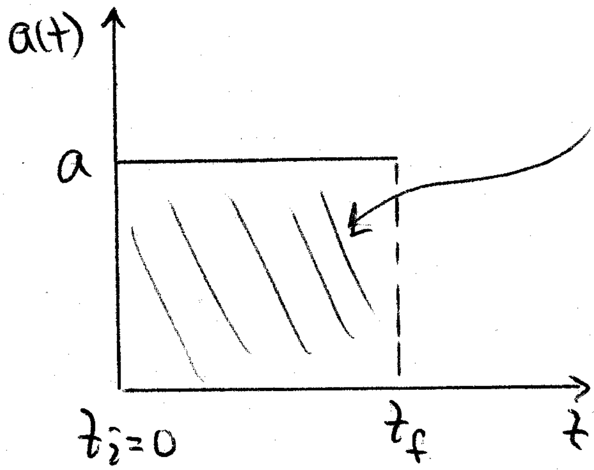
Acceleration: always constant in this course
thus

$$\underbrace{a}_{\text{instantaneous}} = \underbrace{a_{ave}}_{\text{average}} = \underbrace{\frac{\Delta v}{\Delta t}}_{\text{average}} = \underbrace{\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}}_{\text{instantaneous}}$$

Displacement $= \Delta x =$ Area under $v(t)$ curve

change in velocity $= \Delta v =$ Area under $a(t)$ curve

Since in this course: $a(t) = \text{constant} \equiv a$



$$\Delta v = \text{Area} = (\text{base})(\text{height})$$
$$= (t_f - t_i)(a)$$
$$= t_f\, a$$
$$= at$$

$\left(\begin{matrix}\text{Note} \\ t_f \equiv t\end{matrix}\right)$

$$\Delta v = at$$

$$\underset{\downarrow\; v}{v_f} - \underset{\downarrow\; v_0}{v_i} = at \quad \Rightarrow \quad v - v_0 = at$$

$$\Rightarrow \boxed{v = v_0 + at}$$

Equation of motion for velocity

Now let us use the relation for the average velocity

$$v_{ave} = \frac{v_f + v_i}{2} = \underbrace{\frac{v + v_0}{2}}_{\text{(drop "f" subscript)}} = \frac{\Delta x}{\Delta t} = \underbrace{\frac{\Delta x}{t_f - t_i}}_{\text{let } t_i = 0} = \frac{\Delta x}{t_f} = \frac{\Delta x}{t}$$

In this course: $t_i = 0$ always

→ we always start our clocks at time $t = 0$

$$\left.\begin{aligned} t_f &= t \\ v_f &= v \\ a_f &= a \end{aligned}\right\} \text{drop "f" subscripts always}$$

$$\left.\begin{aligned} t_i &= t_0 \\ v_i &= v_0 \\ a_i &= a_0 \end{aligned}\right\}$$

"i" → "0" subscript "i" is always replaced with subscript "0"

Then,

$$v_{ave} = \frac{v + v_0}{2} = \frac{\Delta x}{t}$$

from above: $v = v_0 + at$ → equation of motion for velocity

Solving for "v" in 1st eqn, gives

$$v + v_0 = 2\frac{\Delta x}{t}$$

$$v = 2\frac{\Delta x}{t} - v_0$$

Substitute v into 2nd eqn:

$$2\frac{\Delta x}{t} - v_0 = v_0 + at$$

$$2\frac{\Delta x}{t} = 2v_0 + at$$

$$\frac{\Delta x}{t} = \frac{1}{2}(2v_0 + at) = v_0 + \frac{1}{2}at$$

$$\Delta x = v_0 t + \frac{1}{2}at^2$$

$$x_f - x_i = v_0 t + \frac{1}{2}at^2$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$

$$\boxed{x = x_0 + v_0 t + \frac{1}{2}at^2}$$ equation of motion for position

$$\boxed{a = \text{constant}}$$ equation of motion for acceleration

$$\boxed{v = v_0 + at}$$ equation of motion for velocity

$$\boxed{x = x_0 + \frac{v^2 - v_0^2}{2a}}$$ shortcut equation

With these 4 eqns., one can solve any motion problem in physics for objects undergoing constant acceleration

Equations of motion:

Eq. M1 : $a = \text{constant}$

Eq. M2 : $v = v_0 + at$

Eq. M3 : $x = x_0 + v_0 t + \frac{1}{2} a t^2$

Eq. M4 : $x = x_0 + \dfrac{v^2 - v_0^2}{2a}$
(shortcut eqn.)

