ABSTRACT

ON THE SELF-FORCE PROBLEM OF POINT-LIKE CHARGED PARTICLES IN CLASSICAL ELECTRODYNAMICS

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The problem of a point-like charged particle’s self-interaction force has been a long-standing problem in the field of classical electrodynamics. For over a century, physicists have unsuccessfully developed a satisfactory solution until recently. We expand on the work done by Gralla et. al. [1] in order to establish a smooth and relativistically correct formulation of point-like charged particles in motion and shed light on related problems and applications, particularly the motion of elementary particles with magnetic dipoles through non-uniform static magnetic fields. We conduct numerical simulations in order to quantify the magnitude of the perturbations from this self-force due to the new equations of motion relative to the Lorentz force. These results will be utilized in single particle dynamics experiments in accelerator rings.
ON THE SELF-FORCE PROBLEM OF POINT-LIKE CHARGED PARTICLES IN CLASSICAL ELECTRODYNAMICS

BY

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DEDICATION

This thesis is dedicated to my family. Your undying support and assistance has helped me balance my life in academics and the Army, and none of this would be possible without you.
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CHAPTER 1

INTRODUCTION

1.1 What Is The Self-Force Problem?

For over the past 100 years, the field of classical electrodynamics has been plagued by the question: what happens when a charged particle emits a field at some time and then interacts with that field at some later time? Naively, one might expect that we can treat this the same as any other problem in the field: using Maxwell’s equations, we can calculate the field that the particle emitted and then use that same field that was just calculated to write down the differential equations describing the motion of the particle. However, there are subtleties that make this a very complicated process, and one that has been rife with errant attempts.

Additionally, there is an inconsistency in our most fundamental of electromagnetism classes. First, we are taught (correctly) that magnetic fields do not do any work. We can see this, as the typical Lorentz force that is taught is given as

$$\mathbf{F} = q \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right),$$

and work can be written in the form

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} \, dt.$$
Since the force due to a magnetic field is perpendicular to the velocity, when we take the scalar product of the force with the velocity, we will get zero. However, we are also taught (correctly) that when we place a particle of magnetic dipole $\vec{\mu}$ in a static magnetic field $\vec{B}$ with non-zero gradient, that the particle will start to move. But that movement indicates a change in the particle’s kinetic energy, and therefore work has been done! Is the field of electrodynamics doomed to be plagued by this inconsistency?

In order to answer the questions that we have just asked, it takes a novel approach to the problem of a particle’s self-interaction. Rather than using the typical approach of modeling a point-like charged particle to have zero size while keeping the mass and charge constant, a new approach has been developed[1], whereby each quantity of the particle has been scaled to zero at the same rate, so that the singular point becomes more of a “signature” than a particle. This scaling process then allows for a first order perturbation analysis to be done, yielding equations of motion that are self-consistent and prevent any of the issues seen throughout previous attempts, such as an infinitely negative mass renormalization.

### 1.2 Review of Classical Electrodynamics and Self-Force

**Problem Arises**

Electrodynamics is the field of physics that is concerned with particles that carry electric charge and the different fields they emit (both electric and magnetic).
Electric and magnetic fields are related to each other and to their sources through *Maxwell’s equations*, which we take to be axioms of the theory.

\begin{align}
\nabla \cdot \vec{E}(t, \vec{x}) &= \frac{\rho(t, \vec{x})}{\varepsilon_0}, \\
\nabla \cdot \vec{B}(t, \vec{x}) &= 0, \\
\nabla \times \vec{E}(t, \vec{x}) &= -\frac{\partial \vec{B}(t, \vec{x})}{\partial t}, \\
\nabla \times \vec{B}(t, \vec{x}) &= \mu_0 \vec{j}(t, \vec{x}) + \frac{1}{c^2} \frac{\partial \vec{E}(t, \vec{x})}{\partial t}.
\end{align}

We define each of these quantities as: $\vec{E}$ is the electric field, $\vec{B}$ the magnetic field, $\rho$ is the charge density, and $\vec{j}$ is the charge *current* density. We can rewrite these equations in another, but equivalent form as

\begin{align}
\nabla \cdot \vec{E} &= 4\pi \rho, \\
\nabla \cdot \vec{B} &= 0, \\
\nabla \times \vec{E} &= \frac{1}{c^2} \frac{\partial \vec{B}}{\partial t}, \\
\nabla \times \vec{B} &= \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}.
\end{align}

The difference between these two forms is the unit system that the fields and sources are measured in; (1.1) is written in SI units, the standard unit system for measurement used through most experiments, while (1.2) is written in Gaussian units, which becomes more useful when handling long and arduous calculations. There are different times throughout the paper that these two systems will each be useful, and we will be swapping between them from time to time; from the context and the forms of the equations, it should be clear when we are using one or the other.
All information about classical electromagnetic systems is, at some level, locked
in Maxwell’s equations, and all we need to do is manipulate the equations in order to
find the relevant information that we desire to know. Maxwell’s equations typically
allow us to ask two types of questions in the context of classical electrodynamics:

1. Given a specific source, what are the fields that it emits?

2. Given some external fields, what are the forces that a test charge experience
and what is the trajectory of the particle?

In both of these cases, one can use the model of a point-charge, which makes use of
a Dirac delta, defined as

\[ \delta(\vec{x} - \vec{x}') = \begin{cases} \infty & \vec{x} = \vec{x}', \\ 0 & \vec{x} \neq \vec{x}' \end{cases} \tag{1.3} \]

Additionally, when we integrate the Dirac delta over any region that contains the
point \( \vec{x} = \vec{x}' \), we have

\[ \int_{x-\epsilon}^{x+\epsilon} \delta(x - x') \, dx' = 1. \tag{1.4} \]

What this does is that it keeps the total charge and mass of the particle finite, while
shrinking the physical size of the particle to a single point. The charge distribution,
then, is given as

\[ \rho(t, \vec{x}) = q \delta(\vec{x} - \vec{x}). \tag{1.5} \]

With this model of a point charge, the force on the particle is determined by
making use of the well-known Lorentz force,

\[ \vec{F} = \frac{d\vec{p}}{dt} = q \left( \vec{E} + \vec{v} \times \vec{B} \right). \tag{1.6} \]
This equation has some profound consequences! In the case that a test charge is static, the only force will be that from the electric field. Meanwhile, the force from the magnetic fields is always perpendicular to the direction in which the particle is traveling. This is (at its essence) the principle that governs particle accelerators and has led to some major discoveries throughout the history of physics.

In most cases, the use of a Dirac delta makes calculations much easier, and the two cases have well-defined mathematical and physical meaning. The issue, then, arises when we combine the two cases to form the question that we posed in the first sentence of the paper:

**Given some charged particle traveling along a trajectory that is emitting some electromagnetic field, find the new trajectory of the particle after interacting with the fields emitted at a previous time.**

To see why this causes problems, let’s consider a point-charge emitting fields first, and we utilize potential theory to find the fields emitted by the charged particle. First, recall that we can write the electric and magnetic fields in terms of a scalar potential, \( \Phi \), and a vector potential, \( \vec{A} \), as

\[
\vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t},
\]

\[
\vec{B} = \vec{\nabla} \times \vec{A}.
\]
For a particle along a trajectory, we can then write the potentials of that field using the Liénard-Weichert potentials, defined in [2] as

\[
\Phi(t, \vec{x}) = \left[ \frac{\rho(t, \vec{x})}{\left(1 - \vec{\beta} \cdot \hat{n}\right) R_{\text{ret}}} \right],
\]

\[
\vec{A}(t, \vec{x}) = \left[ \frac{\rho(t, \vec{x})\vec{\beta}}{\left(1 - \vec{\beta} \cdot \hat{n}\right) R_{\text{ret}}} \right],
\]

where we define \( \vec{\beta} = \frac{\vec{v}}{c} \) with \( c \) being the speed of light, while subscript “ret” indicates that the entire quantity must be evaluated at the retarded time. This means that in causal systems, things that have happened in the past must affect the system at later times, and so we are evaluating the quantity at an earlier time. Additionally, since this is valid for a point charge and we know that the charge density for a point charge can be written as in (1.5), these potentials become

\[
\Phi(t, \vec{x}) = \left[ \frac{q\delta(\vec{x} - \vec{x}')}{\left(1 - \vec{\beta} \cdot \hat{n}\right) R_{\text{ret}}} \right], \tag{1.9a}
\]

\[
\vec{A}(t, \vec{x}) = \left[ \frac{q\delta(\vec{x} - \vec{x}') \vec{\beta}}{\left(1 - \vec{\beta} \cdot \hat{n}\right) R_{\text{ret}}} \right]. \tag{1.9b}
\]

It is important to note that while the Dirac delta is not typically written explicitly, as it is already encoded into the the charge \( q \), we have written equation (1.9) in this form to emphasize the fact that these potentials are in fact distributions.
Now, let’s start by looking at just the electric field in this case. Taking our Liénard-Weichert potentials from (1.9 and putting them into (1.7), we can write

\[
\vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t} \]

\[
= \vec{\nabla} \left[ \frac{q \delta(\vec{x} - \vec{x}')}{(1 - \vec{\beta} \cdot \hat{n}) R} \right]_{\text{ret}} - \frac{\partial}{\partial t} \left[ \frac{q \delta(\vec{x} - \vec{x}')}{(1 - \vec{\beta} \cdot \hat{n}) R} \right]_{\text{ret}}. \tag{1.10}
\]

Since our point-charge is constant (it does not change shape, since there is no shape and so it only changes its position, but that does not affect the charge itself), we can pull it out of both of these terms to write:

\[
\vec{E} = q \delta \left( \vec{\nabla} \left[ \frac{1}{(1 - \vec{\beta} \cdot \hat{n}) R} \right]_{\text{ret}} - \frac{\partial}{\partial t} \left[ \frac{\vec{\beta}}{(1 - \vec{\beta} \cdot \hat{n}) R} \right]_{\text{ret}} \right). \tag{1.11}
\]

The same can then be done for the magnetic field in (1.7).

\[
\vec{B} = q \delta \left( \vec{\nabla} \times \left[ \frac{\vec{\beta}}{(1 - \vec{\beta} \cdot \hat{n}) R} \right]_{\text{ret}} \right). \]

The inside of the parentheses is not necessarily important in what we are trying to illuminate, so we will just write these fields as

\[
\vec{E} = q \delta(\ldots), \tag{1.12a}
\]

\[
\vec{B} = q \delta(\bullet). \tag{1.12b}
\]

We can immediately see that the fields that we have found are also distributions, since they depend directly on the Dirac delta used in the potentials. We can now
take these electric and magnetic fields and put them into the force density equation to try to find the force on the same point-like charged particle.

$$\vec{f} = (q\delta)\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

$$= (q\delta)\left[q\delta(\ldots) + q\delta(\bullet)\right]$$

$$= (q\delta)(q\delta)\left[(\ldots) + (\bullet)\right]$$

$$= q^2\delta^2\left[(\ldots) + (\bullet)\right].$$

So what might be the problem here? Well, the Dirac delta, as we defined earlier, is a distribution and not a function. When multiplying two functions together, there is a well-defined mathematical definition of the square of a function. However, no such well-defined mathematical definition exists for distributions. For more information, see Appendix C. It is due to this lack of a mathematical definition that when combining Maxwell’s equations and the self-force problem, taking this approach, as historical attempts have, produces a result that is not mathematically well-posed.

Due to this ill-posed problem, there must be some sort of way to get around this to solve this problem. To date, this is truly the last problem that has yet to be solved satisfactorily within the field of Classical Electrodynamics. While this problem can be solved using a Quantum Mechanical approach to electrodynamics (also known as the field of Quantum Electrodynamics, or QED), this should still be a problem that can be solved using a classical approach. If not, there may be a serious deficiency in this field, which has been tested over and over again, and has provided some of the most powerful results within physics, and a possible reevaluation may be in order.

Luckily, Classical Electrodynamics may have been saved by the paper presented by Robert Wald and his students in 2009 [1] with a new approach to the problem.
It is our goal to shed some light and expand upon this paper then, and show some numerical results that will provide context on the validity of the approach. Before we discuss this approach, though, we will first discuss the history of the problem to give some context to what is novel about it.
CHAPTER 2
HISTORY OF THE SELF-FORCE PROBLEM

2.1 What Is Electromagnetic Radiation?

In order to have some context about the history of the self-interaction force, we should familiarize ourselves with the idea of radiation: what is radiation, why is it important, and how is it relevant?

To start, a loose definition of electromagnetic radiation can be thought of as the components of electric and magnetic fields that reach out infinitely far away from their source with a non-zero contribution. A simple picture is the idea of light coming from a distant star. Here on Earth, we see numerous stars populating the night sky, each one looking like a single point (more or less). At the distance that the stars are from the Earth, the stars appear tiny, yet the light they emit still makes it all the way to our eyes. However, this is just a simple picture in order to have a physical insight into the idea of radiation and electromagnetic fields reaching from far away.

Now that we have a toy model, let’s look at a little more of a formal definition and recall that the Poynting vector, \( \vec{S} \), is defined as the energy flux through a surface per unit time [2], and can be calculated as

\[
\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}. \tag{2.1}
\]
This Poynting vector, then, is used in the conservation of electromagnetic energy (also known as Poynting’s theorem), and can be written as

\[ \frac{\partial u}{\partial t} = -\nabla \cdot \vec{S} - \vec{J} \cdot \vec{E}, \]  

where \( u \) is the energy density and \( \vec{J} \) is the charged current density. The physical interpretation of this equation can be thought of as the change in the amount of energy in a certain volume of space over the course of time must be equal to the amount that the Poynting vector spreads out in space added with how much the electric field interacts with the charged current density. This is still a little dense, so we consider a system with external electromagnetic fields, so that we still have \( \vec{E} \) and \( \vec{B} \) to calculate \( \vec{S} \), but sufficiently far away from the source, such that \( \vec{J} = 0 \). This reduces Equation (2.2) to

\[ \frac{\partial u}{\partial t} = -\nabla \cdot \vec{S}. \]  

So this is where we can really start to see the idea of radiation being the fields that are carried out to infinity. However, it may still be a little obscure, so let’s take this a little further. If we integrate both sides of (2.2) through some volume element, the energy density will simply turn into the energy, \( U \), and we can write

\[ \frac{\partial U}{\partial t} = \int d\vec{a} \cdot \vec{S}. \]  

(2.3)

We now look at the right-hand side of this, and consider what happens as we integrate this Poynting vector over a surface (for example the surface of a sphere) and then take the radius of the sphere to some point infinitely far away. If this evaluation gives a non-zero contribution, it means that the fields are carrying energy out to a
point infinitely far away from their source, and this is what we describe as radiation. However, if the evaluation of this integral gives zero, then the fields do not carry energy infinitely far away from their source and we can say that the source is not radiating.

It is natural to ask why this might be important to consider in the self-force problem, since one would expect that a self-interaction would be more worried about near-fields than far-fields. However, now that we have a more formal definition and understanding of radiation than the toy picture we painted for ourselves, we consider the following: there is a system consisting of some charged particle, which is accelerating (at this point, we will just take it to be true that accelerating particles radiate, while static particles do not). Since the accelerating particle is radiating away energy to infinitely far away, it is losing energy that it will not be able to gain back. If all of its energy is being radiated over the course of time, eventually it would have to stop accelerating and become static.\footnote{In this case, static does not necessarily mean \textit{stationary}. Instead, static simply means “not accelerating.” One can stay static while changing their position in space, as long as they stay along a linear trajectory and do not have forces acting upon them. For example, a car traveling west at 75 mph is still static as long as it stays traveling as such. The moment that the car changes direction, or changes speed, it is no longer static. Being stationary is merely a special case of being static.} It is this reaction of the particle to its own radiation of energy that causes the change in the motion, then. Note that Appendix C is a derivation of the Larmor formula, the generalized, relativistic Larmor formula, which explains why accelerating charges radiate.

It is that idea of the \textit{radiation reaction} that really sets the stage for the self-force problem. As we shall see, this is truly the first step that leads to the possibility of this pesky problem becoming a prominent issue in physics.
2.2 Brief History

In the 1800's, the field of physics focused mostly on electrodynamics, with the famous Maxwell equations championing this field in the latter half, and the field of thermodynamics and statistical mechanics. It was this latter field that first presented the idea of the radiation reaction, with Balfour Stewart giving possibly the first clear statement about this phenomenon. He was considering a system of particles within a confined space, where the particles could bounce off of the walls, losing some of their energy in these non-elastic collisions. [3]

It is not therefore, allowable to suppose that in such an enclosure the moving body retains all of its energy of motion, and consequently such a body will have its energy of motion gradually stopped.

However, this originally did not pose a problem to the field of electromagnetism, as the field was still being formalized into one theory.

About ten years after the consideration of the radiation reaction in thermodynamics, JJ Thomson [4] had been considering the fields of slowly moving charged particles and had described the first example of a mass renormalization, saying that the magnetic field energy should effectively be considered in the mass of the particle with the notion of the relativistic mass scaling,

\[ m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0. \]
He later identified the relationship describing the size of a charged particle in order to attribute it to the mass through the equation

$$a = \frac{e^2}{mc^2},$$

which would be the radius of a charged sphere, implying that the notion of particles were considered to be finite in size, and shaped as spheres, as opposed to the concept of a point-particle at the time.

It was also around this time that Poynting had developed his theory of calculating the energy flow of electromagnetic fields using vectors. Even before he did though, the English physicist by the name of FitzGerald [5] gave the first calculation of the “radiation resistance” in 1883 as

$$R_{\text{rad}} = \frac{2\pi^2}{3c} \left( \frac{L}{\lambda} \right)^2,$$  \hspace{1cm} (2.4)

where \(L\) is the circumference of an electrical loop and \(\lambda = \frac{2\pi c}{\omega}\), with \(\omega\) being the angular frequency of the resulting electromagnetic wave. This was really one of the first definitive statements of the “damping” of electrical circuits due to the emission of radiation. It is important to note that there is a physical difference in the words “resistance” and “reaction.” The resistance is actually a form of reaction, where the electrical loop is “resisting” the change due to the emission of radiation, which changes the dynamics within the loop. However, the “reaction” of a single particle could be drastically different than if it were to stay on its own initial trajectory. There is currently no experimental evidence of what the effects of the classical radiation reaction are on single particles.
It was not until 1892 that Lorentz [6] had truly started to consider the structure of charged particles and he claimed that the charge resided on the body of the particle. He then began to explore the concept of what force the particle would exert on itself as it moved at slow speeds and came out to the form of

$$\vec{F} = \frac{2e^2 \, d^2 \vec{V}}{3c^3 \, dt^2},$$

(2.5)

which is determined for an extended body with finite size. He had no mention of radiation within his derivation, but rather simply used the retarded potentials to obtain his result. Additionally, Lorentz had written the left-hand side of (2.5) in terms of what he defined as the effective mass, where

$$m_{\text{eff}} = m + \frac{ke^2}{r_0c^2},$$

giving rise to the idea that the particle’s electromagnetic properties contributed to the observed mass.

There is a stipulation about these results, however. (2.5) only holds if the motion of the charged particle does not change “sensibly” during the time light takes to cross the charge. What does this mean? Well, in its current form, (2.5) is third-order in time, meaning that it has potential to be plagued by so-called “runaway” solutions, where the particle could continuously accelerate in time, which is not a phenomenon seen in any physical process. This stipulation, then, is meant to prevent these runaway solutions, which is a problem, as it keeps us from being able
to consider it in a general case. One possible solution would be to renormalize the self-force into the “effective” momentum in the form of

$$\vec{F} = \frac{d\vec{p}_{\text{eff}}}{dt} = \frac{d}{dt} \left( \vec{p}_{\text{mech}} - \frac{2e^2}{3c^3} dv \right),$$

but this still leaves us with restricting the physical cases that we can consider.

Around this same time, Max Planck [7] was diving into the relation between electrodynamics and thermodynamics with the emission and absorption of radiation. While working on his own derivation, he discovered that if an electron was perturbed by an electromagnetic wave, that it had the potential to gain indefinite amplitude (or infinite energy) and determined that there was need for a damping force in order to preserve the conservation of energy. In so doing, he arrived at the form

$$\vec{F}_{\text{damp}} = \frac{2e^2}{3c^3} \frac{d^2 \vec{v}}{dt^2},$$

which is exactly the same form as (2.5)! Clearly they were onto something, but there was still some refinement that had to be done in order to prevent the issues discussed above.

Just after the turn of the 20th century, one of Albert Einstein’s four papers of 1905 proposed the initial foundation of special relativity (appropriately named for this paper “On The Electrodynamics of Moving Bodies”). The theory has two main postulates:

1. The laws of physics are invariant to all inertial reference frames.

2. The velocity of light in empty space is a universal constant, and therefore must be the same for all observers.
These two postulates have since changed the course of history for physics, not just in electrodynamics, but all fields. From these two postulates, one can derive the use of the Lorentz transformation, which is how we can analytically write the results of a system for different observers traveling at different speeds. One major part of these transformations is the Lorentz factor, given as

$$\gamma = \frac{1}{\sqrt{1 - \frac{\vec{v} \cdot \vec{v}}{c^2}}}.$$ (2.6)

Armed with this theory, it was only a matter of time until the relativistic vector form of (2.5) was written down, and in 1908 the generalization of the self-force was written down as

$$\vec{F} = \frac{2e^2}{3c^3} \left[ \gamma^2 \vec{v} + \frac{\gamma^4 (\vec{v} \cdot \vec{v})}{c^2} + \frac{3\gamma^4 \vec{v} \cdot \vec{v}}{c^2} + \frac{3\gamma^6 (\vec{v} \cdot \vec{v})^2}{c^4} \right].$$

Note that if we take the non-relativistic limit of $\vec{v} \to 0$, then $\gamma \to 1$, and we recover the same form as (2.5). This result was obtained by Abraham in 1904, [8].

Shortly after the rise of special relativity came the application of covariant notation to physics, which led to Schott calculating the self-force on an extended, accelerating body with an arbitrary velocity [9]. The power obtained in this covariant notation is

$$P = -\frac{2e^2}{3c^3} \frac{du^\mu}{d\tau} \frac{du_\mu}{d\tau},$$ (2.7)

where we are now seeing $u^\mu$ is the 4-velocity, and we are taking the 4-vector form of a scalar product, ensuring that the power radiated by a charged particle is a Lorentz
scalar.\textsuperscript{2} His discussion on the self-force leads to the “acceleration energy,” where the kinetic energy now depends on the acceleration, as well as the velocity.

We now find ourselves considering the work done by W. Pauli in the 1920’s.\textsuperscript{[10]} We note that he was the first to truly derive a description of the self-force (radiation reaction) in covariant notation. Starting with Lorentz’s equation, (2.5), where we remind ourselves that Lorentz had derived his result without any mention of radiation, Pauli managed to derive an equation that described this force \textit{with} a dependence upon the radiation from the body.

\[
F^\mu_{\text{self}} = \frac{2e^2}{3c^3} \frac{d^2u^\mu}{d\tau^2} - P_{\text{rad}} \frac{u^\mu}{c^2}.
\]

Then, by adding this to the typical Lorentz force, the total force on a particle can be written as

\[
F^\mu_{\text{tot}} = F^\mu_{\text{ext}} + F^\mu_{\text{self}} = qF^\mu_{\text{ext}}u_\nu + \frac{2e^2}{3c^3} \frac{d^2u^\mu}{d\tau^2} - P_{\text{rad}} \frac{u^\mu}{c^2}.
\]

In order to prevent any confusion on the notation, we use \( F^\mu \), for the force with the script \( F \) and a single raised index, while the field strength tensor discussed in the introduction is still \( F^{\mu\nu} \). It is important to note that for uniformly accelerated particles, then, the self force vanishes! We can see this from the second time derivative of the 4-velocity. If the acceleration, or \( \frac{d^2u^\mu}{d\tau^2} \), does not change in time, then the first term goes away. Additionally, the power is determined by the acceleration, and with the raised and lowered indexes, this contribution will lead to a null result.

\textsuperscript{2}A Lorentz scalar ensures that the same measurement will be obtained by any observer within any inertial reference frame, regardless of the velocity at which the observer is traveling.
It is useful to note a few key points with regards to this part of the history of the self-force problem. First, almost every calculation was made with the assumption of an extended body with finite size, meaning that the notion of a point-charge was not used. Even when P. Dirac tried his hand at the self-force problem, he made no assumption about the shape or size of the body and still managed to come out to the same result as (2.8). All of these results were plagued by issues of having to put constraints on the system, such as seen in the discussion about Lorentz’s result. Even when a renormalization was considered for the mass of the particle, the only way to prevent runaway solutions was for the renormalized mass to go to an unphysical value of $m \to -\infty$. As it may seem, classical electrodynamics was indeed doomed to be ridden with these small, yet significant issues.

### 2.3 The 4/3 Problem And The Poincare Stress Tensor

There was another question that arose in the field of CED surrounding the issue of the self-force, which came to be known as the 4/3 Problem. It starts with the notion of the “electromagnetic mass,” which was originally proposed by Thomson in 1881. He identified the notion of a purely electromagnetic particle, which gave rise to an observed mass as [11]

$$m_{elm} = f \frac{e^2}{R e^2},$$  \hspace{1cm} (2.9)

where $R$ is the radius of the particle and $f$ is some numerical factor. If the rest-mass of the particle, $m_0$, was 0, then the observed mass would be equal to (2.9), implying that the particle does not consist of any pure matter [11].
In 1903, Abraham used this to be able to calculate the momentum of the Coulomb field of an electron from the Poynting vector, written as

\[ \vec{p} = \frac{1}{c^2} \int \vec{S} \, d^3r. \]  

(2.10)

This gives rise to the result of

\[ \vec{p} = \vec{p}_0 + \vec{p}_{\text{elm}} = m_0 \vec{v} + \frac{4}{3} m_{\text{elm}} \vec{v}. \]

Assuming that the particle is purely electromagnetic in nature, then the first term goes away and we are just left with the second term, and the force would then be written as

\[ \vec{F}_{\text{elm}} = \frac{d}{dt} \left( \frac{4}{3} m_{\text{elm}} \vec{v} \right), \]

or with the correction for the self-force

\[ \vec{F} = \frac{d}{dt} \left( \frac{4}{3} m_{\text{elm}} \vec{v} \right) - \frac{2e^2}{3c^3} \frac{d^2 \vec{v}}{dt^2}. \]

This first term can then be written in terms of the energy,

\[ \vec{F} = \frac{4}{3} \frac{U}{c^2} \vec{v} - \frac{2e^2}{3c^3} \frac{\vec{v}}{\vec{v}}. \]  

(2.11)

We now look to this first term and see that the coefficient of the acceleration is what we would usually write as the total mass. But, this gives rise to a problem when we
consider the mass-energy relation that is well-known from relativity. In relativity, we learn the fundamental relation

\[ U = mc^2. \] (2.12)

But, if we look to the first term of (2.11), we would get

\[ mc^2 = \frac{4}{3} U. \]

This is where the name of the 4/3 Problem originates. We should be getting a result identical to that known from relativity, but yet we see that the mass is calculated to be more than the energy of the particle itself, assuming that it is purely electromagnetic in nature. Why?

The solution is to realize that the definition of the momentum in (2.10) is incorrect for bound fields, even though it is valid for the radiation fields themselves. Poincare recognized the instability of purely electromagnetic particles, and so he proposed a new stress-energy tensor defined to be

\[ S^{\alpha\beta} = \Theta^{\alpha\beta} + P^{\alpha\beta}, \] (2.13)

where \( \Theta^{\alpha\beta} \) is the known classical symmetric stress tensor, while \( P^{\alpha\beta} \) is something new [2]. To see the instability that Poincare recognized, we start by refreshing our memories that the classical symmetric stress tensor is defined as

\[ \Theta^{\alpha\beta} = \frac{1}{4\pi} \left( g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right). \]
This can be a lot to unpack, so we write out the individual components by summing over the repeated indexes and find

\begin{align}
\Theta^{00} &= \frac{1}{4\pi} (E^2 + B^2) , \\
\Theta^{0i} &= \frac{1}{4\pi} (\vec{E} \times \vec{B})_i , \\
\Theta^{ij} &= -\frac{1}{4\pi} \left[ E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (E^2 + B^2) \right].
\end{align}

We can then explicitly calculate the divergence, which can be written as \( \partial_\alpha \Theta^{\alpha\beta} \), which in a source-free region gives

\[ \partial_\alpha S^{\alpha\beta} = 0. \]  
(2.15)

If we separate \( \beta \) into time (\( \beta = 0 \)) and spatial (\( \beta = i \)) components, this can be written in vector form

\begin{align*}
\partial_\alpha \Theta^{\alpha0} &= \frac{1}{c} \left( \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right) = 0, \\
\partial_\alpha \Theta^{\alpha i} &= \frac{1}{c^2} \frac{\partial S_i}{\partial t} - \frac{\partial}{\partial x^j} T_{ij} = 0,
\end{align*}

which provide the three conservation laws of energy, momentum, and charge.

Moving onto a region with some localized source, this divergence becomes less trivial.

\begin{align*}
\partial_\alpha \Theta^{\alpha\beta} &= \frac{1}{4\pi} \left[ \partial^\mu (F_{\mu\lambda} F^{\lambda\beta}) + \frac{1}{4} \partial^\beta (F_{\mu\lambda} F^{\mu\lambda}) \right] \\
&= \frac{1}{4\pi} \left[ \partial^\mu (F_{\mu\lambda} F^{\lambda\beta}) + F_{\mu\lambda} \partial^\mu F^{\lambda\beta} + \frac{1}{2} F_{\mu\lambda} \partial^\beta F^{\mu\lambda} \right].
\end{align*}
(2.16)
The first term can be found by using Maxwell’s equations,

\[(\partial^\mu F_{\mu\lambda})F^{\lambda\beta} = \frac{1}{c}F^{\beta\lambda}J_\lambda,\]

where \(J_\lambda\) is the 4-vector associated with the charge density and charged current density. The other two terms cancel out due to symmetries and contraction of indexes. This leaves us with the divergence of the stress-energy tensor for fields interacting with charged particles as

\[\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{c}F^{\beta\lambda}J_\lambda,\]  \hspace{1cm} (2.17)

where we have absorbed the factor of \(\frac{1}{4\pi}\) into the field-strength tensor. In the same way that we did for the source-free region, we then write this divergence in vector form by separating out the time and space components.

\[\partial_\alpha \Theta^{\alpha0} = \frac{1}{c} \left( \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} \right) = -\frac{1}{c} \vec{J} \cdot \vec{E},\]  \hspace{1cm} (2.18a)

\[\partial_\alpha \Theta^{\alpha i} = \frac{1}{c^2} \frac{\partial S_i}{\partial t} - \frac{\partial}{\partial x^j} T_{ij} = -\rho E_i - \frac{1}{c} \left( \vec{J} \times \vec{B} \right)_i\]  \hspace{1cm} (2.18b)

This second equation is the same form as the force density in (B.10), and so the 4-vector form of the force density is what results from the divergence of a region with a localized source.

\[\partial_\alpha \Theta^{\alpha\beta} = f^\beta.\]  \hspace{1cm} (2.19)

However, this leaves us at a cross-roads, as the self-force from a charged particle would lead to an instability from electrical forces pushing away from each other. The solution, then, is to make use of the Poincare stress tensor (2.13).
Conceptually, the Poincare stress tensor is meant to counteract the electrical forces of a charged particle of finite size and stabilize said particle. To do this, the divergence must be zero, which will ensure that the total stress-energy does not tear itself apart.

$$\partial_\alpha S^{\alpha\beta} = \partial_\alpha (\Theta^{\alpha\beta} + P^{\alpha\beta}) = 0.$$  \hspace{1cm} (2.20)

This additional tensor of $P^{\alpha\beta}$ holds the particle together while the electrical forces from $\Theta^{\alpha\beta}$ tries to force the particle apart. It is important to note that the spatial integrals of $S^{\alpha0}$ then transform as a 4-vector, giving us a covariant and stable model for the particle.

J. Schwinger was then able to take the Poincare stress tensor into account and develop a model for it in order to keep $S^{\alpha\beta}$ as a divergence-free tensor that stabilizes charged particles. With this tensor that he modeled, it then becomes possible to redefine the momentum in terms of a covariant theory (as opposed to the vector formulation leading to the 4/3 problem), where

$$cP^\alpha = \int S^{\alpha0} \, d^3x,$$  \hspace{1cm} (2.21)

and since the Poynting vector can be defined in terms of a momentum density, $\vec{g}$, a relativistically correct model for the extended particle, maintaining

$$\vec{P}_{elm} = \frac{U_{elm}}{c^2} \vec{v},$$

$$U_{elm} = m_{elm}c^2.$$

However, this was still done while considering the notion of an extended body, rather than a point-charge, and we have yet to fully address the notion of a point-like charged particle. As highlighted in the introduction and Appendix B, the mathe-
matical formulation of a point-like charged particle does not hold water when attempting to calculate its self-force while modeling it with a Dirac delta. This process has highlighted a crucial point for us though: we need to have a model that conserves energy, momentum, and charge. If we neglect to consider all three of those, then we will still have a theory that may be prone to giving runaway solutions or unphysical renormalizations.

It was with this history in mind that Wald and his students then attempted a novel approach to calculating the self-force of a point-charge. Before going in-depth with the summary of the paper leading to the main work done here, it would be good to paint a physical picture of why this new model is so different. Typically when considering a point-charge, as discussed before, we model it with a Dirac delta, so that we have

\[
\rho(\mathbf{r}) = q\delta(\mathbf{r}).
\]

However, this process merely scales down the physical size of the particle, while leaving the mass and the charge constant, which is what leads to these inconsistencies. This novel approach, though, looks at the particle in a new lens where the size is scaled down to nothing while also scaling down the mass and charge of the particle at an equal rate. With this scaling down, it may be helpful to maybe consider the point-charge to be more of a signature of the particle, rather than the particle itself. In much the same way, consider the expression

\[
\frac{\sin(x)}{x},
\]
which is undefined when evaluated at $x = 0$. However, we can make use of l’Hospital’s rule when using taking the limit of this expression,

$$
\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \cos(x) = 1,
$$

which is defined. It is this same manner that will be allow us make an estimate of the particle as it shrinks to zero.

\section{2.4 Gralla Et. Al. Summary}

As we have already seen why a point-like charged particle is ill-posed, we should begin the summary of this approach by mentioning that considering a particle of finite size is additionally a non-trivial task. In addition to the electromagnetic considerations in the nature of the problem, one would also need to consider all of the aspects of the body’s motion, which includes all of the body’s internal degrees of freedom, such as the forces holding the body together. One way to make this process easier on the solver is by considering a rigid body, which will reduce the degrees of freedom for the body, as it will not be free to change shape and contort as it travels along its path. However, in so doing, this approach severely restricts the systems that can be considered, in a similar manner to the restrictions on the Lorentz solution of the self-force.\footnote{This section is drawn heavily from [1], and all information for this section is assumed to be cited from here, with few explicit exceptions when noted.}

It was with this fact in mind that the paper begins its introduction and assumptions. As previously stated, the purpose of the paper is to eliminate any mathematical and physical deficiencies by using an approach that takes a limit of
the body, charge, and mass all scaling to zero. This scaling allows us to eliminate
the electromagnetic self-energy that we saw causes instabilities in the body itself.
The effects of any self-interaction then arises as a perturbation to the center-of-mass
of the particle and the Lorentz force law is then derived as a consequence (along
with the additional terms).

The assumptions of the paper are simple: in consideration is some “one-parameter
family of charged bodies” which are described by the charge-current density 4-vector
and a matter stress-energy tensor. The parameter used is λ, which will be our
main scaling factor throughout. The electromagnetic fields considered can be writ-
ten in the form of the field-strength tensor, \( F_{\mu\nu} \) and are assumed to be solutions
to Maxwell’s equations. All conservation laws are assumed to hold true, and the
only equations that are assumed to be true are Maxwell’s equations, conservation
of charge (which is implicit in Maxwell’s equations already), and conservation of
stress-energy. There are two stress-energy tensors that will be used, one for the
matter of the particle, \( T_{\mu\nu}^{M} \), and one for the electromagnetic stress-energy, \( T_{\mu\nu}^{EM} \). In
the formalism of the paper, these equations are

\[
\nabla^\nu F_{\mu\nu} = 4\pi J_\mu, \quad (2.22a)
\]
\[
\nabla_{[\mu} F_{\nu\rho]} = 0, \quad (2.22b)
\]
\[
\nabla_\nu J^\nu = 0, \quad (2.22c)
\]
\[
\nabla^\nu (T_{\mu\nu}^{M} + T_{\mu\nu}^{EM}) = 0. \quad (2.22d)
\]

For the duration of this summary, this will be the notation used for each quantity.
Notice that (2.20) is the same as what we wrote for (2.22d), only replacing the \( \Theta \)
and \( P \) with \( T \), while keeping superscripts to specify the appropriate tensor. It would
be useful to note at this time that we will be following the notation of the paper
with natural units where \( c = 1 \). We will go back later and replace all factors of \( c \) just before we conduct our own numerical calculations.

In addition to these assumptions, the paper makes use of something called “Fermi normal coordinates,” which are used to justify the use of a flat metric. The important thing to note is that we can write this coordinate system as \((T, X^i)\), and at any point of constant time \( T \), the plane is orthogonal to the worldline of the particle \( \gamma \) and \( X^i \) measures the spacial distances in the neighborhood of \( \gamma \). A much more in-depth discussion of Fermi normal coordinates can be found in the paper “The Motion Of Point Particles In Curved Spacetime” by Eric Poisson, [12]. The important thing for us is just to understand that this allows us to work with our flat metric, while allowing for the possibility to extend the work in a more complicated non-flat metric.

In order to stay as general as possible, such that the body is not necessarily rigid (and therefore obeying special relativity), the quantities of size, mass, and charge must all shrink to zero proportionately to each other. This allows the particle to appear the same, regardless of the scale of the observer. To do this, the charge and mass of the particle are scaled as

\[
J^\mu(\lambda, T, X^i) = \frac{1}{\lambda^2} \tilde{J}^\mu(\lambda, T, \frac{X^i}{\lambda}), \tag{2.23a}
\]

\[
T^M_{\mu\nu} = \frac{1}{\lambda^2} \tilde{T}^M_{\mu\nu}(\lambda, T, \frac{X^i}{\lambda}), \tag{2.23b}
\]

where \( \tilde{J}^\mu \) and \( \tilde{T}^M_{\mu\nu} \) can be any arbitrary, smooth function. These conditions depend on a scaling of \( \frac{1}{\lambda^2} \), since any faster scaling would then be similar to a Dirac delta, which we do not want, while a slower scaling would require higher-order corrections.
Lastly, the total electromagnetic tensor, $F_{\mu\nu}$, consists of both the external electromagnetic fields and the fields produced by the body, given as

$$F_{\mu\nu} = F^\text{ext}_{\mu\nu} + F^\text{self}_{\mu\nu}. \quad (2.24)$$

The self-field is given as the retarded solution to Maxwell’s equations from the source described by the 4-vector $J^\mu$, while the external field is assumed to be a homogeneous solution for Maxwell’s equations.