For an object falling or rising in a gravitational field, it experiences an acceleration

$$a = -g = -9.8\,\text{m/sec}^2$$

Then:

Eq. M1 : $a = -g$

Eq. M2 : $v = v_0 - gt$

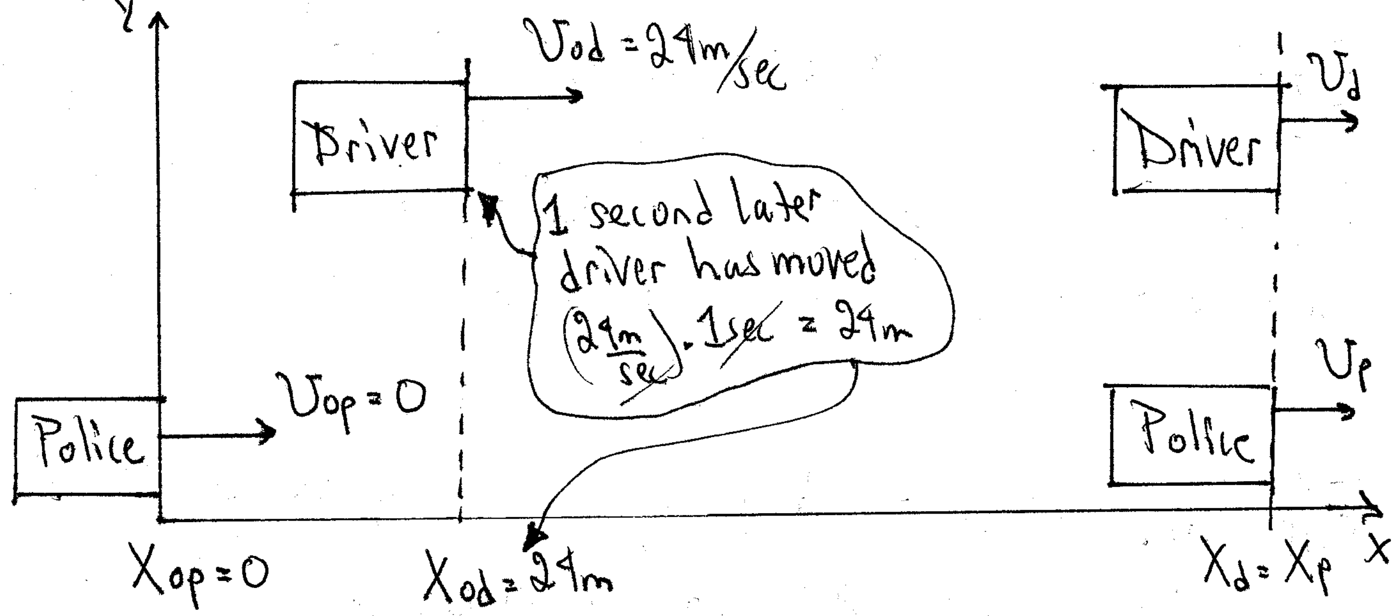
Eq. M3 : $x = x_0 + v_0 t - \frac{1}{2} g t^2$

Eq. M4 : $x = x_0 + \dfrac{v^2 - v_0^2}{2(-g)}$

Example 2.5 in Serway:

A car traveling at constant speed of 24 m/sec passes a police car. One second later police chases after it with constant acceleration of $3\ m/sec^2$. (a) How long does it take for police to catch the speeder?