### 2.4.1 Far-Zone Limit

#### 2.4.1.1 Conservation of Charge

There are two limits that are considered in the analysis of the paper, which are referred to as the “far-zone” and “near-zone” limits. The far-zone limit, which is considered first, corresponds with an observer at a fixed radius from the worldline of the particle as the body shrinks down. In other words, the coordinates $x^\mu$ are fixed. The goal here is to show that to first order in $\lambda$ that the body is described by a point particle and obeys the typical Lorentz force law. However, it is important to recognize that the Lorentz force law comes as a consequence of this and is not a postulate of the theory, as we have previously mentioned.

With the scaling of the charged current density and mass stress tensor in (2.23), we can describe these mathematically using distributions, which allows them and their derivatives with respect to the parameter $\lambda$ to have well-defined limits as $\lambda$ approaches zero. To do this, we consider that for any locally integrable function,
which we will call $H(t, x^i)$, there is an associated distribution $\mathcal{D}_H$ defined by the action on a smooth test function

$$\mathcal{D}_H[f] = \langle H, f \rangle = \int H(t, x^i) f(t, x^i) \sqrt{-g} \, d^4x, \quad (2.25)$$

where the factor $\sqrt{-g}$ stands for the diagonal of the metric\(^4\) and $d^4x$ tells us that we need to integrate over all time and space. This integrable function can then be replaced by $J^\mu$, for which we will use the definition from (2.23) and rewrite the spatial components as $X^i = x^i - z^i(t)$. The distributional form of $J^\mu$ is then

$$\mathcal{D}_{J^\mu}[f] = \int \frac{1}{\lambda^2} \tilde{J}^\mu \left( \lambda, t, \frac{x^i - z^i(t)}{\lambda} \right) f_\mu(t, x^i) \, dt \, d^3\bar{x}. \quad (2.26)$$

We can now make use of a change of variables, where we define

$$\bar{x}^i = \frac{x^i - z^i(t)}{\lambda},$$

so that the integrating differential becomes $d^3x = \lambda^3 d^3\bar{x}$. This allows us to rewrite the distribution of $J^\mu$ (and subsequently why the scaling of $\lambda^{-2}$ makes sense) as

$$\mathcal{D}_{J^\mu}[f] = \lambda \int \tilde{J}^\mu(\lambda, t, \bar{x}^i) f_\mu(t, z^i(t) + \lambda\bar{x}^i) \, dt \, d^3\bar{x},$$

and we can now take the limit of $\lambda$ approaching zero inside of the integral, which yields

$$\mathcal{D}_{J^\mu}[f] = \lambda \int dt \, d^3\bar{x} \tilde{J}^\mu(0, t, \bar{x}^i) \int d^3\bar{x} \tilde{J}^\mu(0, t, \bar{x}^i). \quad (2.27)$$

\(^4\)We have added this factor in to (2.25), as this factor is in all of the calculations for the paper. The $g$ without any indexes implies that we are taking a multiplication between all of the diagonal components of the metric, which gives us a factor of -1. So when we take the negative and then square root, the factor becomes 1. Due to this, we will be dropping this and any other factor of the form of $\sqrt{g}$ from our calculations.
It is through this process of viewing $J^\mu$ as a family of distributions that we can take the limit in the 0th order for $\lambda$, so that we can see

$$J^{(0)\mu} \equiv \lim_{\lambda \to 0} J^\mu(\lambda) = 0.$$  \hspace{1cm} (2.28)

Next, we need to look to first order in $\lambda$, and so we define $\mathcal{J}^\mu(t)$ as the spatial integral of (2.27), and we get

$$J^{(1)\mu} \equiv \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} J^\mu(\lambda) = \mathcal{J}^\mu(t) \delta\left(x^i - z^i(t)\right) \frac{d\tau}{dt}. \hspace{1cm} (2.29)$$

This is all well, but what does this do for us? It sets us up to approach the problem for when we apply the conservation of charge (2.22c). We can apply this same approach to the conservation law, as the main difference between it and what we have just done is that we have now added the condition that the divergence is zero, from which we will be able to gain some helpful insight.

With this idea of viewing $J^\mu$ as a distribution, instead of just the charge current density, we apply the conservation of charge to (2.25), giving us the result of

$$\int \nabla_\nu J^\nu(\lambda) f \, d^4x = 0.$$  \hspace{1cm} (2.30)

We will then need to apply the tool of integrating by parts. To remind ourselves of this process, we recall that for any integral of two functions, we can write

$$\int u \, dv = uv - \int v \, du,$$
and so we define $dv = \nabla_\nu J^\nu d^4x$ and $u = f$, such that $v = J^\nu$ and $du = \nabla_\nu f d^4x$. Since the conservation of charge means that the divergence of $J^\mu$ is zero, the factor of $uv$ goes away, and we are left with

$$0 = \int J^\nu \nabla_\nu f d^4x = \int \frac{1}{\lambda^2} \tilde{J}^\mu(\lambda, t, \frac{X^i}{\lambda}) \nabla_\mu f d^4x$$

$$= \lambda \int \tilde{J}^\mu(\lambda, t, \tilde{x}) \nabla_\mu f(t, z^i(t) - \lambda \tilde{x}^i) dt d^3x,$$

which comes directly from the definition in (2.23). In the usual way, we take the limit of the integrand and split this total integral over $d^4x$ into the time and spatial components, so that we have

$$0 = \lambda \int \tilde{J}^\mu(\lambda = 0, t, \tilde{x}) d^3\tilde{x} \int \nabla_\mu f(t, z^i(t)) dt,$$

$$= \int J^\mu \nabla_\mu f(t, z^i(t)) d\tau.$$  \hspace{1cm} (2.32)

Notice that this is true for any test function, as we have kept $f$ as general as possible, only imposing that it be some smooth function. Therefore, we are free to use the Fermi normal coordinates as our function, with

$$f = X^i c(t),$$  \hspace{1cm} (2.33)

where we simply assume that we are in a neighborhood of the origin as $\lambda$ approaches 0. This means that the integral in (2.32) can be written as

$$\int J^\mu(t) \nabla_\mu X^i c(t) d\tau = 0.$$   \hspace{1cm} (2.34)
What this tells us is that the projection of $J^\mu$ is orthogonal to the worldline $\gamma$ and vanishes for all times, which allows us to rewrite $J^\mu$ as a scalar times the 4-velocity, $J^\mu \rightarrow J(t)u^\mu$. Putting this back integral, we can then make use of integration by parts again, much the same as before, so that we get $dv = \dot{J}(t)\,d\tau$, which then implies that $\dot{J}(t) = 0$, and so $J$ is constant in time. The scalar form of $J$ must give us the total charge of the particle, then, $q$. Using this result, we can go back to (2.29) and write

$$J^{(1)}_\mu = qu^\mu \delta\left(x^i - z^i(t)\right) \frac{d\tau}{dt}. \quad (2.35)$$

This is exactly the form of a point particle’s 4-vector charge current density. Perhaps it is useful to note that the charge itself, $q$, can be written as

$$q \equiv \int \tilde{J}^0(\lambda = 0, t, \bar{x}^i) \, d^3\bar{x}, \quad (2.36)$$

where $\tilde{J}^0$ is the temporal component of the charge current density that we had in the scaling. What we have seen, then, is that as the particle scales down, an observer at some constant radius away from the worldline of the particle will begin to see the particle itself as a point charge. While this is perhaps not a profound result, it does give us some assurance that this is the correct path to go down.

Before going any further into the far-zone limit, it would be beneficial to discuss Appendix A of the paper in order to have a better understanding of the processes done for the electromagnetic and matter tensors.
2.4.1.2 Scaling Of The Self-Field

Appendix A begins by writing the 4-potential in terms of the scaled current in the Lorentz gauge as

\[ A^\text{self}_\mu = \int d^3x' \left[ \frac{J_\mu(\lambda, t - |\bar{x}' - \bar{x}^i|, \bar{x}'^j)}{|\bar{x}' - \bar{x}^i|} \right], \]

where the coordinate system is defined as

\[ \bar{x}^i = \frac{x^i - z^i(t)}{\lambda}, \quad (2.37a) \]
\[ \bar{x}'^i = \frac{x'^i - z^i(t)}{\lambda}, \quad (2.37b) \]
\[ \bar{y}^i = \bar{x}'^i - \bar{x}^i, \quad (2.37c) \]

which allow us to write the potential in terms of \( \bar{y}^i \) and the parameter \( \lambda \).

\[ A^\text{self}_\mu = \int d^3\bar{y} \left[ \frac{\bar{J}_\mu(\lambda, t - \lambda|\bar{y}^i|, \bar{x}^j + \bar{y}^j + \left[ z^j(t) - z^j(t - \lambda|\bar{y}^k|) \right])}{|\bar{y}^i|} \right], \]

where the third component of the charged current density has been explicitly written in all three spatial components. Additionally, \( z^i(t) \) is a smooth function, which allows us to define another smooth function \( V^i \), such that

\[ z^i(t) - z^i(t - \lambda|\bar{y}^i|) = \lambda|\bar{y}^i|V^i(t, \lambda|\bar{y}^i|), \quad (2.38) \]
which in turn helps to neaten up the form of (2.4.1.2) and allow $A_{\mu}$ to converge when integrating over all space. Due to this, we can write

$$A_{\mu}^{self}(\lambda, t, x^i) = \tilde{A}_\mu\left(\lambda, t, \frac{x^i - z^i(t)}{\lambda}\right). \quad (2.39)$$

We can then apply this form of the self-potential to the definition of the self-field strength tensor.

$$F_{\mu\nu} = \partial_\mu A_\nu(\lambda, t, x^i) - \partial_\nu A_\mu(\lambda, t, x^i)$$
$$= \partial_\mu \tilde{A}_\nu\left(\lambda, t, \frac{x^i - z^i(t)}{\lambda}\right) - \partial_\nu \tilde{A}_\mu\left(\lambda, t, \frac{x^i - z^i(t)}{\lambda}\right). \quad (2.40)$$

With this form, due to the nature of taking the derivative of the potential, it becomes clear to see that the self-field can be scaled as

$$F_{\mu\nu}^{self} = \frac{1}{\lambda} F_{\mu\nu}^{\tilde{self}}\left(\lambda, t, \frac{x^i - z^i(t)}{\lambda}\right). \quad (2.41)$$

We now look to the fall-off behavior for the self-field. First, we need to identify two parameters as

$$\alpha \equiv \left| x^i - z^i(t) \right|, \quad (2.42a)$$
$$\beta \equiv \frac{\lambda}{\alpha}. \quad (2.42b)$$

From this, it is clear that the product of $\alpha$ and $\beta$ results in just the scaling parameters, $\lambda$. Using our scaling function for $J^\mu$, the self potential can be written as

$$A_{\mu}^{self}(\lambda, t, x^i) = \int d^3x \frac{1}{\lambda^2} \tilde{J}_\mu\left(\lambda, t - \left| x^i - x'^i \right|, \frac{x^i - z^i(t)}{\lambda}, \frac{x'^i - z'^i(t)}{\lambda}\right). \quad (2.43)$$
Before continuing, and seeing how any of this helps us, we need to define one more parameter.

\[
\begin{align*}
    w^i &= x^{i,i} - z^i(t - |x^j - x'^j|) \\
        &= x^{i,i} - z^i + |x^j - x'^j| V^i(t, |x^j - x'^j|) \\
        &= \lambda \bar{w}^i \\
        &= \alpha \beta \bar{w}^i.
\end{align*}
\] (2.44)

Note that the second line comes from the use of (2.38). With this definition of the new parameter, we can clean up the form of (2.43) to get

\[
A^\text{self}_\mu = \int d^3x \frac{1}{\lambda^2} \tilde{J}_\mu(\lambda, t - |x^i - x'^i|, \bar{w}^i) |x^i - x'^i|.
\] (2.45)

Before moving to the next part of the derivation, a word of warning: while what we are doing may seem convoluted an unnecessary, the relevance will show itself by the end.

We now have the ability to define the quantity \(x^i - x'^i\) in terms of \(\beta, V^i, \bar{w}^i\), and a parameter that we define as \(n^i = \frac{x^i - z^i(t)}{\alpha}\). Additionally, in a similar manner to the definition of \(\bar{y}^i\), we define \(y \equiv |x^i - x'^i|\). With these two definitions, we have

\[
x^i - x'^i = \alpha(n^i - \beta \bar{w}^i) + y V^i(t, y).
\]

By taking the absolute square of this, we will get a quadratic in \(y\), for which we find the solution

\[
y = \frac{\alpha |n^i - \beta \bar{w}^i|}{1 - |V^j|^2} \left( \frac{V^i(n^i - \beta \bar{w}^i)}{|n^j - \beta \bar{w}^j|} + \sqrt{1 - |V^k|^2 + \left( \frac{V^k(n^k - \beta \bar{w}^k)}{|n^j - \beta \bar{w}^j|} \right)^2} \right). \] (2.46)
This is a serious mess. However, as we see in (2.38), $V^i$ depends on $y$, which means that we will have factors of $y$ on both sides of this equation above. Thanks to this, there must be a function satisfied by our remaining parameters $(t, \alpha, \beta, n^i, \bar{w}^i)$, such that we can write $y$ in the form of

$$y = \alpha L(t, \alpha, n^i, \beta \bar{w}^i), \quad (2.47)$$

where $L$ is a smooth, positive function of its arguments. With this form of $y$ dependent upon the function $L$, we can rewrite $\bar{w}$ in terms of this function, and so $x'^i$ also has a dependence on this arbitrary, smooth function.

$$x'^i = \alpha \beta \bar{w}^i + z^i(t - \alpha L). \quad (2.48)$$

What have we just gone through, and why? Well, what we have done is find a possible coordinate transformation for $x'^i$, which is going to help us simplify the integral for the self-potential, $A^\text{self}_\mu$. The reason for doing this is that we are looking for a convenient form for the field strength tensor, $F_{\mu\nu}$. This procedure, while long, does indeed get us there.

Now, from (2.48), we can get the Jacobian, $J_{ij} = \frac{\partial f_i}{\partial x_j}$ as

$$\left| \frac{\partial x'^i}{\partial \bar{w}^j} \right| = (\alpha \beta)^3 W(t, \alpha, \beta, n^i, \bar{w}^i). \quad (2.49)$$

Recall that this factor of $\alpha \beta$ is the same as $\lambda$, which we can plug back into (2.45) along with the change of coordinates to find the result of

$$A^\text{self}_\mu = \beta \int d^3 \bar{w} \tilde{J}_\mu(\alpha \beta, t - \alpha L, \bar{w}^k) \frac{W(t, \alpha, \beta, n^i, \bar{w}^h)}{L(t, \alpha, n^i, \beta \bar{w}^j)} W(t, \alpha, \beta, n^i, \bar{w}^h). \quad (2.50)$$
Using this same argument to find $\mathcal{J}$ (recall that we had defined it in the previous section), since the integral is smooth in its arguments of $\alpha$, $\beta$, $t$, and $n^i$, there exists another smooth function $\mathcal{A}$ that is dependent upon these arguments and we can write

$$A^{\text{self}}_{\mu} = \beta \mathcal{A}(t, \alpha, \beta, n^i), \quad (2.51)$$

This form of the potential allows us to write the field strength tensor in the form

$$F^{\text{self}}_{\mu\nu} = \frac{\beta}{\alpha} F_{\mu\nu}(t, \alpha, \beta, n^i),$$

which leads us to find

$$\lambda F^{\text{self}}_{\mu\nu} = \beta^2 F_{\mu\nu}(t, \alpha, \beta, n^i). \quad (2.52)$$

This form, where $F^{\text{self}}_{\mu\nu}$ is proportional to $\beta^2$, is what we will use as our convenient form mentioned earlier. This is will be important in the following discussion with regards to the field tensors. Note that this whole section was done such that it is true in all coordinate systems and metrics, not just the flat metric on which we are focusing. Now that we have followed the derivation of find $F_{\mu\nu}$ proportional to this parameter of $\beta^2$, we can move back to the discussion about the far-zone limit.

### 2.4.1.3 Return to Far-Zone Limit: Field Tensors And Resulting Motion Of Particle

We return to the discussion of the far-zone limit by reminding ourselves that the total electromagnetic field strength tensor is a sum of the external and self fields,

$$F_{\mu\nu} = F_{\mu\nu}^{\text{ext}} + F_{\mu\nu}^{\text{self}}. \quad (2.53)$$
To begin this discussion, we recognize that the external fields are already assumed to be a homogeneous solution for Maxwell’s equations. Due to this assumption, it is implied that for any field of this form has a defined perturbation expansion in $\lambda$.

Moving onto the self-field, we recall that from the previous discussion of Appendix A from the paper, we redefine the two parameters $\alpha$ and $\beta$ as

$$\alpha = r,$$
$$\beta = \frac{\lambda}{r}. $$

We can now look to the form of (2.52) and with these redefined parameters, we get that the self-field can be written as

$$F_{\text{self}}^{\mu\nu} = \lambda \frac{r}{2} \mathcal{F}_{\mu\nu}(t, r, \frac{\lambda}{r}, \theta, \phi).$$

(2.54)

This is smooth near zero for $r$ and $\lambda r$, so can make use of a Taylor expansion to any finite order $N, M$, where we will expand for $r$ to order $N$ and $\frac{\lambda}{r}$ to order $M$.

$$F_{\text{self}}^{\mu\nu} = \lambda \frac{r}{2} \sum_{n=0}^{N} \sum_{m=0}^{M} r^{n} \left( \frac{\lambda}{r} \right)^{m} (\mathcal{F}_{\mu\nu})_{nm}(t, \theta, \phi) + \mathcal{O}(r^{N+1}) + \mathcal{O}\left( \left[ \frac{\lambda}{r} \right]^{M+1} \right).$$

Pulling the factors of $\lambda$ and $r^2$ into the sums (while simultaneously dropping the higher order terms) gives us

$$F_{\text{self}}^{\mu\nu} = \sum_{n=0}^{N} \sum_{m=0}^{M} (\lambda^{m+1}) \left( r^{n-m-2} \right) (\mathcal{F}_{\mu\nu})_{nm},$$

(2.55)

which is the general form of the far-zone expansion near $r = 0$. Since the self-field is the retarded solution to the fields with the associated charged current distribution,
it is clear the $F^{\text{self}(n)}_{\mu\nu}$ is the solution to the $n$th order perturbation for $J^{\mu}$. In terms of the Maxwell’s equations, we can write this as

$$\nabla^\nu F^{(n)}_{\mu\nu} = \mu_0 J^{(n)}_{\mu}.$$  \hfill (2.56)

We have seen previously that in this far-zone limit that the 0th order perturbation to $J^{\mu}$ is zero, as seen in (2.28), which means that the 0th order perturbation of the self-field is also zero,

$$F^{\text{self}(0)}_{\mu\nu} = 0.$$  \hfill (2.57)

Additionally, since $J^{(1)}$ is the same description as what is expected for a point-like charged particle, $F^{\text{self}}_{\mu\nu}$ results in the fields associated with the standard Lienard-Weichart potentials.

Now that we have zeroth-order and first-order solutions for $J^{\mu}$ and the electromagnetic field strength tensor, we must concern ourselves with the matter stress-energy tensor. However, rather than going through an entire derivation, as we had with the charged current density, we see that the matter stress-energy tensor is scaled exactly the same way as $J^{\mu}$, as seen in (2.23). It therefore follows the exact same procedure as what we did for $J^{\mu}$, and it becomes immediately clear that

$$T^{\text{M},(0)}_{\mu\nu} = \lim_{\lambda \to 0} T^{\text{M}}_{\mu\nu} = 0,$$  \hfill (2.58a)

$$T^{\text{M},(1)}_{\mu\nu} = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} T^{\text{M}}_{\mu\nu} = T^{\text{M}}_{\mu\nu}(t) \delta(x^{i} - z^{i}(t)) \frac{d\tau}{dt}.$$  \hfill (2.58b)

Again, to first order, the mass and the matter of the particle has the same description as a point particle in the far-zone limit.