Always draw a coordinate system describing the situation:



For police, use subscript "p" : $x_{op} = 0$ = initial position of police

v_{op} = initial velocity of police $= 0$

For driver, use subscript "d" : $x_{od} = 24m$ = initial position of driver

v_{od} = initial velocity of driver $= 24\ m/sec$

The final position and velocity of police: x_p, v_p

The final position and velocity of driver: x_d, v_d

Note: when police catches up with driver: $x_p = x_d$

Equations of Motion for the driver:

Eq. M1 : $a_d = 0$ → driver moves at constant velocity

Eq. M2 : $v_d = v_{od} + \cancel{a_d}t$ (→ 0)

since $a_d = 0$

$\Rightarrow v_d = v_{od}$ → again, driver moves at constant velocity

Eq. M3 : $x_d = \underbrace{x_{od}}_{24m} + \underbrace{v_{od}}_{24m/sec}t + \cancel{\frac{1}{2}\underbrace{a_d}_{a_d = 0}t^2}$ (→ 0)

$\Rightarrow \boxed{x_d = x_{od} + v_{od}t}$ → the only useful eqn. for the driver

Equations of Motion for the police:

Eq. M1 : $a_p = 3m/sec^2$ → given by problem

Eq. M2: $v_p = \cancel{v_{op}} + a_pt$ (→ 0)

initially at rest $\Rightarrow v_{op} = 0$

$\Rightarrow \boxed{v_p = a_pt}$

Eq. M3: $x_p = \cancel{x_{op}} + \cancel{v_{op}t} + \frac{1}{2}a_pt^2$ (→ 0, → 0)

initially at $x = 0$

$\Rightarrow \boxed{x_p = \frac{1}{2}a_pt^2}$

Summarizing results: we have 3 useful equations

$$\left\{\begin{array}{ll} v_p = a_p t & (1) \\ x_p = \frac{1}{2} a_p t^2 & (2) \\ x_d = x_{0d} + v_{0d} t & (3) \end{array}\right.$$

When the police catches up with the driver: $x_p = x_d$ and we can set Eqns. (2) and (3) equal to each other

$$x_{0d} + v_{0d} t = \frac{1}{2} a_p t^2$$

$$x_{0d} + v_{0d} t - \frac{1}{2} a_p t^2 = 0$$

To solve for time, t, apply the solution to a quadratic equation:

Recall: $ax^2 + bx + c = 0$

Solution for x: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

So for: $\underset{\downarrow\; c}{x_{0d}} + \underset{\downarrow\; b}{v_{0d}}\, t \underbrace{-\tfrac{1}{2} a_p}_{\downarrow\; a} t^2 = 0$

$$t = \frac{-v_{0d} \pm \sqrt{(v_{0d})^2 - 4\left(-\frac{1}{2} a_p\right) x_{0d}}}{2\left(-\frac{1}{2} a_p\right)}$$

$$t = \frac{-24\,\text{m/sec} \pm \sqrt{(24\,\text{m/sec})^2 + 2(3\,\text{m/sec}^2)(24\,\text{m})}}{-3\,\text{m/sec}^2}$$

$$= \frac{-24\,\text{m/sec} \pm \sqrt{720\,\text{m}^2/\text{sec}^2}}{-3\,\text{m/sec}^2} = \frac{-24\,\text{m/sec} \pm 26.8328\,\text{m/sec}}{-3\,\text{m/sec}^2}$$

$$= \left(\frac{-24 + 26.8328}{-3}\right)\text{sec} \quad \text{OR} \quad \left(\frac{-24 - 26.8328}{-3}\right)\text{sec}$$

$$= \left(-0.94\,\text{sec}\right) \quad \text{OR} \quad \left(16.94\,\text{sec}\right)$$

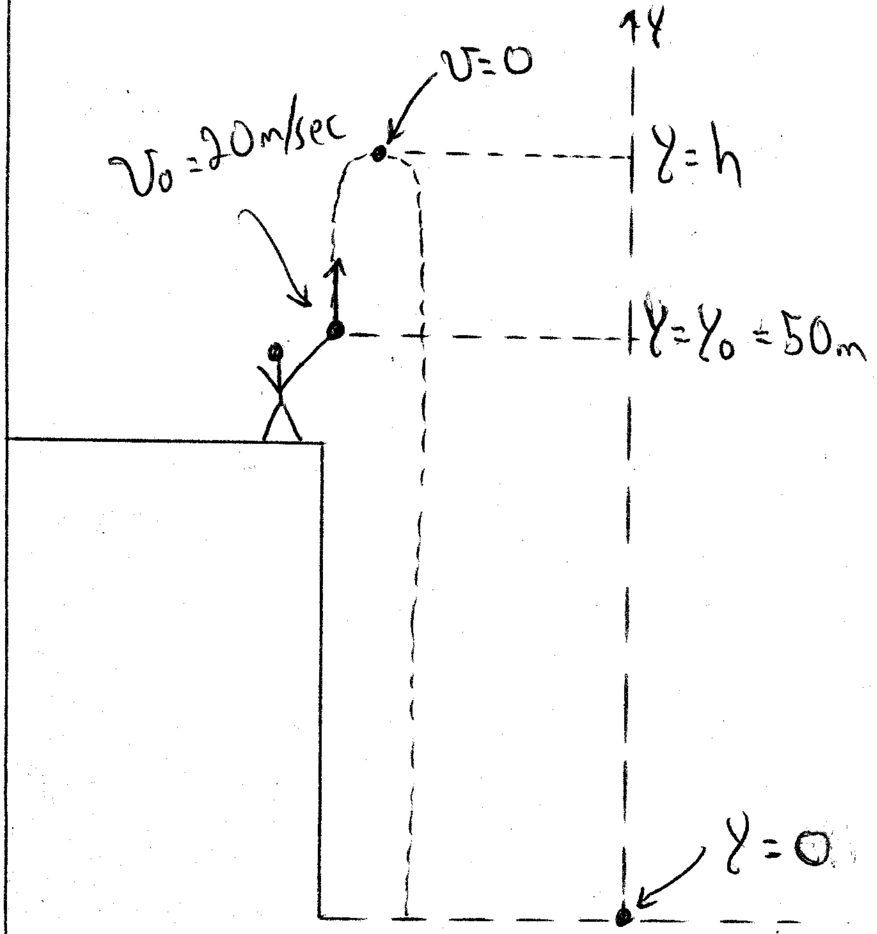
ignore negative times

$\Rightarrow$ Answer: $t = 16.94\,\text{sec}$ for police to catch the speeder

Example 2.8 in Serway:

A ball is thrown from the top of a building with an initial velocity of 20 m/sec straight up, at an initial height of 50 m above the ground. The ball just misses the edge of the roof on the way down.

(a) What is the time needed for the ball to reach its maximum height?



Eq. M1: $a = -g$

Eq. M2: $v = v_0 + at$

Eq. M3: $y = y_0 + v_0 t + \frac{1}{2}at^2$

Eq. M4: $y = y_0 + \frac{v^2 - v_0^2}{2a}$

(a) to find time it take to reach $y=h$:
Note that at the top, $v=0$. Then Eq. M2 gives

$$\underbrace{v}_{\substack{0 \text{ at top} \\ \text{(final velocity)}}} = \underbrace{v_0}_{20\,m/sec} - gt \quad \Rightarrow \quad 0 = v_0 - gt$$

$$gt = v_0$$

$$t = \frac{v_0}{g} = \frac{20\,m/sec}{9.8\,m/sec^2} = 2.04\,sec$$

(b) To find the maximum height use Eq. M4:

$$\underbrace{y}_{h} = \underbrace{y_0}_{50m} + \frac{\cancelto{0}{v^2} - v_0^2}{2a} \qquad v_f = v = 0$$

$$v_0 = 20\,m/sec$$

$$h = y_0 - \frac{v_0^2}{2(-g)} = y_0 + \frac{v_0^2}{2g} = 50m + \frac{(20m/sec)^2}{2(9.8m/sec^2)}$$

$$= (50 + 20.408)m = 70.408m \text{ above the ground}$$

(c) time needed to return to original height: Use Eq. M3:

$$y = y_0 + v_0 t + \tfrac{1}{2}at^2 \Rightarrow y_0 = y_0 + v_0 t + \tfrac{1}{2}gt^2$$

$y \downarrow$ final position $\downarrow$ y_0; $y_0 \downarrow$ initial position $\downarrow$ 50m; $v_0 \downarrow$ 20m/sec; $a \downarrow$ $-g$

$$0 = v_0 t - \tfrac{1}{2}gt^2$$

$$0 = t\left(v_0 - \tfrac{1}{2}gt\right)$$

The two solutions are: $t = 0$ (not of much use)

$$v_0 - \tfrac{1}{2}gt = 0$$

$$v_0 = \tfrac{1}{2}gt$$

$$t = \frac{2v_0}{g} = \frac{2(20m/sec)}{9.8m/sec^2} = 4.08sec$$

Notice that $t = 2 \times (\text{"}t\text{" in part (a)}) = 2 \times (2.04sec) = 4.08sec$
By symmetry (it takes the same amount of time to come down as it took to go up)