Now that we have seen what the particle itself will look like to an observer at a constant radius from the worldline as the particle shrinks down, we must now look
to the total stress-energy tensor in order to determine what the trajectory of the
particle will be. To do this, we start by discussing the electromagnetic stress-energy
tensor. We again remind ourselves of the definition of this tensor as

$$T_{\mu\nu}^{EM} = \frac{1}{4\pi} \left( F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right). \quad (2.59)$$

We can see that when we recall that the total electromagnetic field strength tensor
is a sum of the external and self fields, (2.53), and so we can see that we should get
an energy-stress tensor of the form

$$T_{\mu\nu}^{EM} = T_{\mu\nu}^{ext} + T_{\mu\nu}^{self} + T_{\mu\nu}^{cross}.$$

We first wish to explicitly show that $T_{\mu\nu}^{EM}$ is indeed of this form. We apply the form
of the total field strength tensor and move the factor of $4\pi$ to the left-hand side.

$$4\pi T_{\mu\nu}^{EM} = \left( F_{\mu\alpha} F_{\nu}^{ext,\alpha} + F_{\mu\alpha} F_{\nu}^{self,\alpha} \right) - \frac{1}{4} g_{\mu\nu} \left( F_{\alpha\beta} F_{\mu\alpha} F_{\nu}^{ext,\alpha} + F_{\alpha\beta} F_{\mu\alpha} F_{\nu}^{self,\alpha} \right)$$

$$= F_{\mu\alpha} F_{\nu}^{ext,\alpha} + F_{\mu\alpha} F_{\nu}^{self,\alpha} + F_{\mu\alpha} F_{\nu}^{ext,\alpha} + F_{\mu\alpha} F_{\nu}^{self,\alpha}$$

$$- \frac{1}{4} g_{\mu\nu} \left( F_{\alpha\beta} F_{\mu\alpha} F_{\nu}^{ext,\alpha} + F_{\alpha\beta} F_{\mu\alpha} F_{\nu}^{self,\alpha} \right) - \frac{1}{4} g_{\mu\nu} \left( F_{\alpha\beta} F_{\nu}^{self,\alpha} + F_{\alpha\beta} F_{\nu}^{ext,\alpha} \right).$$

We then rearrange the terms so that we get just the external field strength tensor
components, just the self-field components, and then the resulting cross terms.

$$4\pi T_{\mu\nu}^{EM} = \left( F_{\mu\alpha} F_{\nu}^{ext,\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F_{\nu}^{ext,\alpha} \right) + \left( F_{\mu\alpha} F_{\nu}^{self,\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F_{\nu}^{self,\alpha} \right)$$

$$+ \left[ F_{\mu\alpha} F_{\nu}^{ext,\alpha} + F_{\mu\alpha} F_{\nu}^{self,\alpha} - \frac{1}{4} g_{\mu\nu} \left( F_{\alpha\beta} F_{\nu}^{self,\alpha} + F_{\alpha\beta} F_{\nu}^{ext,\alpha} \right) \right]. \quad (2.60)$$
The first term is exactly that of the stress tensors of the external and self fields, and two cross terms. Combining those two cross terms into a single stress tensor, we have guaranteed that $T_{\mu\nu}^{\text{EM}}$ is the combination of the three stress tensors.

Now, one could expect that since $F_{\mu\nu}^{\text{self},(0)} = 0$, the lowest order contribution by substituting $\lambda F_{\mu\nu}^{\text{self},(1)}$ into $T_{\mu\nu}^{\text{EM}}$. However, by going about this method, we would obtain a scaling of $\lambda^2$, rather than $\lambda$, and the first order contribution would vanish. The solution, then, is to show that $T_{\mu\nu}^{\text{self}}$ has a non-zero distributional limit in a similar manner to that which we have used for $J_{\mu}$ and $T_{\mu\nu}^{\text{M}}$. This is really where the discussion on the scaling of the self-field becomes useful. Earlier, we defined $\alpha$ and $\beta$ in terms of $r$, and we have

$$\alpha = r = \sqrt{\sum (x^i - z^i(t))},$$

$$\beta = \frac{\lambda}{\alpha},$$

$$\lambda F_{\mu\nu}^{\text{self}} = \beta^2 F_{\mu\nu}^{\text{self}}.$$ 

Now, since we are scaling $F_{\mu\nu}^{\text{self}}$ as $\frac{1}{\lambda} \tilde{F}_{\mu\nu}$, this means the field tensor $\tilde{F}_{\mu\nu}$ can be written as

$$\tilde{F}_{\mu\nu}^{\text{self}} = \beta^2 F_{\mu\nu}.$$ 

This means that the self-field, in terms of this $F_{\mu\nu}$ can be written as

$$F_{\mu\nu}^{\text{self}} = \frac{1}{\lambda} \beta^2 F_{\mu\nu}.$$ 

(2.62)
which we will use to find the scaling of the stress tensor for the self-field. For the following calculation, we will be dropping the “self” label, out of convenience and with the understanding that this is only for the self-field.

\[
T_{\mu\nu} = \frac{1}{4\pi} \left( F_{\mu\alpha} F_{\nu}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \\
= \frac{1}{4\pi} \left( \frac{1}{\lambda^2} \beta^2 F_{\mu\alpha}^\beta F_{\nu}^\alpha - \frac{1}{4} g_{\mu\nu} \frac{1}{\lambda^2} \beta^2 F_{\alpha\beta}^\alpha F^{\alpha\beta} \right) \\
= \frac{1}{\lambda^2} \frac{1}{4\pi} \left( \beta^2 F_{\mu\alpha}^\beta F_{\nu}^\alpha - \frac{1}{4} \beta^2 F_{\alpha\beta}^\alpha F^{\alpha\beta} \right) \quad (2.63) \\
= \frac{1}{\lambda^2} \frac{1}{4\pi} \left( \tilde{F}_{\mu\alpha} \tilde{F}_{\nu}^\alpha - \frac{1}{4} g_{\mu\nu} \tilde{F}_{\alpha\beta} \tilde{F}^{\alpha\beta} \right) \\
= \frac{1}{\lambda^2} \tilde{T}_{\mu\nu}.
\]

So, as we see, the self-field stress tensor is actually scaled in exactly the same way as \( J^\mu \) and \( T_{\mu\nu}^M \). It is clear, then, that we can get the zeroth-order and first-order perturbative contributions of the self-field are

\[
T_{\mu\nu}^{\text{self},(0)} \equiv \lim_{\lambda \to 0} T_{\mu\nu}^{\text{self}} = 0, \quad (2.64a) \\
T_{\mu\nu}^{\text{self},(1)} \equiv \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} T_{\mu\nu}^{\text{self}} = T_{\mu\nu}^{\text{self}} \delta(x^i - z^i(t)). \quad (2.64b)
\]

Now that we have the proper scaling for the self-field stress tensor, we can define

\[
T_{\mu\nu} = T_{\mu\nu}^M + T_{\mu\nu}^{\text{self}}.
\]
and so from the conservation of total stress energy, we have

\[ 0 = \nabla^\nu T^{\text{tot}}_{\mu \nu} \]

\[ = \nabla^\nu (T^M_{\mu \nu} + T^E_{\mu \nu}) \]

\[ = \nabla^\nu (T^M_{\mu \nu} + T^{\text{self}}_{\mu \nu} + T^{\text{ext}}_{\mu \nu} + T^{\text{cross}}_{\mu \nu}) \]

\[ = \nabla^\nu T^M_{\mu \nu} + \nabla^\nu T^{\text{ext}}_{\mu \nu} + \nabla^\nu T^{\text{cross}}_{\mu \nu}. \]

As we saw in the discussion of the 4/3 problem, for a source-free region, the divergence of the electromagnetic is zero, \((2.15)\), making that term go away. This means that the divergence of the tensor we defined as \(T_{\mu \nu}\) is

\[ \nabla^\nu T^M_{\mu \nu} = \nabla^\nu T^{\text{cross}}_{\mu \nu} = F^{\text{ext}}_{\mu \nu} J^\nu. \quad (2.65) \]

This second equality comes from making use of the definition of the cross tensor in \((2.60)\) and Maxwell’s equations.

As we can see above, the divergence of this \(T_{\mu \nu}\) is proportional to the charged current density, and so we can again go to the distributional view. The exact same process then produces the result

\[ T^{(1)}_{\mu \nu}(t) = m u_\mu u_\nu \delta(x^i - z^i(t)) \frac{d\tau}{dt}, \quad (2.66) \]

where \(m\) can be interpreted as the mass and is given by

\[ m = \int \left( T^M_{00}(\lambda = 0, t, \bar{x}^i) + T^{\text{self}}_{00}(\lambda = 0, t, \bar{x}^i) \right) d^3 \bar{x}. \quad (2.67) \]
What is interesting about this result is that even in the far-zone limit, the self stress-energy has a contribution to the mass of the body that is finite. We no longer have to worry about the negatively infinite mass renormalizations of old!

In addition to this result, when applying this same procedure to the divergence of $T_{\mu\nu}$, we also find that $u_\mu$ must obey the relationship that

$$ma_\mu = mu^\nu \nabla_\nu u_\mu = qu^\nu F^\text{ext}_{\mu\nu} \left( \lambda = 0, t, z^i(t) \right).$$

(2.68)

What we see, then, is that this is exactly the form of the Lorentz force law, which is a result of the approach and assumptions.

Before we move onto the near-zone limit, let’s just run through a short summary of what the paper had unlocked in this section. First, the far-zone limit corresponds to an observer at constant radius, $r = \sqrt{x^i - z^i(t)}$, who is looking in towards the shrinking body as $\lambda$ goes to zero. In this limit, we have seen that the charge and the mass of the body shrinks down to a Dirac delta, reflecting the form of a point-like charged particle, just as had been done in the past. However, unlike the previous attempts at this problem, we have seen that the first order correction of the self-field contributes a non-zero, while also non-infinite, correction to the mass of the particle. While this self-field results in a correction to the mass, the body itself will still follow along the path governed by the Lorentz force law due to the external electromagnetic field.
2.4.2 Near-Zone Limit

2.4.2.1 Properties Of The Near-Zone Limit

As a big-picture reminder, we desire to find first order (in $\lambda$) corrections to the motion of the particle. As we take the limit of $\lambda$ going to zero, the body shrinks down to a worldline, so that the lowest order description of the motion is determined by that worldline.

When $\lambda > 0$, the body is of finite size, and so we need to have a representative worldline to describe the motion of that particle. This means that it makes sense to define the worldline based on the center of mass. However, this poses a problem for us, since we saw that the mass of the body has a contribution by the electromagnetic self-energy, and so we would need to take this self-energy into account when defining our center of mass. But, when $\lambda > 0$, and so the body is of finite size, if we define $T_{\mu\nu}^{\text{self}}$ based on the retarded solution for $F_{\mu\nu}^{\text{self}}$, we would need to consider everything that has happened before, long into the distant past. It is clear that this can lead to serious complications when trying to solve for the motion of the particle.

The solution to this problem that we encounter is by considering the body in the near-zone limit, rather than the far-zone, which allows us to write a definition for the center of mass of charged bodies to include the self-energy, while still excluding radiation that had been emitted in the distant past. In order to do this, everything
needs to be properly rescaled in order to have a well-defined limit at a point along the worldline.

\[
\bar{g}_{ab} = \frac{1}{\lambda^2} g_{ab}, \quad (2.69a)
\]
\[
\bar{J}^a = \lambda^3 J^a, \quad (2.69b)
\]
\[
\bar{T}_{ab}^M \equiv T_{ab}^M, \quad (2.69c)
\]
\[
\bar{F}_{ab} = \frac{1}{\lambda} F_{ab} \quad (2.69d)
\]

This rescaling of our quantities allows us to write the Maxwell equations and the matter equations as

\[
\nabla^b \bar{F}_{ab} = 4\pi \bar{J}^a,
\]
\[
\nabla_{(a} \bar{F}_{b)\mu} = 0,
\]
\[
\nabla_a \bar{J}^a = 0,
\]
\[
\nabla^b (\bar{T}_{ab}^M + \bar{T}_{ab}^{EM}) = 0.
\]

Additionally, the time and space coordinates are rescaled so that we get

\[
\bar{t} \equiv \frac{t - t_0}{\lambda},
\]
\[
\bar{x}^i = \frac{x^i - z^i(t)}{\lambda}, \quad (2.70)
\]

so that as \( \lambda \to 0 \), an observer at some fixed \( \bar{x}^\mu(\bar{t}, \bar{x}^i) \) will approach a point on the worldline, \((t_0, z^i(t_0))\) at the same rate as the body will. In other words, the spacetime intervals between the particle and the observer will stay finite due to the
observer using the rescaled metric to measure distances. For convenience and ease, we can define
\[ z_i(t_0) = 0. \quad (2.71) \]
When looking at these rescaled quantities, we can see that the barred quantities are essentially the same as the quantities with tildes on them in the far-zone limit and we can write
\[
\begin{align*}
\bar{J}^\mu(\lambda, t_0; \bar{t}, \bar{x}^i) &= \tilde{J}^\mu(\lambda, t_0 + \lambda \bar{t}, \bar{x}^i), \\
\bar{T}_{\mu\nu}^M(\lambda, t_0; \bar{t}, \bar{x}^i) &= \tilde{T}_{\mu\nu}^M(\lambda, t_0 + \lambda \bar{t}, \bar{x}^i), \\
\bar{F}_{\mu\nu}(\lambda, t_0; \bar{t}, \bar{x}^i) &= \tilde{F}_{\mu\nu}(\lambda, t_0 + \lambda \bar{t}, \bar{x}^i) + \lambda F_{\mu\nu}^{\text{ext}}(\lambda, t_0 + \lambda \bar{t}, \lambda \bar{x}^i).
\end{align*}
\]
If we look to second term of this third equation, we see that as \( \lambda \) goes to zero, all effects of the external universe (which includes the body’s own past) reduces in importance and eventually completely goes to zero at the worldline.

Now, for any generic barred quantity, \( \bar{f} \), we can take any order of perturbation such that we have
\[ \bar{f}^{(n)}(\bar{x}^\mu) = \frac{1}{n!} \lim_{\lambda \to 0} \frac{\partial^n}{\partial \lambda^n} \bar{f}(\lambda, \bar{x}^\mu), \quad (2.72) \]
and since \( \bar{x}^\mu \) depends on \( t_0 \) and \( z^i(t_0) = 0 \), each of these quantities in the near-zone limit depends on \( t_0 \) and \( \bar{t} \) as \( \bar{f}(\lambda, t_0 + \lambda \bar{t}, \bar{x}^i) \). It is also important to note that we require each \( \bar{f} \) to be a smooth function of this dependence in time. One important consequence of this smooth dependence is that any \( n \)th order perturbation of \( \bar{f}^{(n)} \) can be at most an \( n \)th order polynomial in \( \bar{t} \),
\[ \bar{f}^{(n)}(\bar{t}, \bar{x}^i) = \sum_{m=0}^{n} A_m^{(n)}(\bar{x}^i) \bar{t}^m. \quad (2.73) \]
If we look to the zeroth order, then, we can see that the power of $\bar{t}$ is zero, and therefore all zeroth order near-zone limit quantities are stationary, or independent of time. We also impose something called the “consistency condition,” where each near-zone limit perturbation series at each time $t_0$ are defined with reference to the single one-parameter family in the far zone,

$$\frac{\partial}{\partial \bar{t}} \bar{f}^{(n+1)} = \frac{\partial}{\partial t_0} \bar{f}^{(n)}.$$  

(2.74)

This ensures for us that near-zone quantities at different scaling times, $t_0$, cannot be specified independently.

If we look to the electromagnetic field in the near-zone limit now, we have already seen the scaling for the external field as

$$\bar{F}^{\text{ext}}_{\mu\nu} (\lambda, t_0; \bar{t}, \bar{x}^i) \equiv \lambda F_{\mu\nu}^{\text{ext}} (\lambda, t_0 + \lambda \bar{t}, \lambda \bar{x}^i).$$  

(2.75)

Here, $F_{\mu\nu}$ is a smooth function of its arguments as usual, and the $n$th order perturbative expansion of $\bar{F}_{\mu\nu}$ must depend on $\bar{x}^i$ polynomially and to order of $n - 1$. We can then see that when relating the near-zone quantities to the far-zone quantities, we have

$$\bar{F}^{\text{ext},(0)}_{\mu\nu} (\lambda, t_0; \bar{t}, \bar{x}^i) = 0,$$  

(2.76a)

$$\bar{F}^{\text{ext},(1)}_{\mu\nu} (\lambda, t_0; \bar{t}, \bar{x}^i) = F_{\mu\nu}^{\text{ext},(0)} \bigg|_{t=t_0, x^i=0},$$  

(2.76b)

$$\bar{F}^{\text{ext},(2)}_{\mu\nu} (\lambda, t_0; \bar{t}, \bar{x}^i) = F_{\mu\nu}^{\text{ext},(1)} \bigg|_{t=t_0, x^i=0} + \bar{x}^i \partial_i F_{\mu\nu}^{\text{ext},(0)} \bigg|_{t=t_0, x^i=0} + \bar{t} \partial_0 F_{\mu\nu}^{\text{ext},(0)} \bigg|_{t=t_0, x^i=0}.$$  

(2.76c)
The first thing to notice is that the zeroth order of the near-zone external fields is zero, which corresponds to the effects of the universe vanishing. Meanwhile, we also notice that the first order near-zone external field is a constant and given by the zeroth order far-zone limit external fields evaluated on the worldline at time $t_0$.

We have now covered the external fields, so it is now time to discuss the self field. Recall that in the far-zone limit we discussed the Taylor series expansion of the self field which resulted in (2.55). In a similar manner to that process, we change the coordinates such that we have $r = \lambda \bar{r}$ and $t = t_0 + \lambda \bar{t}$ to get

$$
F_{\mu\nu}^{\text{self}}(\bar{t}, \bar{r}, \theta, \varphi) = \sum_{n=0}^{N} \sum_{m=0}^{M} (\lambda^n)(\bar{r}^{n-m-2})(F_{\mu\nu})_{nm}(t + \lambda \bar{t}, \theta, \varphi) \\
= \sum_{n=0}^{N} \sum_{m=0}^{M} \sum_{p=0}^{P} (\lambda^{n+p})(\bar{r}^{n-m-2})(F_{\mu\nu})_{nmp}(t_0),
$$

and we can write $(F_{\mu\nu})_{nmp}$ as the $p$th perturbation on $(F_{\mu\nu})_{nm}$,

$$(F_{\mu\nu})_{nmp} \equiv \frac{1}{p!} \frac{\partial^p}{\partial t^p} (F_{\mu\nu})_{nm} \bigg|_{t=t_0}.$$
We can then get the zeroth order, first order, and second order pertubative expansions for the self field by using (2.77) and comparing to the far-zone limit’s own expansion to get

\[
\bar{F}_{0}^{\text{self},(0)} = \frac{q}{r^2} n_i + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{2.78a}
\]

\[
\bar{F}_{ij}^{\text{self},(0)} = \mathcal{O}\left(\frac{1}{r^3}\right) \tag{2.78b}
\]

\[
\bar{F}_{0}^{\text{self},(1)} = \frac{q}{\bar{r}} \left(-\frac{1}{2} a_j n^j n_i - \frac{1}{2} a_i\right) + \mathcal{O}\left(\frac{1}{\bar{r}^2}\right) \tag{2.78c}
\]

\[
\bar{F}_{ij}^{\text{self},(1)} = \mathcal{O}\left(\frac{1}{\bar{r}^2}\right) \tag{2.78d}
\]

\[
\bar{F}_{0}^{\text{self},(2)} = q \left(\frac{3}{8} (a_j n^j)^2 n_i + \frac{3}{4} (a_j n^j) a_i + \frac{1}{2} a_j a^j n_i + \frac{1}{2} \dot{a}_0 n_i + \frac{2}{3} \dot{a}_i\right) + \mathcal{O}\left(\frac{1}{\bar{r}}\right) \tag{2.78e}
\]

\[
\bar{F}_{ij}^{\text{self},(2)} = -\frac{1}{4} q (\dot{a}_i n_j - \ddot{a}_j n_i) + \mathcal{O}\left(\frac{1}{\bar{r}}\right) \tag{2.78f}
\]

where we have used a shorthand of \( n_i = \frac{x_i}{r} \) and \( a_i \) is the acceleration, and each of these perturbative expansions are evaluated at \( \bar{t} = 0 \). We have now discussed the essentials of the properties of the near-zone and are now free to move onto center of mass in the near-zone and the parameters of the body.