(d) The time needed to reach the ground: Use Eq. M3

$y = y_0 + v_0 t + \frac{1}{2}at^2 \Rightarrow 0 = y_0 + v_0 t - \frac{1}{2}gt^2$

y ↓ final position $= 0$; y_0 ↓ 50m; v_0 ↓ 20m/sec; a ↓ $-g$

Solution using quadratic eqn:

$$t = \frac{-v_0 \pm \sqrt{v_0^2 - 4(-\frac{1}{2}g)(y_0)}}{2(-\frac{1}{2}g)}$$

$$= \frac{-20\,\text{m/sec} \pm \sqrt{(20\,\text{m/sec})^2 + 2(9.8\,\text{m/sec}^2)(50\,\text{m})}}{-9.8\,\text{m/sec}^2}$$

$$= \left(\frac{-20 \pm 37.148}{-9.8}\right)\text{sec} = (5.83\,\text{sec}) \text{ OR } (-1.75\,\text{sec})$$

↓ ignore negative times

$\Rightarrow t = 5.83\,\text{sec}$

The velocity of the ball when it reaches the ground is then; using Eq. M2:

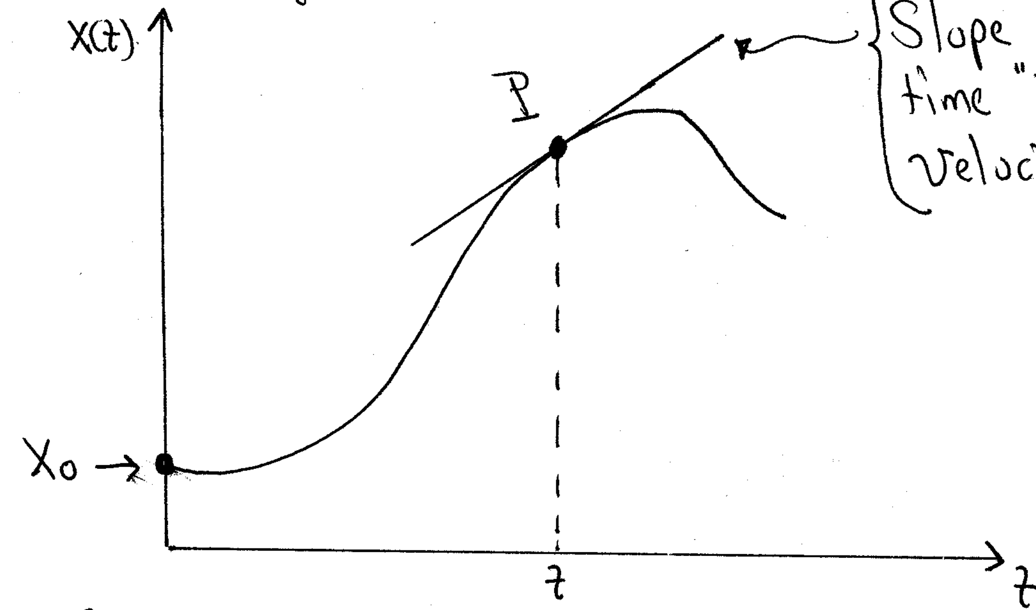
$v = v_0 + at \Rightarrow v = 20\,\text{m/sec} - (9.8\,\text{m/sec}^2)(5.83\,\text{sec})$