### 2.4.2.2 Center Of Mass And Body Parameters

We need to define a center of mass along the worldline of a body at some finite \( \lambda \), as we pointed out in the previous section. In order to define the first order correction to Lorentz force motion, we only need to define a notion of the center of mass to the zeroth order in the near-zone limit, and so we again emphasize that we are working
with Fermi normal coordinates about the worldline which corresponds to working in the rest frame of the body. Our scaled metric, in this frame then, becomes

\[
\bar{g}_{00} = -1 - 2\lambda a_i(t_0)\bar{x}^i - \lambda^2 \left[ 2\dot{a}_i\dot{\bar{x}}^i + (a_i\bar{x}^i)^2 + 2(\delta a_i)\bar{x}^i \right] + \mathcal{O}(\lambda^3),
\]

\[
\bar{g}_{i0} = \mathcal{O}(\lambda^3),
\]

\[
\bar{g}_{ij} = \delta_{ij},
\]

so the spatial components of the metric stay the same, and the zeroth diagonal component is truly that which becomes more complicated. However, just mentioned that we need concern ourselves with only the zeroth order in the near-zone in order to find the first order correction to Lorentz force motion, and so our metric stays flat. If we were to go to higher order corrections, we would then need to concern ourselves with the terms that contain factors of \(\lambda\). It is also useful to note that for these higher orders that \(a^i \equiv a^i(\lambda = 0, t)\) is the far-zone acceleration of the unperturbed worldline, while \(\delta a^i \equiv \partial_\lambda a^i\) is the far-zone perturbed acceleration of the worldline to first order in \(\lambda\).

Now, in the same manner as we did with the far-zone, we can define a new stress tensor,

\[
\bar{T}_{ab} \equiv \bar{T}^M_{ab} + \bar{T}^{\text{self}}_{ab}.
\]

(2.80)

As we just saw, to zeroth order, the metric becomes flat, or Minkowskian, and the total body stress-energy is zero. So, the total mass at zeroth order should be

\[
m(t_0) = \int \bar{T}_{00} d^3\bar{x}.
\]

(2.81)

Notice that this mass is still including the electromagnetic self-energy as we had discovered in the far-zone limit. Additionally, \(\bar{F}_{\mu\nu}^{\text{self}}\) is independent of the time, as
we have seen, and falls of at large $\bar{r}$ as $\frac{1}{\bar{r}^2}$, which means that the stress-energy tensor $\bar{T}^{\text{self}}_{\mu\nu}$ falls off as $\frac{1}{\bar{r}^4}$, and so this integral converges. This is good because we can immediately see from this fact that we do not need to concern ourselves with negatively infinite mass renormalizations.

Armed with this definition of the mass, we are free to define the center of mass in much the same manner as we do in classical mechanics

$$\bar{X}_i^{\text{CM}} = \frac{1}{m} \int \bar{T}^{(0)}_{00} \bar{x}^i \, d^3\bar{x},$$

(2.82)

Notice that since $\bar{T}^{(0)}_{00}$ is proportional to $\bar{r}^{-4}$, then $\bar{X}_i^{\text{CM}}$ is proportional to and falls off as $\bar{r}^{-3}$, which means that it does not converge absolutely. However, the angular average of $\bar{r}^{-3}$ part of the integrand vanishes, which makes this definition for the center of mass well-defined as a limit $\bar{R} \to \infty$ for the integral over a sphere of radius $\bar{R}$.

Now that we have a well-defined center of mass, we see that it depends on $\bar{x}^i$, which means that it can change under a change of origin of $\bar{x}^i$, which corresponds to a change in first order in $\lambda$ of the far-zone limit origin of $x^i$. So, we can define the perturbed motion to first order in $\lambda$ in the far zone limit by the demand that the zeroth order near-zone limit center of mass must vanish.

$$\bar{X}_i^{\text{CM}}(t_0) = 0.$$  

(2.83)

An observer moving along the worldline $\gamma$ is free to assign their own position as the center of mass of the body at each time $t_0$! We have now eliminated the effects of the exterior universe and set a well-defined form for the center of mass of the
body, which helps to set the conditions for finding the corresponding equations of the motion.

Before setting out to find these equations of motion, we must first define some other body parameters, the first being the spin tensor. This spin tensor corresponds to the intrinsic spin that depends on the magnetic moment of the particle (which we will discuss shortly) and the angular momentum of the particle. The tensor itself is defined as

\[
S^{0j}(t_0) = -S^{j0} = \int \bar{T}^{(0),00} \bar{x}^j \, d^3\bar{x},
\]

\[
S^{ij}(t_0) = -S^{ji} = 2 \int \bar{T}^{(0),i} \bar{x}^j \, d^3\bar{x}.
\]

We will be able to confirm the anti-symmetric property seen above later. We also have that thanks to the center of mass condition in (2.83), we see that the integral in (2.84a) must go to zero, leaving just \(S^{ij}\) to be non-zero. The spin vector can be defined by using this non-zero contribution to the spin tensor as

\[
S_i = \frac{1}{2} \epsilon_{ijk} S^{jk},
\]

where \(\epsilon_{ijk}\) is the standard Levi-Cevita tensor, which we define as

\[
\epsilon_{ijk} = \begin{cases} 
+1 & (ijk, jki, kij), \\
-1 & (jik, ikj, kji), \\
0 & \text{otherwise}.
\end{cases}
\]

We will derive the exact components of the spin vector and spin tensor later.
We can now discuss a first order correction to the mass, which we start by mentioning a concept from the field of general relativity. Killing’s equation is given as

$$\nabla_{(\mu} K_{\nu)} = 0,$$

and if the metric is independent of some coordinate $x^{\sigma^*}$, then the vector $\partial_{\sigma^*}$ will satisfy Killing’s equation and it will always be possible to find a coordinate system in which $K = \partial_{\sigma^*}$. The Killing vector is the vector field that satisfies this equations. Using this idea of the Killing vector, and looking to our metric in (2.79), to first order in $\lambda$, the metric is no longer Minkowskian, but it is still independent of time, and so we have a Killing field of $\partial_{\bar{t}}$ and the stress-energy current $\bar{T}_{\mu\nu} \left( \frac{\partial}{\partial \bar{t}} \right)^\nu$ is conserved. We can then define the first order correction to the mass, $\delta m$ as

$$\delta m(t_0) = \int_{\Sigma} \frac{\partial}{\partial \lambda} \left( \bar{T}_{ab} \left( \frac{\partial}{\partial \bar{t}} \right)^b d\Sigma^a \right) \bigg|_{\lambda=0}. $$

Since we have our center of mass condition, the zeroth order drops out and our time drops out because of the Killing equation, and so we have

$$\delta m = \int_{t=0} T_{00}^{(1)} \bigg|_{t=0} d^3 \bar{x}. $$

(2.87)

From this and (2.78), we can compute the first order correction to $\bar{T}_{00}$. This computation gives us (again using the same definitions for $n^j$ and $a_j$)

$$\bar{T}_{00}^{(1)} \bigg|_{t=0} = -\frac{1}{4\pi} \frac{q^2}{r^3} a_j n^j + O \left( \frac{1}{r^4} \right).$$

(2.88)

This means that the integral in (2.87) is still well-defined as $R$ approaches $\infty$. 
In a similar manner to the mass, the total charge of the particle can be written as

\[ q(\lambda) = \int_{\Sigma} J^{\alpha} \, d\Sigma_{\alpha}, \]

which gives the zeroth order in the near-zone limit and the first order in the far-zone limit,

\[ q \equiv \int \bar{J}^{(0),0} \, d^3\bar{x}. \quad (2.89) \]

Additionally, we have the ability to use the charged current density to define the electromagnetic dipole tensor, which is of the same form as the spin tensor.

\[ Q^{\mu j}(t_0) \equiv \int \bar{J}^{(0)\mu} \bar{x}^j \, d^3\bar{x}. \quad (2.90) \]

Much like the spin tensor, this is an anti-symmetric tensor which means each component is a negative of itself under a permutation of the indexes, \( Q^{0j} = -Q^{j0} \). The time-space components are the electric dipole, while the purely spatial components give the magnetic dipole,

\[ p^i = Q^{0i}, \quad (2.91a) \]
\[ \mu_i = \frac{1}{2} \epsilon_{ijk} Q^{jk}. \quad (2.91b) \]

We can then use this form when finding the first order correction to the charge. Note that since this has a similar form to the mass, we can consider these two quantities as analogs of each other, and so in a manner similar to that when finding the first order correction to the mass, we get

\[ \delta q \equiv \int \bar{J}^{(1)0} \bigg|_{\bar{t}=0} \, d^3\bar{x} + a_i Q^{0i}. \quad (2.92) \]
This second term is not in the mass analog for this correction due to the center of mass condition. We now have our fundamental understanding of the near zone, its properties, and the parameters of the body in this limit. With the understanding that we have gained in these two sections, we can now go and summarize the derivation of the motion of the particle in this near-zone limit.

### 2.4.2.3 Derivation Of The Motion

We start the derivation of the motion of this particle by recalling that the divergence of the total stress-energy tensor is zero. When we write the total stress-energy tensor in terms of the matter and the electromagnetic components, we can then write the electromagnetic tensor in terms of the external and self fields, as well as a cross term. Then, taking the divergence of the sum of the matter and self-field components, we find a result that depends on the charged current density and the external field.

\[
0 = \bar{\nabla} b (T_{ab}^M + T_{ab}^{EM})
\]

\[
= \bar{\nabla} b (T_{ab}^M + T_{ab}^{self} + T_{ab}^{ext} + T_{ab}^{cross})
\]

\[
= \bar{\nabla} b (\bar{T}_{ab} + \bar{T}_{ab}^{ext} + \bar{T}_{ab}^{cross}).
\]

Notice that the divergence of the external electromagnetic fields still go away since we are still assuming that the source for those fields is far away and so we can assume we are in a source-free region. Again, we can go back to the definition of
$T_{ab}^{\text{cross}}$, having the same exact form as that in the far-zone limit, and we will find the same sort of relationship as we previously had:

$$\nabla^b T_{ab} = J^b F_{\mu ab}.$$

(2.94)

This relationship is extremely important for us to remember, as we will later find that the first order correction to Lorentz force motion is void of any mention of the self-field and it is a fair question to ask why. However, we can relate the self-field to the external field through this relationship here.

Recall that at zeroth order in $\lambda$, the metric $\bar{g}$ is flat and $\bar{F}_{\mu \nu}^{\text{ext}} = 0$, and so all zeroth order near-zone limit quantities are independent of time. We can then write the divergence in terms of just the spatial components,

$$\partial_j \bar{T}_{ij}^{(0)} = 0,$$

which leads to

$$\int \bar{T}_{i0}^{(0)} d^3 \bar{x} = 0,$$

(2.95a)

$$\int \bar{T}_{ij}^{(0)} d^3 \bar{x} = 0,$$

(2.95b)

where we have multiplied both sides by $\bar{x}^i$ and then integrated the time and spatial components of $T_{ij}$ separately. Putting in a second factor of $\bar{x}^k$ into this integral will
still yield the same result, but we also confirm the anti-symmetric property of the spin tensor.

\[
\int \bar{T}^{(0),i}_{0i} \bar{x}^k d^3 \bar{x} = - \int \bar{T}^{(0),k}_{0i} \bar{x}^i d^3 \bar{x} = 0,
\]

\[
\int \bar{T}^{(0)}_{ij} \bar{x}^k d^3 \bar{x} = 0.
\]

In a similar manner, we can use the conservation of charge to write

\[
\int \bar{J}^{(0),i} d^3 \bar{x} = 0,
\]  
(2.96a)

\[
\int \bar{J}^{(0),i} \bar{x}^j d^3 \bar{x} = - \int \bar{J}^{(0),j} \bar{x}^i d^3 \bar{x}.
\]  
(2.96b)

These equations justify the reason that both the dipole tensor and the spin tensor have anti-symmetric properties.

In order to discuss the evolution of mass, we start by going back to the divergence of the total stress-energy tensor in (2.94) and expand it by taking the definition of \( \bar{\nabla}^b \) to get

\[
\bar{\partial}^0 \bar{T}^{(1)}_{00} + \bar{\partial}^i \bar{T}^{(1)}_{0i} + a^i \bar{T}^{(0)}_{0i} - \bar{J}^{(0)} \mu F_{0\mu} = 0,
\]  
(2.97)

\[
\bar{\partial}^0 \bar{T}^{(1)}_{k0} + \bar{\partial}^i \bar{T}^{(1)}_{kj} + a_k \bar{T}^{(0)}_{00} + a^i \bar{T}^{(0)}_{kj} - \bar{J}^{(0)} \mu F_{k\mu} = 0.
\]  
(2.98)

By taking a spatial integral over (2.97), we will recover the time derivative of \( m \),

\[
\int \left[ \bar{\partial}^0 \bar{T}^{(0)}_{00} + \bar{\partial}^i \bar{T}^{(0)}_{0i} + a^i \bar{T}^{(0)}_{0i} - \bar{J}^{(0)} \mu F_{0\mu} \right] d^3 \bar{x} = - \frac{dm(t_0)}{dt_0}.
\]  
(2.99)

This comes about because the third term goes to zero from (2.95a), the second terms goes to zero by using an integration by parts, and the fourth term goes to zero from
the conservation of charge. Since we know that the first term is also zero from the center of mass condition, we find

\[-\frac{dm(t_0)}{dt_0} = 0. \tag{2.100}\]

So, to zeroth order, the mass does not evolve in time in the near-zone limit.

We can then follow a similar approach by integrating (2.98) over space after multiplying the whole thing by the coordinates \(\bar{x}^i\),

\[
\int \left[ \bar{\partial}^0 \bar{T}_{k0}^{(1)} \bar{x}^i + \bar{\partial}^j \bar{T}_{kj}^{(1)} \bar{x}^i + a_k \bar{T}_{00}^{(0)} \bar{x}^i + a^j \bar{T}_{kj}^{(0)} \bar{x}^i - \bar{J}^{(0)\mu} F_{k\mu}^\text{ext} \bar{x}^i \right] d^3 \bar{x} = 0.
\]

Here, the third term evaluates to zero due to the center of mass condition, while the fourth term goes to zero due to (2.95b). Meanwhile, from the consistency condition discussed earlier, the first term evaluates to one half of the time derivative of the spin tensor.

\[
\int \bar{\partial}^0 \bar{T}_{k0}^{(1)} \bar{x}^i d^3 \bar{x} = -\frac{1}{2} \frac{dS_{ik}(t_0)}{dt_0}. \tag{2.101}\]

Meanwhile, we can use the definition of the dipole tensor to rewrite the last term in the integral to find

\[
\frac{1}{2} \frac{d}{dt_0} S_{ik} = Q^\mu_{[i} F_{k\mu}^\text{ext} + \int \bar{T}_{ik}^{(1)} d^3 \bar{x}.
\tag{2.102}\]

It is at this point that we can split this into anti-symmetric and symmetric parts. The anti-symmetric part if this result gives

\[
\frac{d}{dt_0} S_{ij} = 2Q^\mu_{[i} F_{j\mu}^\text{ext}, \tag{2.103}\]

while the symmetric part gives

\[ \int \bar{T}^{(1)}_{ij} \, d^3 \bar{x} = Q^{\mu} (i F^\mu_{\bar{j}j}) \mu. \] (2.104)

For a reference in notation, the bracketed indexes indicate that we need to take a difference between the two quantities with permuted indexes, while the parentheses indicates an addition between the two quantities. [14]

\[ A[\alpha B\beta] = \frac{1}{2} (A_\alpha B_\beta - A_\beta B_\alpha), \] (2.105)

\[ A(\alpha B\beta) = \frac{1}{2} (A_\alpha B_\beta + A_\beta B_\alpha). \] (2.106)

We have now determined the lowest order evolutions of mass and spin, and we can now move onto the equation of motion itself.

We start by integrating (2.98) over space again, but this time we do not multiply each term by the coordinates. Instead, we have

\[ \int \left[ \bar{\partial}^{\bar{\mu}} \bar{T}^{(1)}_{\bar{k}0} + \bar{\partial}^{j} \bar{T}^{(1)}_{kj} + a_k \bar{T}^{(0)}_{00} + a^j \bar{T}^{(0)}_{kj} - \bar{J}^{(0)}_{\mu} F^\mu_{\bar{k}k} \right] \, d^3 \bar{x} = 0. \]

The first term vanishes from the consistency condition, the second term vanishes as it falls off by \( \frac{1}{\bar{r}^3} \) and the fourth term vanishes from (2.95b). This leaves us with just the third and fifth terms, which we can immediately see

\[ m a_i = q F^\text{ext}_{\bar{i}0}. \] (2.107)

Notice that we find this thanks to the definition of the mass of the body and the charge of the body as seen when we discussed the parameters of the body. This is the same form as Lorentz force motion when under the influence of an external
electric field. We have now derived the lowest order equation of motion for the particle in the near-zone limit. However, instead of leaving it here, we will go on to find the first order correction to each of the quantities that we have gone through.

For the first order corrections, we consider the conservation of charge again which takes the form
\[
\bar{\partial}_0 \bar{J}^{(1),0} + \bar{\partial}_j \bar{J}^{(1),j} + a^{(0)}_j \bar{J}^{(1),j} = 0. \tag{2.108}
\]
In the same manner as before, we can multiply this by $\bar{x}^i$ and integrate over space. The first term evaluates to the time derivative of $Q_i^0$, the third term evaluates to the spatial components $a_j Q^{ij}$, and we can write that the integral over the spatial components of $J^{(1)\mu}$ gives
\[
\int \bar{J}^{(1)i} \, d^3\bar{x} = \frac{d}{dt} Q_i^0 + a_j Q^{ji}. \tag{2.109}
\]
Note that the indexes are flipped due to the asymmetry of $Q^{\mu\nu}$, resulting in the right-hand side being positive.

Next, in order to find equations similar to the zeroth order (2.97) and (2.98), we look back to the metric, perturbed mass, and conservation of stress-energy in (2.79), (2.87), and (2.93), respectively. Again, we also note that the acceleration is $a^i = a^i(t) \equiv a^i(\lambda = 0, t)$ and evaluated on the worldline, and $\delta a^i$ is
\[
\delta a^i = \partial_\lambda a^i(\lambda = 0, t).
\]
It is then possible to see that conservation of stress-energy makes our two equations become

\[ 0 = \partial_0 T_{00}^{(2)} + a_i T_{0i}^{(1)} + \left( \delta a^i \right) T_{0i}^{(0)} + \dot{a}_i \bar{T}_{0i}^{(0)} + 2 a_i \bar{x}^i \partial_0 \bar{T}_{00}^{(1)} + 2 \dot{a}_i \bar{x}^i \bar{T}_{00}^{(0)} \]

\[ - a_i \bar{x}_i a^j \bar{T}_{0j}^{(0)} - \bar{J}^{(0),\mu} \left( \bar{x}_i \partial_\mu F_{00}^{\text{ext}} + F_{00}^{\text{ext},\mu} + \delta F_{00}^{\text{ext}} \right) - \bar{J}^{(1),\mu} F_{00}^{\text{ext}} , \]

\[ (2.110) \]

\[ 0 = \bar{\partial}_0 T_{00}^{(2)} + \bar{\partial}_i T_{0i}^{(2)} + a_k T_{00}^{(1)} + a_i T_{0i}^{(1)} + \delta a_k T_{00}^{(0)} + \delta a^i T_{0i}^{(0)} + \dot{a}_k T_{00}^{(0)} \]

\[ + a_i \bar{x}^i \bar{T}_{00}^{(1)} + 2 a_i \bar{x}^i \bar{T}_{00}^{(1)} + \dot{a}_i \bar{x}^i \bar{T}_{00}^{(0)} - a_i \bar{x}^i \bar{T}_{00}^{(0)} - 3 a_k a_i \bar{x}^i \bar{T}_{00}^{(0)} \]

\[ - \bar{J}^{(0),\mu} \left( x^i \partial_\mu F_{k\mu}^{\text{ext}} + F_{k\mu}^{\text{ext},\mu} + \delta F_{k\mu}^{\text{ext}} \right) - \bar{J}^{(1),\mu} F_{k\mu}^{\text{ext}} . \]

\[ (2.111) \]

Here, it is taken that \( \delta F_{\mu\nu}^{\text{ext}} \) is the first order external field from the far-zone limit evaluated at \( t_0 \) and \( x^i = 0 \). These two equations are quite daunting, but we will still be able to go through the same process as before.

To find the perturbed acceleration, \( \delta a^i \), we can evaluate (2.111) at \( \bar{t} = 0 \) and integrate over space. We will also need to apply some relationships that we have already seen: the consistency condition, the center of mass condition, the spatial integrals over \( (\bar{T}_{ij}^{(0)}) \) and \( (\bar{T}_{ij}^{(0)} \bar{x}^k) \) evaluate to zero, the definition of the spin tensor, the definition of the dipole tensor, and the definition of mass. While it may not be immediately obvious, the application of each of these to the spatial integral of (2.111) yields

\[ 0 = \frac{d}{dt_0} \int T_{00}^{(1)} \, d^3 \bar{x} + \int r^2 n^i T_{ki}^{(1)} \, d\Omega + a_k \delta m + a^i \int T_{ki}^{(1)} \, d^3 \bar{x} + \delta a_k m + a^i \frac{d}{dt_0} S_{ki} \]

\[ + \frac{1}{2} \partial_k S_{ki} - Q^{\mu i} \partial_\mu F_{k\mu}^{\text{ext}} - \delta F_{k\mu}^{\text{ext}} \int \bar{J}^{(0),\mu} \, d^3 \bar{x} - \int \bar{J}^{(1),\mu} F_{k\mu}^{\text{ext}} \, d^3 \bar{x} . \]

\[ (2.112) \]

This is already much more manageable. Note that the second term comes from integration by parts of \( \partial^i \bar{T}_{00}^{(2)} \, d^3 \bar{x} \), where we have seen before \( n^i = \frac{\bar{x}_i}{r} \). This has
potential to clean up nicely, but first we need to find $\bar{T}_{ij}^{(2)}$, we can find by making use of the definition of $\bar{T}_{\mu\nu} = \bar{T}_{\mu\nu}^M + \bar{T}_{\mu\nu}^{self}$. We also make use of (2.78), from which we recover

$$
\left(-\frac{4\pi r^2}{q^2}\right)\bar{T}_{ij}^{(2)} \bigg|_{\bar{t}=0} = n_in_j \left[ 6(a_kn^k)^2 + \frac{1}{4}a_k a^k + \dot{a}_0 \right] + 4a_kn^ka_{(i}n_{j)} + \frac{4}{3}\dot{a}_{(i}n_{j)}
+ \delta \left[ 4(a_kn_k)^2 + \frac{1}{4}a_k a^k + \frac{1}{2}a_0 + \frac{2}{3}\dot{a}_kn^k \right] + \frac{1}{4}a_i a_j
$$

(2.113)

We make use of the fact that the integral over a sphere of the product of an odd number of normal vectors (in this case $n^i$) vanishes. So, when integrating this as part of (2.112), we are only left with one term.

$$
\int \bar{r}^2 n^i \bar{T}_{ki}^{(2)} \, d\Omega = \frac{2}{3}q^2 \dot{a}^i.
$$

(2.114)

We can then put this back into (2.112) and we use our previous definitions that we have seen to evaluate all remaining terms, yielding

$$
m\delta a_i = -(\delta m)a_i + (\delta q)F_{i0}^{ext} + q(\delta F_{i0}^{ext}) + \frac{2}{3}q^2 \dot{a}^i
- \frac{1}{2}Q^{\mu\nu}\partial_\mu F_{i\nu}^{ext} + \frac{d}{dt}\left(a^j S_{ji} - 2Q^j \left[ F_{i0}^{ext} \right]_{[i} \right).$

(2.115)

We then follow the same steps for (2.110) by using the same definitions, evaluating it at $\bar{t} = 0$, and integrating over space to obtain the time evolution of the perturbed mass to first order:

$$
\frac{d}{dt_0} (\delta m) = \frac{1}{2}Q^{\mu\nu} \frac{d}{dt_0} F_{i\mu}^{ext} - \frac{d}{dt} \left( Q^{\mu0} F_{i\mu}^{ext} \right).
$$

(2.116)
We now have equations to describe the motion according to a perturbed acceleration and the time evolution of the perturbed mass.

Before continuing, we should a couple of comments about these results so far. First, on the right side of (2.115), we have the lowest order mass, charge, and acceleration as well as the perturbed mass and charge, as well as the spin and electromagnetic dipole tensors. As we have seen before, $m$, $q$, and $\delta q$ are all conserved, and $a_i$ is described by the Lorentz force law. Second, we cannot obtain an evolution equation for the electromagnetic dipole tensor because of its “non-universal” property. Each particle has different dipole moments, and so it is impossible to write a general form of the time evolution of this quantity, which means that additional conditions and assumptions are required in order to obtain a deterministic evolution. With this in mind, when we arrive at the numerical calculations, we will make use of elementary particles that do not have any electric dipole moments (or at least electric dipole moments that are the significantly small enough to allow us to neglect them). Appendix B of the paper we are currently summarizing follows the same procedure.

To write our results in covariant form, that in Fermi normal coordinates, we can write each factor that has a 0 as one of the indexes as the total tensor multiplied with the 4-velocity. For example, we have

$$F_{\text{ext}}^{ia} \rightarrow u^a F_{\text{ext}}^{ia}.$$

Then, for the spatial components of quantities on the worldline, for example what we have above, correspond to the spacetime tensor

$$(\delta^c_b + u^c u_b) u^a F_{\text{ext}}^{ca} = u^a F_{\text{ext}}^{ba}.$$
Lastly, the time derivative of \( \frac{d}{dt_0} \) corresponds to the Fermi derivative, which we write as the covariant time derivative

\[
\frac{d}{dt_0} \rightarrow D = u^a \nabla_a. \tag{2.117}
\]

Out of convenience, we also redefine the perturbed mass \( \delta m \) as \( \delta \hat{m} \) to include the dipole interaction energy,

\[
\delta \hat{m} = \delta m - u_b u^c Q^{bd} F_{cd}^{\text{ext}}. \tag{2.118}
\]

We finally put all of this together to find the perturbed equation of motion, the evolution of spin, and the evolution of the perturbed mass as

\[
\delta [\hat{m} a_a] = \delta [q F_{ab}^{\text{ext}} u^b] + (g_a^b + u_a u^b) \left[ \frac{2}{3} q^2 \frac{D}{d\tau} a_b - \frac{1}{2} Q^{cd} \nabla_b F_{cd}^{\text{ext}} \right. \\
\left. + \frac{D}{d\tau} \left( a^c S_{cb} - 2 u^d Q^c_{[b} F_{d]e}^{\text{ext}} \right) \right], \tag{2.119}
\]

\[
\frac{D}{d\tau} S_{ab} = 2(g_a^c + u_a u^c) \left( g_b^d + u_b u^d \right) Q^{e} \left[ F_{de}^{\text{ext}} - 2a^c S_{[a} u_{b]} \right], \tag{2.120}
\]

\[
\frac{D}{d\tau} \delta \hat{m} = \frac{1}{2} Q^{ab} F_{ab}^{\text{ext}} + 2Q_a^b F_{bc} a^{[c} u^{a]} \tag{2.121}
\]

These three results now give us a complete look at the perturbed equations of motion, evolution of spin of the body, and evolution of perturbed mass for a particle of shrinking size in the near-zone limit. The process has made no mention of the exact makeup of the body, and the only things that were assumed were conservation of stress-energy, conservation of charge, and Maxwell’s equations. We now move onto the next discussion, which is that of self-consistent motion.
It is at this time that we would like to also point out that (2.120) already includes what is known as the *Thomas precession* in its final term. Recall that the Thomas precession corresponds to a relativistic correction to the change of angular momentum of the body. In Chapter 11 of Jackson, recall that he ran through a derivation which resulted in the spin of the body as

\[
\frac{dS^\alpha}{d\tau} = \frac{ge}{2mc} \left[ F^\alpha\beta S_\beta + \frac{1}{c^2} U^\alpha \left( S_\lambda F^{\lambda\mu} U_\mu \right) \right] - \frac{1}{c^2} U^\alpha \left( S_\lambda \frac{dU^\lambda}{d\tau} \right),
\]

where it is explicitly mentioned that the final term is the Thomas precession. We can immediately see the correlation between the two terms. Additionally, we can recognize that this also has a form similar to that of the BMT equation, which we see by starting with the result from Jackson,

\[
\frac{dS^\alpha}{d\tau} = \frac{e}{mc} \left[ \frac{g}{2} F^\alpha\beta S_\beta + \frac{1}{c^2} \left( \frac{g}{2} - 1 \right) U^\alpha \left( S_\lambda F^{\lambda\mu} U_\mu \right) \right],
\]

which arises if gradient force terms can be neglected and there are no other appreciable forces on the particle outside of uniform electromagnetic fields. Recall that when dealing with elementary particles, then the dipole and spin tensors are directly proportional to each other, and it becomes straightforward to see that in this case, these two results correspond closely. So, it comes as a result of this analysis that we are able to include the BMT equation and Thomas precession.

### 2.5 Self-Consistent Equations Of Motion

As can be seen above, we have solutions that still depend on the time derivatives of accelerations, making this of the same order in time as before, and so we look at
the paper’s attempt at making a “self-consistent” theory from these results. In this manner, it is considered that the lowest order quantities of mass, charge, and the external electromagnetic fields are viewed to be given. Meanwhile, the worldline, velocity, acceleration, and time derivative of acceleration are to be determined by solving (2.68). The additional requirements for a self-consistent theory are

1. The equations must have a well-posed initial value formulation,

2. They must have the same number of degrees of freedom as the first-order perturbative system,

3. For corresponding initial data, the solutions to the self-consistent perturbative equations should be close to the solutions of the first-order perturbative system over the time interval for which the first-order perturbative description should be accurate.

In other words, we would expect that the solutions to the perturbative equations do not vary wildly from the well-known unperturbed solutions. It is with this in mind that it is considered that a “reduction of order” process is best suited to fulfill these three conditions. In order to do this, we will need to replace any factor of $a_\alpha$ by

$$a_\alpha = \frac{q}{m} F_{\alpha \beta}^\text{ext} u^\beta,$$

(2.122)

which is the unperturbed Lorentz force acceleration. Taking the time derivative of each side allows us to replace any factor of $\dot{a}_\alpha$ in our perturbed equation,

$$\dot{a}_\alpha = u^\sigma u^\gamma \nabla_\gamma \left( \frac{q}{m} F_{\alpha \sigma}^\text{ext} \right) + \left( \frac{q}{m} F_{\alpha \sigma} \right) \frac{q}{m} F^\sigma_{\gamma \delta} u^\delta.$$  

(2.123)
This will be a process that we will go through once we complete writing out the results of perturbed motion in vector notation.

It is mentioned that even if each term on the right-hand side of (2.119) is small compared to the Lorentz force at all time, one might expect that the overall effect over a large period of time may eventually cause deviations that add up and wildly vary from the expected unperturbed solutions. In order to avoid this, one could go to higher order in perturbation theory in order to lead to more accurate results over longer times. The addition of extra terms from this higher order method could still break down over sufficiently long time scales.

Our way of avoiding these large deviations over sufficiently long time scales is then to consider a family of Lorentz trajectories. Each of these world lines would have small deviations at each point due to the first order corrections that have been calculated. We can then piece each of the relevant trajectory pieces together according to the three principles listed above, giving a Lorentz trajectory that should stay valid over long time scales.

Lastly, we just rewrite our the results of the paper using Greek indexes out of comfort and recognizability. This form is what we will use when we calculate the relativistically correct vector form.

\[
\delta \tilde{m}_a = \delta \left[ q F^\text{ext}_{a\beta} u^\beta \right] + (g_\alpha^\beta + u_\alpha u^\beta) \left[ \frac{2}{3} q^2 \frac{D}{d\tau} t_a - \frac{1}{2} Q^\gamma_\delta \nabla_\beta F^\text{ext}_{\gamma\delta} \right.
\]
\[
\left. + \frac{D}{d\tau} \left( a^\gamma S_{\gamma\delta} - 2u^\delta Q^\gamma_{[\beta} F^\text{ext}_{\delta]} \right) \right],
\]

(2.124a)

\[
\frac{D}{d\tau} S_{\alpha\beta} = 2(g_\alpha^\gamma + u_\alpha u^\gamma) \left( g_\beta^\delta + u_\beta u^\delta \right) Q^\epsilon_{[\gamma} F^\text{ext}_{\delta]\epsilon} - 2a^\gamma S_{\gamma[a} u_{b]},
\]

(2.124b)

\[
\frac{D}{d\tau} \delta \tilde{m} = \frac{1}{2} Q^\alpha_{\beta\gamma} F^\text{ext}_{\alpha\beta} + 2Q^\beta_{\alpha\gamma} a^{[\gamma} u^{\alpha]}.
\]

(2.124c)
CHAPTER 3
WRITING OUT THE RESULTS IN VECTOR FORM

Our objective is to rewrite the three resulting equations from the near-zone limit, (2.124), in terms of the temporal and spatial quantities. There are some important points that we need to discuss before doing out our calculations first. We need to derive the 4-velocity and 4-acceleration of the particle in a relativistically-correct form. The paper produced a form similar to that which we are looking for, but is admittedly non-relativistic, which is why we have neglected to discuss it. In order to keep this analysis as general and applicable as possible, we cannot neglect special relativity.

Before we go through that derivation, we should first remind ourselves of the other quantities and how they are defined.

1. Spacetime metric: Thanks to working in Fermi-Normal Coordinates, the region of the particle can always be approximated to being in a flat spacetime metric in this first order. Additionally, we take the time-component to be negative, and so the metric has the form

\[
g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
2. Electromagnetic Field-Strength Tensor:

\[ F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \]  \hspace{1cm} (3.2)

Notice that there is an “anti-symmetric” property to the tensor, so that if we were to flip each quantity around diagonal, we would get the same result, but the electric field components would swap (which, when multiplying by the metric, we can see why this would be the case)

\[ F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \]  \hspace{1cm} (3.3)

3. Next, we move onto the spin tensor, \( S^{\alpha\beta} \). Recall that we had defined the spin tensor as (2.84),

\[ S^{0j}(t_0) = -S^{j0}(t_0) = \int T^{(0)00} \bar{x}^j d^3\bar{x}, \]

\[ S^{ij} = 2 \int \bar{T}^{(0)i} \bar{x}^j d^3\bar{x}. \]
However, when the center-of-mass condition, $\bar{X}_{\text{CM}}^i = \frac{1}{m} \int \bar{T}_{00}^{(0)} \bar{x}^i \, d^3 \bar{x} = 0$, is applied, it is clear that the zeroth components go to zero, and we are left with just the spatial components, which we saw in (2.85),

$$S_i = \frac{1}{2} \epsilon_{ijk} S^{ij},$$

where $\epsilon_{ijk}$ is the Levi-Cevita tensor, which we defined in (2.86). Here, it is perhaps useful to reiterate that we are still considering a classical theory. This “spin tensor” and “spin vector” do not mean the quantum mechanical spin of particles, but rather the angular momentum of these particles. Let’s just do one of these components, just to see how this works. Let’s say $S_i = S_1$. Then we have

$$S_1 = \frac{1}{2} \epsilon_{1jk} S^{jk}$$
$$= \frac{1}{2} \epsilon_{123} (S^{23} - S^{32})$$
$$= \frac{1}{2} (S^{23} - S^{32}).$$

From this, since each component has to be equal to $S_1$, and in order for these two components to add to $2S_1$ to cancel with the factor of $\frac{1}{2}$, we see that the two components of the tensor become:

$$S^{23} = S_1, \quad (3.4a)$$
$$S^{32} = -S_1. \quad (3.4b)$$
If we then follow this same procedure, we would get exactly the same form as before.

\[
S^{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & S_z & -S_y \\
0 & -S_z & 0 & S_x \\
0 & S_y & -S_x & 0
\end{pmatrix}.
\] (3.5)

4. In a similar manner, we can then go through each component for the dipole tensor. However, since the spatial components are analog to those of the spin tensor and the time-space components directly correspond to the electric dipole, it becomes immediately obvious when looking to (2.91) that we get

\[
Q^{\alpha\beta} = \begin{pmatrix}
0 & p_x & p_y & p_z \\
-p_x & 0 & -\mu_z & \mu_y \\
-p_y & \mu_z & 0 & -\mu_x \\
-p_z & -\mu_y & \mu_x & 0
\end{pmatrix}.
\] (3.6)

Now that we have those definitions out of the way, we can now go ahead to properly define the 4-vectors that we will need.

To start, we introduce the concept of the *proper time*. In order to introduce this concept, we consider something known as the “space-time interval,” which we define as

\[
(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.
\] (3.7)
So, this appears to be related to the change in spatial and temporal coordinates of an object.\(^1\) We can then define the proper time as the negative of (3.7),

\[
(\Delta \tau)^2 = - (\Delta s)^2.
\]  

(3.8)

This can be interpreted as the time elapsed for an observer traveling in a straight line between two events. Another way to consider this is in the context of a charged particle in an accelerator. If we were to compare the passage of time in the reference frame of the stationary observer to that of the particle in the pipe, then the particle trajectory is determined by its proper time compared to our frame. However, we can relate the proper time in its frame to the time of the observer by using the Lorentz invariant quantity,

\[
d\tau = \frac{dt}{\gamma(t)} \rightarrow \gamma \frac{d}{dt} = \frac{d}{d\tau},
\]

(3.9)

where \(\gamma\) is the Lorentz factor,

\[
\gamma = \frac{1}{\sqrt{1 - \frac{\vec{v} \cdot \vec{v}}{c^2}}} = \frac{1}{\sqrt{1 - \beta \cdot \beta}}.
\]

(3.10)

We will now need to bring this idea of the proper time to determining the 4-acceleration of a particle. Recall that in 4-vectors, there is always a 0th (or time)

\(^1\)Note that there could be mixing between the different terms, so that we could see an interval (which is really what makes up metrics) of the form

\[
ds^2 = dt^2 + dx \, dy - dy \, dx + dz^2.
\]

Here, we can see a mixing of the \(x\) and \(y\) components. However, we will be concerning ourselves with a flat space-time metric, so there will be no mixing between the different coordinates.
component to the vector, and then three spatial dimensions. The most useful for us, then, would be the 4-velocity, which we can get from

\[
    u^\mu = \frac{dx^\mu}{d\tau},
\]  

(3.11)

where \( x^\mu \) is the position 4-vector. Notice that we are still taking the derivative with the respect to the proper time because this is meant to be the velocity that the particle is experiencing in its frame. The 4-position can be written as

\[
    x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},
\]  

(3.12)

To find the 4-velocity, we need to take the derivative of these quantities with respect to the proper time, and so we can use (3.9) to find the 4-velocity that we would measure in our stationary frame,

\[
    u^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = \gamma \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix},
\]  

(3.13)
Notice that this now has a factor of $\gamma$ out front of the column vector. To get this into the same natural units as Gralla and Wald’s paper, we factor out the $c$ and set $c = 1$, to find

$$u^\mu = \gamma \begin{pmatrix} 1 \\ \beta_x \\ \beta_y \\ \beta_z \end{pmatrix}$$

(3.14)

Now, if we were to try to find the 4-acceleration, we would see that the temporal component is in fact non-zero, because we now need to find it through the derivative of the 4-velocity with respect to the proper time.