v ↓ final velocity; v_0 ↓ 20m/sec; a ↓ $-9.8\,\text{m/sec}^2$

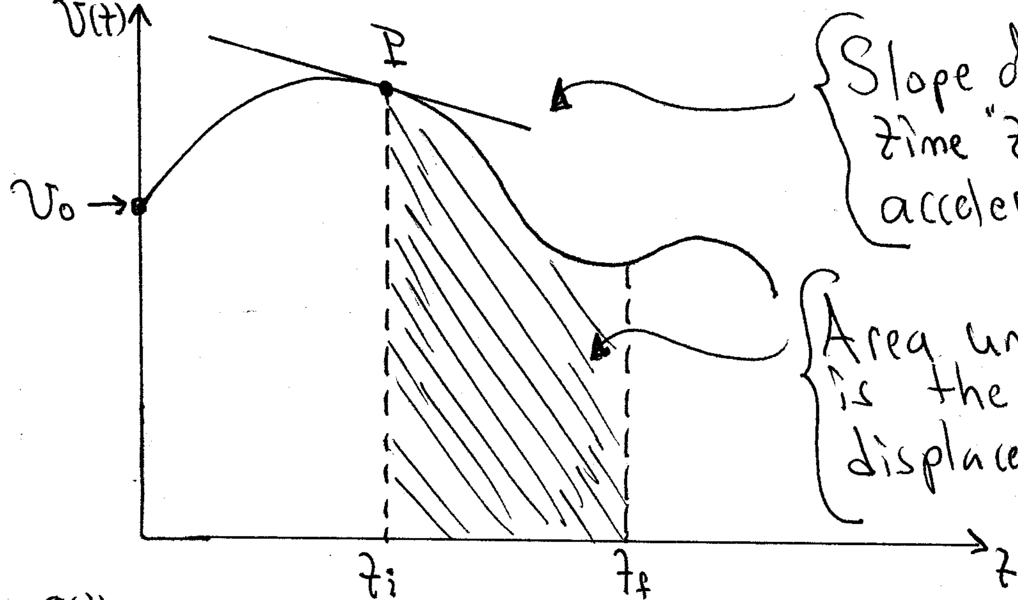
$= -37.1\,\text{m/sec}$

↑ minus sign because ball is moving in the negative y-direction

Summary:

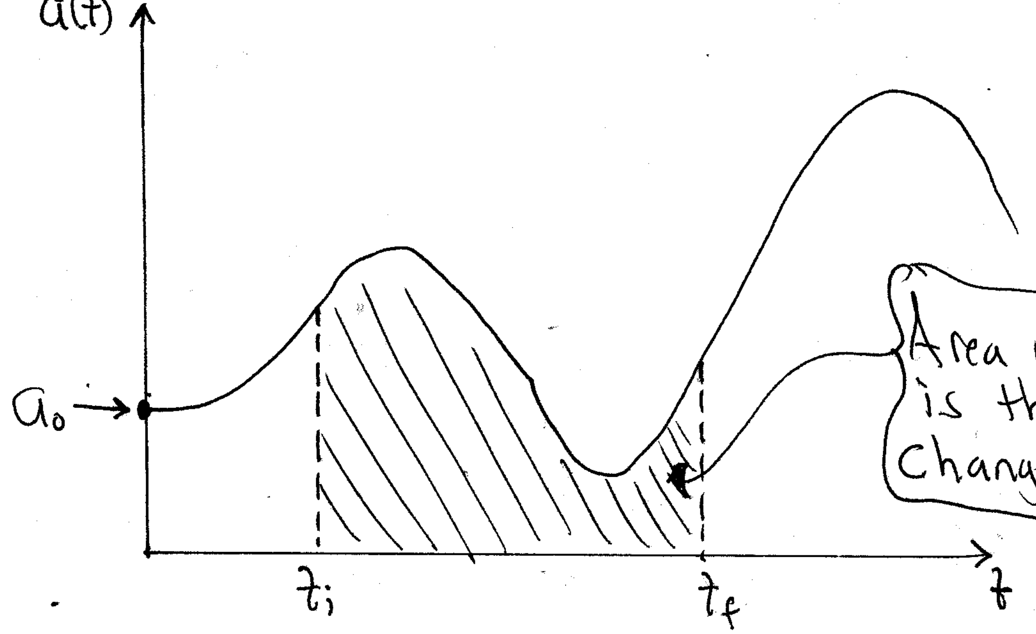


Slope of tangent line at time "t" is the velocity: $v(t)$



Slope of tangent line at time "t" is the acceleration: $a(t)$

Area under curve $v(t)$ is the displacement: $\Delta x = x_f - x_i$



Area under curve $a(t)$ is the change in velocity: $\Delta v = v_f - v_i$