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma \frac{du^\mu}{dt},$$

(3.15)

and so we need to calculate the time derivative of $\gamma$ and $\gamma \beta_i$. Let’s start with calculating just the time derivative of $\gamma$.

$$\frac{d\gamma}{dt} = \frac{d}{dt} \left(1 - \vec{\beta} \cdot \vec{\beta}\right)^{-1/2}$$

$$= -\frac{1}{2} \left(1 - \vec{\beta} \cdot \vec{\beta}\right)^{-3/2} \left(-2 \vec{\beta} \cdot \dot{\vec{\beta}}\right)$$

$$= \gamma^3 \left(\vec{\beta} \cdot \dot{\vec{\beta}}\right).$$

(3.16)

We can then apply this to the spatial components of the the 4-velocity, which will simply give us a nice product rule result of

$$\frac{d\gamma \beta_i}{dt} = \beta_i \frac{d\gamma}{dt} + \gamma \frac{d\beta_i}{dt} = \gamma^3 \left(\vec{\beta} \cdot \dot{\vec{\beta}}\right) \beta_i + \gamma \dot{\beta}_i.$$  

(3.17)
If we put all of this back into our definition of the 4-acceleration, we get
\[
a^\mu = \gamma^2 \begin{pmatrix}
\gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}}) \\
\gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x + \dot{\beta}_x \\
\gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_y + \dot{\beta}_y \\
\gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_z + \dot{\beta}_z
\end{pmatrix}.
\]
(3.18)

Here, we see that if the acceleration is perpendicular to the velocity (for example in centripetal motion) then the temporal component evaluates to zero, while all of the spatial components are left with just the acceleration in their respective directions. Additionally, if the particle is non-relativistic, this reduces further still to just
\[
a^\mu_{\text{non-rel}} = \begin{pmatrix}
0 \\
\dot{\beta}_x \\
\dot{\beta}_y \\
\dot{\beta}_z
\end{pmatrix}.
\]

The last thing that we should mention before heading into calculating each and every single component (both spatial and temporal) is the covariant derivative, \(\frac{D}{d\tau}\). This is defined to be
\[
\frac{D}{d\tau} = u_\alpha \nabla^\alpha,
\]
(3.19)
which implies summation over the indexes \(\alpha\) from 0 to 3. Instead of having to continuously do this process within the context of each term, let’s just calculate this now, so that we will just be able to put the result back in. To start, this summation obviously becomes
\[
u_\alpha \nabla^\alpha = u_0 \nabla^0 + u_1 \nabla^1 + u_2 \nabla^2 + u_3 \nabla^3.
\]
(3.20)
Now, if we use that the definition of the 4-gradient is (in natural units) $\nabla^\alpha = \left(-\frac{\partial}{\partial t}, \vec{\nabla}\right)$, then (3.20) becomes

$$u_\alpha \nabla^\alpha = -\gamma \frac{\partial}{\partial t} + \gamma \beta_x \frac{\partial}{\partial x} + \gamma \beta_y \frac{\partial}{\partial y} + \gamma \beta_z \frac{\partial}{\partial z}$$

$$= \gamma \left(-\frac{\partial}{\partial t} + \beta_x \frac{\partial}{\partial x} + \beta_y \frac{\partial}{\partial y} + \beta_z \frac{\partial}{\partial z}\right)$$

$$= \gamma \left(\frac{\partial}{\partial t} + \vec{\beta} \cdot \vec{\nabla}\right)$$

Notice that the result above is really just the product rule resulting in the total time derivative. From this, we can really see that the covariant derivative is the time derivative with respect to the proper time.

$$\frac{D}{d\tau} = \frac{d}{d\tau} = \gamma \frac{d}{dt}. \quad (3.21)$$

Armed with this knowledge, we are now ready to take on calculating each resulting term from (2.124). Additionally, notice that the three times the electromagnetic field-strength tensor arise, it is only the external field that is playing an effect on the particle to first order, and so we will drop the superscript “ext” in the understanding that the resulting fields that we see are purely from the external contributions.

### 3.1 The Metric Multiplied to Each Term

There are two sets of terms that we truly need to calculate, the first with the metric $g_{\alpha\beta}$ multiplying the terms in parentheses, while the second is an additional component arising from the velocity of the particle, $u_\alpha u^\beta$. If we expand the time
derivative in the parentheses of (2.124a), we will have four terms to write out in vector form.

\[
\frac{2}{3} q^2 g_\alpha^\beta \frac{D}{d\tau} a_\beta, \quad (3.22a)
\]

\[-\frac{1}{2} g_\alpha^\beta \nabla_\beta F_{\gamma\delta}, \quad (3.22b)\]

\[g_\alpha^\beta \frac{D}{d\tau} a^\gamma S_{\gamma\beta}, \quad (3.22c)\]

\[-2 g_\alpha^\beta \frac{D}{d\tau} \left( u^\delta Q^\gamma \nabla_\delta F_{\gamma\delta} \right), \quad (3.22d)\]

In order to do this, we see that there are a number of terms with multiple indexes that are repeated throughout the term. Rather than trying to sum over all of them at once, we will take our time, specify a certain index’s value, and sum over the remaining indexes. Once we complete the proper summations, we can then add up all of the results to get back to the original form that we wanted. Additionally, we need both the temporal component and the spatial components, so we will do those out separately.

### 3.1.1 Equation (3.22a): Temporal Component

We shall start on the simplest case that we have in front of use before moving onto the more difficult things. Since we are specifically looking for the temporal component, we start by setting \( \alpha = 0 \), as the left hand-side of our equation is \( m a_\alpha \). This then gives us

\[
\frac{2}{3} q^2 \ g_0^\alpha^\beta \frac{D}{d\tau} = \frac{2}{3} q^2 \left[ g_0^0 \frac{d}{d\tau} a_0 + g_0^1 \frac{d}{d\tau} a_1 + g_0^2 \frac{d}{d\tau} a_2 + g_0^3 \frac{d}{d\tau} a_3 \right] \quad (3.23)
\]
Here, we need to be careful. There is a rule about raising and lowering indexes of tensors, which follows

$$T_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}T^{\mu\nu}. \quad (3.24)$$

Since we have our metric defined in (3.1), and we are trying to raise an index, we need to take

$$g_{\alpha\beta} = g_{\alpha\mu}g^{\mu\beta}. \quad (3.25)$$

Luckily, since this metric is for a flat space-time, we can take it for granted that $g_{\alpha\beta} = g^{\alpha\beta}$. However, if we were to do this matrix multiplication out, we would indeed find

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.26)$$

If we look to (3.26) then, we see that only the $g_{00}$ component will contribute with a factor of 1, since all other components of the metric are zero, and we are left with

$$\frac{2}{3}q^2g_{00}^0 \frac{D}{d\tau} = \frac{2q^2}{3} (1) \gamma \frac{d\alpha_0}{dt}.$$
If we look back to the definition of the 4-acceleration, we see $a_0 = -\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}})$, since we have lowered the index, which introduces the factor of -1. Explicitly taking the time derivative of this, we get

$$-rac{da_0}{dt} = (\vec{\beta} \cdot \dot{\vec{\beta}}) \frac{d}{dt} (\gamma^4) + \gamma^4 \frac{d}{dt} (\vec{\beta} \cdot \dot{\vec{\beta}})$$

$$= (\vec{\beta} \cdot \dot{\vec{\beta}}) \frac{d}{dt} \left[ (1 - \vec{\beta} \cdot \vec{\beta})^{-1/2} \right]^4 + \gamma^4 \frac{d}{dt} (\vec{\beta} \cdot \dot{\vec{\beta}})$$

$$= (\vec{\beta} \cdot \dot{\vec{\beta}}) \frac{d}{dt} (1 - \vec{\beta} \cdot \vec{\beta})^{-2} + \gamma^4 \frac{d}{dt} (\vec{\beta} \cdot \dot{\vec{\beta}})$$

$$= (\vec{\beta} \cdot \dot{\vec{\beta}}) \left[ -2 (1 - \vec{\beta} \cdot \vec{\beta})^{-3} (-2 \vec{\beta} \cdot \dot{\vec{\beta}}) \right] + \gamma^4 \frac{d}{dt} (\vec{\beta} \cdot \dot{\vec{\beta}})$$

$$= 4\gamma^6 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 + \gamma^4 (\vec{\beta} \cdot \ddot{\vec{\beta}}) + \gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}})$$.

Dimensionally, each term will add nicely, as $\vec{\beta}$ is dimensionless, $\dot{\vec{\beta}}$ has dimensions of Time$^{-1}$, and $\ddot{\vec{\beta}}$ has dimensions of Time$^{-2}$. Each term has this dimension of Time$^{-2}$, so we are good (for now). Lastly, we put this result back into (3.22a), and get for the temporal component,

$$\frac{2}{3} q^2 g_0 \beta D \frac{D}{dt} a_\beta = -\frac{2}{3} q^2 \gamma^5 \left[ 4 \gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 + \vec{\beta}^2 + (\vec{\beta} \cdot \vec{\beta}) \right].$$

### 3.1.2 Equation (3.22a) Spatial Components

We can now move onto the spatial component of the Abraham-Lorentz-Dirac (ALD) force, where we set $\alpha = 1, 2, 3$. Explicitly writing this out,

$$\frac{2}{3} q^2 g_i \beta D \frac{D}{d\tau} a_\beta = \frac{2}{3} q^2 \gamma \left[ g_1 \beta \frac{d}{dt} a_\beta + g_2 \beta \frac{d}{dt} a_\beta + g_3 \beta \frac{d}{dt} a_\beta \right].$$
As seen with the temporal component, we can immediately see that the only non-zero contributions will be when \( \beta = \alpha \), since \( g_{\alpha}{}^\beta = \delta_{\alpha}^\beta \). This allows us to write (3.29) as
\[
\frac{2}{3} q^2 g_i{}^\beta \frac{D}{d\tau} a_\beta = \frac{2}{3} q^2 \gamma \left[ \frac{da_1}{dt} + \frac{da_2}{dt} + \frac{da_3}{dt} \right].
\]
(3.30)

Since each of these spatial components are the same, we can write the sum of the components in brackets as \( \frac{da_i}{dt} \), and taking the definition from the 4-acceleration, we find
\[
\frac{da_i}{dt} = \frac{d}{dt} \left[ \gamma^4 (\bar{\beta} \cdot \dot{\bar{\beta}}) \bar{\beta} + \gamma^2 \ddot{\bar{\beta}} \right]
\]
\[
= \bar{\beta} (\bar{\beta} \cdot \dot{\bar{\beta}}) \frac{d\gamma^4}{dt} + \gamma^4 \dot{\bar{\beta}} \frac{d}{dt} (\bar{\beta} \cdot \dot{\bar{\beta}}) + \gamma^4 (\bar{\beta} \cdot \dot{\bar{\beta}}) \ddot{\bar{\beta}} + \dot{\bar{\beta}} \frac{d\gamma^2}{dt} + \gamma^2 \dddot{\bar{\beta}}
\]

We can calculate each derivative separately here, which we will be able to reference later on.

\[
\frac{d\gamma^2}{dt} = \frac{d}{dt} \left[ (1 - \bar{\beta} \cdot \bar{\beta})^{-1} \right]
\]
\[
= -\left(1 - \bar{\beta} \cdot \bar{\beta}\right)^{-2} \left(-2\bar{\beta} \cdot \dot{\bar{\beta}}\right)
\]
\[
= 2\gamma^4 (\bar{\beta} \cdot \dot{\bar{\beta}}).
\]
(3.31)

\[
\frac{d\gamma^4}{dt} = \frac{d}{dt} \left[ (1 - \bar{\beta} \cdot \bar{\beta})^{-1/2} \right]^4
\]
\[
= \frac{d}{dt} \left[ (1 - \bar{\beta} \cdot \bar{\beta})^{-2} \right]
\]
\[
= -2 \left(1 - \bar{\beta} \cdot \bar{\beta}\right)^{-3} \left(-2\bar{\beta} \cdot \dot{\bar{\beta}}\right)
\]
\[
= 4\gamma^6 (\bar{\beta} \cdot \dot{\bar{\beta}}).
\]
(3.32)

\[
\frac{d}{dt} (\bar{\beta} \cdot \dot{\bar{\beta}}) = \bar{\beta} \cdot \ddot{\bar{\beta}} + \dot{\bar{\beta}}^2.
\]
(3.33)
With all of these calculated, we put this back into the time derivative of $a_i$ to find
\[
\frac{da_i}{dt} = 4\gamma^6 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \vec{\beta} + \gamma^4 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2 \right) \vec{\beta} + \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \ddot{\vec{\beta}} + \gamma^2 \dddot{\vec{\beta}} = 4\gamma^6 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \vec{\beta} + \gamma^4 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2 \right) \vec{\beta} + 3\gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \ddot{\vec{\beta}} + \gamma^2 \dddot{\beta}.
\]
(3.34)

Lastly, we can put this back into (3.30) to get the spatial components of this term as
\[
2 \frac{q}{3} g^\beta D a_\beta = \frac{2q^2 \gamma^3}{3} \left[ 4\gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \vec{\beta} + \gamma^2 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2 \right) \vec{\beta} + 3\gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \ddot{\vec{\beta}} + \dddot{\vec{\beta}} \right].
\]
(3.35)

### 3.1.3 Equation (3.22b): Temporal Component

We now move to Equation (3.22b), and again starting with the temporal component, which will help point us in the correct direction. Since $\alpha = 0$, we know immediately that $\beta = 0$, thanks to the identity property of the metric in this form. This now brings us to
\[
-\frac{1}{2} g_0^\gamma F_{\gamma\delta},
\]
so we need only sum over $\gamma$ and $\delta$. Since this is a simultaneous summation, let’s specify each value for $\gamma$, and then sum over $\delta$. Once we have all four components for $\gamma$, we will add them all back up.
1. $\gamma = 0$:
\[
-\frac{1}{2} (1) Q^{0\delta} \nabla_0 F_{0\delta} = -\frac{1}{2} \left( Q^{00} \frac{\partial}{\partial t} F_{00} + Q^{01} \frac{\partial}{\partial t} F_{01} + Q^{02} \frac{\partial}{\partial t} F_{02} + Q^{03} \frac{\partial}{\partial t} F_{03} \right)
\]
\[
= -\frac{1}{2} \left( 0 + p_x \frac{\partial}{\partial t} (-E_x) + p_y \frac{\partial}{\partial t} (-E_y) + p_z \frac{\partial}{\partial t} (-E_z) \right)
\]
\[
= \frac{1}{2} \mathbf{P} \cdot \frac{\partial \mathbf{E}}{\partial t}.
\]

2. $\gamma = 1$:
\[
-\frac{1}{2} Q^{1\delta} \nabla_0 F_{1\delta} = -\frac{1}{2} \left( Q^{10} \frac{\partial}{\partial t} F_{10} + Q^{11} \frac{\partial}{\partial t} F_{11} + Q^{12} \frac{\partial}{\partial t} F_{12} + Q^{13} \frac{\partial}{\partial t} F_{13} \right)
\]
\[
= -\frac{1}{2} \left[ (-p_x) \frac{\partial}{\partial t} E_x - \mu_z \frac{\partial}{\partial t} (B_z) + (\mu_y) \frac{\partial}{\partial t} (-B_y) \right]
\]
\[
= \frac{1}{2} \left( p_x \frac{\partial E_x}{\partial t} \right) + \frac{1}{2} \left( \mu_y \frac{\partial B_y}{\partial t} \right) + \frac{1}{2} \left( \mu_z \frac{\partial B_z}{\partial t} \right).
\]

3. $\gamma = 2$:
\[
-\frac{1}{2} Q^{2\delta} \nabla_0 F_{2\delta} = -\frac{1}{2} \left( Q^{20} \frac{\partial}{\partial t} F_{20} + Q^{21} \frac{\partial}{\partial t} F_{21} + Q^{22} \frac{\partial}{\partial t} F_{22} + Q^{23} \frac{\partial}{\partial t} F_{23} \right)
\]
\[
= -\frac{1}{2} \left[ (-p_y) \frac{\partial}{\partial t} E_y + (\mu_z) \frac{\partial}{\partial t} (-B_z) - \mu_x \frac{\partial}{\partial t} (B_x) \right]
\]
\[
= \frac{1}{2} \left( p_y \frac{\partial E_y}{\partial t} \right) + \frac{1}{2} \left( \mu_x \frac{\partial B_x}{\partial t} \right) + \frac{1}{2} \left( \mu_z \frac{\partial B_z}{\partial t} \right).
\]

4. $\gamma = 3$:
\[
-\frac{1}{2} Q^{3\delta} \nabla_0 F_{3\delta} = -\frac{1}{2} \left[ Q^{30} \frac{\partial}{\partial t} F_{30} + Q^{31} \frac{\partial}{\partial t} F_{31} + Q^{32} \frac{\partial}{\partial t} F_{32} + Q^{33} \frac{\partial}{\partial t} F_{33} \right]
\]
\[
= -\frac{1}{2} \left[ (-p_z) \frac{\partial}{\partial t} E_z - \mu_y \frac{\partial}{\partial t} (B_y) + (\mu_x) \frac{\partial}{\partial t} (-B_z) \right]
\]
\[
= \frac{1}{2} \left( p_z \frac{\partial E_z}{\partial t} \right) + \frac{1}{2} \left( \mu_x \frac{\partial B_x}{\partial t} \right) + \frac{1}{2} \left( \mu_y \frac{\partial B_y}{\partial t} \right).
\]
When we look at all these results, we see that we have two factors of the interaction of a dipole with its respective field (electric dipole and electric field or magnetic dipole in a magnetic field), and when we add up all of the components, it is clear that we get

$$\frac{1}{2}g_0^\beta Q^{\gamma\delta}\nabla_\beta F_{\gamma\delta} = \vec{p} \cdot \frac{d\vec{E}}{dt} + \vec{\mu} \cdot \frac{d\vec{B}}{dt}. \quad (3.36)$$

### 3.1.4 Equation (3.22b): Spatial Components

Moving onto the spatial components, again (as we will have for all four of these terms involving the metric) $\beta = \alpha$. Since this is going to stay the same for each of the components of $\alpha = 1, 2, 3$, we will keep $\alpha = \beta = i$ and then sum over them at the end. Again, we will set the values for $\gamma$, sum over $\delta$, and sum over each component at the end.

1. $\gamma = 0$:

$$-\frac{1}{2}Q^{0\delta}\nabla_i F_{0\delta} = -\frac{1}{2}[Q^{00}\nabla_i F_{00} + Q^{01}\nabla_i F_{01} + Q^{02}\nabla_i F_{02} + Q^{03}\nabla_i F_{03}]$$

$$= -\frac{1}{2}[p_x \nabla_i (-E_x) + p_y \nabla_i (-E_y) + p_z \nabla_i (-E_z)]$$

$$= \frac{1}{2}p_j \nabla E_j.$$

Here, we imply a sum over $j$ in the last line for each dipole moment and electric field component. This means that for the $x$-component of the force, we will have something of the form

$$F_x = p_x \frac{\partial E_x}{\partial x} + p_y \frac{\partial E_y}{\partial x} + p_z \frac{\partial E_z}{\partial x},$$

and this will repeat for the $y$ and $z$ components as well.
2. $\gamma = 1$:

$$-\frac{1}{2}Q^{1\delta} \nabla_i F_{1\delta} = -\frac{1}{2}(Q^{10} \nabla_i F_{10} + Q^{11} \nabla_i F_{11} + Q^{12} \nabla_i F_{12} + Q^{13} \nabla_i F_{13})$$

$$= -\frac{1}{2} [(-p_x) \nabla_i E_x - \mu_z \nabla_i (B_z) + (\mu_y) \nabla_i (-B_y)]$$

$$= \frac{1}{2} p_x \vec{\nabla} E_x - \frac{1}{2} \mu_y \vec{\nabla} B_y + \frac{1}{2} \mu_z \vec{\nabla} B_z$$

We can immediately see the pattern that emerges from this calculation, as it mirrors that seen in the temporal component ($\gamma = 0$), and we can then write the end result of this term for the spatial components.

$$-\frac{1}{2} g_i^{\beta} Q^{\gamma \delta} \nabla_\beta F_{\gamma \delta} = p_j \vec{\nabla} E_j + \mu_j \vec{\nabla} B_j. \quad (3.37)$$

### 3.1.5 Equation (3.22c): Temporal Component

We are now onto the third equation from (3.22c), with $\beta = \alpha$ yet again, and as the temporal component is defined to be when $\alpha = 0$, this means that we will be left with

$$g_0^{\beta} \frac{D}{d\tau} a^\gamma S_{\gamma \beta} = \frac{D}{d\tau} a^\gamma S_{\gamma 0}. \quad (3.38)$$

However, if we look back to when we had the definition of the spin tensor in equation (3.5), we see that all of the temporal components of $S$ are zero, which means that we will have no contribution from this term.

$$g_0^{\beta} \frac{D}{d\tau} a^\gamma S_{\gamma \beta} = 0. \quad (3.39)$$
3.1.6 Equation (3.22c): Spatial Components

Moving onto the spatial components, we again are left with $\alpha = i = 1, 2, 3$. Since we do not have multiple sums, as $\alpha = \beta$ for these four terms, let’s go ahead and explicitly calculate at least the first term, $\alpha = \beta = 1$. We can then take that result and apply it to the other two spatial components to find our final result.

\[
g_1 \frac{1}{\delta}a^\gamma S_{y1} = \gamma \frac{d}{dt} \left[ a_0 S_{y1} + a_1 S_{y1} + a_2 S_{y1} + a_3 S_{y1} \right] = \gamma \frac{d}{dt} \left[ \gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_y + \beta_y \right] (-S_z) + \gamma^2 \left[ \gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_z + \beta_z \right] S_y
\]

This could use a little unpacking. The first thing that we see on the inside is two vector products. Recall that we can write vector products as

\[
\vec{a} \times \vec{b} = \hat{x}(a_y b_z - a_z b_y) + \hat{y}(a_z b_x - a_x b_z) + \hat{z}(a_x b_y - a_y b_x), \tag{3.40}
\]

and we are seeing the $x$-component of this here. We can then rewrite the spatial components in terms of this vector product as

\[
g_1 \frac{1}{\delta}a^\gamma S_{y1} = \gamma \frac{d}{dt} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left( \vec{S} \times \vec{a} \right)_x + \gamma^2 \left( \vec{\bar{S}} \times \vec{\bar{\beta}} \right)_x \right].
\]

Now, if we were to add up all the components, which will have the same form as this, we would immediately see that this would result in just the total vector product
between the spin vector and the velocity (or acceleration, depending on the term), and so the overall form of the spatial components for this term becomes

\[ g_i \frac{D}{dt} a^\gamma S_{\gamma i} = \gamma \frac{d}{dt} \left[ \gamma^4 (\bar{\beta} \cdot \ddot{\beta}) (\bar{S} \times \dot{\beta}) + \gamma^2 (\bar{S} \times \ddot{\beta}) \right]. \] (3.41)

However, instead of keeping everything inside of the time derivatives, we should distribute the time derivative inside of the brackets, which will result in a number of terms from the product rule. Starting with the first term, we have

\[
\frac{d}{dt} \left[ \gamma^4 (\bar{\beta} \cdot \ddot{\beta}) (\bar{S} \times \dot{\beta}) \right] = (\bar{\beta} \cdot \ddot{\beta})(\bar{S} \times \dot{\beta}) \frac{d\gamma^4}{dt} (\bar{S} \times \dot{\beta}) + \gamma^4 (\bar{S} \times \dot{\beta}) \frac{d}{dt} (\bar{\beta} \cdot \ddot{\beta}) + \gamma^4 (\bar{\beta} \cdot \ddot{\beta}) (\bar{S} \times \dot{\beta}) \frac{d\gamma^4}{dt} \\
= 4\gamma^6 (\bar{\beta} \cdot \ddot{\beta})^2 (\bar{S} \times \dot{\beta}) + \gamma^4 (\bar{\beta} \cdot \ddot{\beta})(\bar{S} \times \dot{\beta}) + \gamma^4 (\bar{\beta} \cdot \ddot{\beta}) (\ddot{\beta} \cdot \bar{S}) (\frac{d\bar{S}}{dt} \times \dot{\beta}) \]

As for the second term, we have

\[
\frac{d}{dt} \left[ \gamma^2 (\bar{S} \times \dot{\beta}) \right] = (\bar{S} \times \dot{\beta}) \frac{d\gamma^2}{dt} (\bar{S} \times \dot{\beta}) + \gamma^2 \frac{d}{dt} (\bar{S} \times \dot{\beta}) \]

\[
= 2\gamma^4 (\bar{\beta} \cdot \ddot{\beta})(\bar{S} \times \dot{\beta}) + \gamma^2 \left[ (\frac{d\bar{S}}{dt} \times \dot{\beta}) + (\bar{S} \times \ddot{\beta}) \right].
\]

Putting these results together, we find a rather ugly mess for this term,

\[ g_i \frac{D}{dt} a^\gamma S_{\gamma i} = 4\gamma^6 (\bar{\beta} \cdot \ddot{\beta})^2 (\bar{S} \times \dot{\beta}) + \gamma^4 \left[ (\bar{\beta} \cdot \ddot{\beta} + \ddot{\beta}^2)(\bar{S} \times \dot{\beta}) + 3(\bar{\beta} \cdot \ddot{\beta})(\bar{S} \times \ddot{\beta}) \right. \]

\[ + (\bar{\beta} \cdot \ddot{\beta}) \left( \frac{d\bar{S}}{dt} \times \dot{\beta} \right) + \gamma^2 \left[ (\frac{d\bar{S}}{dt} \times \dot{\beta}) + (\bar{S} \times \ddot{\beta}) \right] \]

\[(3.42)\]
We can really start to see why this covariant notation is nice to use.

### 3.1.7 Equation (3.22d)

Before calculating either the temporal component or the spatial components, we should expand the form seen in the parentheses. Notice that there are the brackets around the subscripts of $\beta$ and $\gamma$, which tells us that this is actually an anti-symmetric sum between the two. The general form to find this is

$$A_{[\alpha B_{\beta]}} = \frac{1}{2!}(A_{\alpha B_{\beta}} - A_{\beta B_{\alpha}}).$$  \hspace{1cm} (3.43)

Since the factor of $2!$ is simply a constant, we can pull it out of the parentheses involving the time derivative and the term becomes

$$2g_{\alpha} \beta \frac{D}{d\tau} \left( u^\delta Q^{\gamma} \left[_{\beta F_{\delta \gamma}} \right] \right) = g_{\alpha} ^{\beta} \frac{D}{d\tau} \left[ u^\delta \left( Q^{\gamma} \left[_{\beta F_{\delta \gamma}} \right] - Q^{\gamma} \left[_{\delta F_{\beta \gamma}} \right] \right) \right].$$  \hspace{1cm} (3.44)
Additionally, we now have our dipole tensor with the second index lowered, while the first index is still raised. We can calculate this using the relation to lower indexes,

\[ Q^\gamma_{\beta} = Q^\gamma_\alpha g_{\alpha\beta} \]

\[
\begin{pmatrix}
0 & p_x & p_y & p_z \\
-p_x & 0 & -\mu_z & \mu_y \\
-p_y & \mu_z & 0 & -\mu_x \\
-p_z & -\mu_y & \mu_x & 0
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(3.45)

Here, we can see that this operation multiplies the 0th column of the \(Q^\alpha_{\beta}\) column by -1 while keeping the magnetic dipole components the same. One might naively expect that it would be the opposite way, but if we look at this through the summation of indexes, we know that for non-zero contributions, \(\alpha = \beta\). We would then have

\[ Q^\gamma_{\beta} = Q^\gamma_0 g_{0\beta} + Q^\gamma_1 g_{1\beta} + Q^\gamma_2 g_{2\beta} + Q^\gamma_3 g_{3\beta}, \]

and so the first term will have to be multiplied by -1. If we look at the order of the indexes, we see \(Q^\gamma_j\), where \(j = 0, 1, 2, 3\). This means that for each component in the 0th column will need to gain the factor of -1, rather than the 0th row.
3.1.8 (3.22d): Temporal Component

For the temporal component, we have $\beta = 0$ as usual, and so we get

$$g_0 \frac{D}{d\tau} u^{(\delta)} (Q^{\gamma}_0 F_{\delta\gamma} - Q^{\gamma}_0 F_{0\gamma}).$$

We will now need to sum over $\delta$ and $\gamma$, so we can do this by specifying the values for $\delta$, sum over $\gamma$, and then add those results together at the end.

1. $\delta = 0$:

$$g_0 \frac{D}{d\tau} [u^0 (Q^{\gamma}_0 F_{0\gamma} - Q^{\gamma}_0 F_{0\gamma})] = 0. \quad (3.46)$$

We can immediately see that this evaluates to zero since we have two terms that are equivalent under the swapping of $\gamma$ and $\delta$.

2. $\delta = 1$:

$$2g_0 \frac{D}{d\tau} [u^1 (Q^{\gamma}_0 F_{1\gamma})] = \frac{D}{d\tau} [u^1 (Q^2_0 F_{12} - Q^2_1 F_{02} + Q^3_0 F_{13} - Q^3_1 F_{03})]$$

$$= \frac{d}{dt} [u^1 (p_y B_z - \mu_z E_y + p_z (-B_y) - (-\mu_y) E_z)]$$

$$= \frac{d}{dt} [u^1 (p_y B_z - p_z B_y + \mu_y E_z - \mu_z E_y)]$$

$$= \frac{d}{dt} (\gamma \beta_x \left[ (\vec{p} \times \vec{B})_x + (\vec{\mu} \times \vec{E})_x \right]). \quad (3.47)$$

Notice that we immediately dropped the terms that involved a diagonal component of either $Q^{\gamma}_\beta$ or $F_{\delta\gamma}$, as all diagonal components are zero, anyway.
3. $\delta = 2$

$$g_0 \frac{D}{d\tau} \left[ u^2(Q^{\gamma} F_{2\gamma} - Q^{\gamma} F_{0\gamma}) \right] = \frac{D}{d\tau} \left[ u^2(Q^1 F_{21} - Q^1 F_{01} + Q^3 F_{23} - Q^3 F_{03}) \right]$$

$$= \gamma \frac{d}{dt} \left[ u^2(p_x(-B_z) - (-\mu_z)E_x + p_z B_x - \mu_x E_z) \right]$$

$$= \gamma \frac{d}{dt} \left[ u^2(p_x B_z - p_x B_z + \mu_z E_x - \mu_x E_z) \right]$$

$$= \gamma \frac{d}{dt} \left( \gamma \beta \left[ (\pmb{p} \times \pmb{B})_z + (\pmb{\mu} \times \pmb{E})_z \right] \right).$$

$$\text{(3.48)}$$

4. $\delta = 3$: Looking to the $\delta = 1$ and $\delta = 2$ cases, we can immediately see that this term will result in

$$g_0 \frac{D}{d\tau} \left[ u^3(Q^{\gamma} F_{3\gamma} - Q^{\gamma} F_{0\gamma}) \right] = \gamma \frac{d}{dt} \left( \gamma \beta \left[ (\pmb{p} \times \pmb{B})_z + (\pmb{\mu} \times \pmb{E})_z \right] \right).$$

$$\text{(3.49)}$$

Now that we have all of the components, we can see that when we add up all of the components, we will have a scalar product between $\pmb{\beta}$ and the vector product of the dipole moments with the fields.

$$2g_0 \frac{D}{d\tau} \left[ u^3(Q^{\gamma} F_{3\gamma} - Q^{\gamma} F_{0\gamma}) \right] = \gamma \frac{d}{dt} \left( \gamma \beta \left[ (\pmb{p} \times \pmb{B}) + (\pmb{\mu} \times \pmb{E}) \right] \right).$$

We have one last piece for the temporal component, and that is to distribute this time derivative. We will keep the terms involving the derivatives of the dipole moments and the fields as $\frac{d}{dt} (\pmb{p} \times \pmb{B})$.

$$\frac{d}{dt} \left[ \gamma \beta \cdot (\pmb{p} \times \pmb{B}) \right] = \gamma^3 \left( \beta \cdot \dot{\beta} \right) \left[ \pmb{\beta} \cdot (\pmb{p} \times \pmb{B}) \right] + \gamma \beta \cdot (\pmb{p} \times \pmb{B}) + \gamma \beta \cdot \frac{d}{dt} (\pmb{p} \times \pmb{B}),$$

$$\frac{d}{dt} \left[ \gamma \beta \cdot (\pmb{\mu} \times \pmb{E}) \right] = \gamma^3 \left( \beta \cdot \dot{\beta} \right) \left[ \pmb{\beta} \cdot (\pmb{\mu} \times \pmb{E}) \right] + \gamma \beta \cdot (\pmb{\mu} \times \pmb{E}) + \gamma \beta \cdot \frac{d}{dt} (\pmb{\mu} \times \pmb{E}).$$
Putting this all back together, we have

\[
2g_0 D \frac{d}{d\tau} \left( u^\delta Q^\gamma \left[ F_\delta \right] \right) = \gamma^4 \left( \vec{\beta} \cdot \vec{\beta} \right) \left[ \vec{\beta} \cdot (\vec{p} \times \vec{B}) + \vec{\beta} \cdot (\vec{\mu} \times \vec{E}) \right] \\
+ \gamma^2 \left[ \vec{\beta} \cdot (\vec{p} \times \vec{B}) + \vec{\beta} \cdot (\vec{\mu} \times \vec{E}) \right] \\
+ \frac{d}{d\tau} \left[ \gamma \left[ (\vec{E} \times \vec{\mu})_x + (\vec{B} \times \vec{p})_x \right] \right].
\]  

(3.50)

3.1.9 (3.22d): Spatial Components

Moving back to the spatial components of the fourth term, we have to now sum over \( \beta \) in addition to \( \delta \) and \( \gamma \). We will start by focusing on the \( \beta = 1 \) terms and attempt to extract the results of the other two components. We will follow the exact same procedure as during the temporal component.

1. \( \delta = 0 \):

\[
2g_1 \frac{D}{d\tau} \left[ u^0 Q^\gamma \left[ F_0 \right] \right] = \frac{D}{d\tau} \left[ u^0 \left( Q^2 F_{02} - Q^2 F_{01} + Q^3 F_{03} - Q^3 F_{13} \right) \right] \\
= \gamma \frac{d}{d\tau} \left[ \gamma \left[ \mu_z E_y - p_y B_z - \mu_y E_z - p_z (-B_y) \right] \right] \\
= \gamma \frac{d}{d\tau} \left[ \gamma \left[ \mu_y E_z - \mu_z E_y + p_z B_y \right] \right] \\
= \gamma \frac{d}{d\tau} \left[ \gamma \left[ (\vec{E} \times \vec{\mu})_x + (\vec{B} \times \vec{p})_x \right] \right] 
\]

(3.51)

2. \( \delta = 1 \):

\[
2g_1 \frac{D}{d\tau} \left[ u^1 Q^\gamma \left[ F_1 \right] \right] = 0.
\]

(3.52)
This is something that we had seen in temporal component of this term, as the interchange of the same index results in a difference of zero.

3. \( \delta = 2 \):

\[
2g_1 \frac{D}{d\tau} \left[ u^2 Q^\gamma [1 F_2] \right] = \frac{D}{d\tau} \left[ u^2 (Q^0_1 F_{20} - Q^0_2 F_{10} + Q^3_1 F_{23} - Q^3_2 F_{13}) \right] \\
= \gamma \frac{d}{dt} \left[ \gamma_\beta \beta_y (p_x (-E_y) - p_y (-E_x) + (-\mu_y) B_x - \mu_x (-B_y)) \right] \\
= \gamma \frac{d}{dt} (\gamma_\beta \beta_y [p_y E_x - p_x E_y + B_y \mu_x - \mu_y B_x]) \\
= \gamma \frac{d}{dt} \left( \gamma_\beta \beta_y \left( \vec{E} \times \vec{p} \right)_y + \left( \vec{B} \times \vec{\mu} \right)_y \right). 
\]

(3.53)

4. \( \delta = 3 \):

\[
2g_1 \frac{D}{d\tau} \left[ u^3 Q^\gamma [1 F_3] \right] = \frac{D}{d\tau} \left[ u^3 (Q^0_1 F_{30} - Q^0_3 F_{10} + Q^2_1 F_{32} - Q^2_3 F_{12}) \right] \\
= \gamma \frac{d}{dt} \left[ \gamma_\beta \beta_z (p_x (-E_z) - p_z (-E_x) + \mu_z (-B_x) - (-\mu_x) B_z) \right] \\
= \gamma \frac{d}{dt} \left( \gamma_\beta \beta_z \left( \vec{p} \times \vec{E} \right)_y + \left( \vec{B} \times \vec{\mu} \right)_y \right). 
\]

(3.54)

Looking at (3.49) and (3.50), we naively may have a discrepancy. However, this is okay, since we can rewrite each of these results in a form that helps us, using the property

\[
\vec{A} \times \vec{B} = -\left( \vec{B} \times \vec{A} \right). 
\]
When we add these two results together then, we apply this property and come out to the result (after going through the rigors of adding in the other two components for $\beta$)

$$2g_i \frac{D}{d\tau} (u^\delta Q^\gamma [i F_\delta]_\gamma) = \gamma \frac{d}{dt} (\gamma \left[ (\vec{E} \times \vec{\mu}) + (\vec{B} \times \vec{p}) \right. + \gamma \vec{\beta} \times (\vec{E} \times \vec{p}) + \gamma \vec{\beta} \times (\vec{B} \times \vec{\mu}) \left. \right]).$$

Lastly, we can again distribute the time derivative to get

$$2g_i \frac{D}{d\tau} (u^\delta Q^\gamma [i F_\delta]_\gamma) = \gamma^4 \left( \vec{\beta} \cdot \vec{\beta} \right) \left[ (\vec{E} \times \vec{\mu}) + (\vec{B} \times \vec{p}) \right] + \gamma^2 \left[ \frac{d}{dt} (\vec{E} \times \vec{\mu}) + \frac{d}{dt} (\vec{B} \times \vec{p}) \right]$$

$$+ \gamma^4 \left( \vec{\beta} \cdot \vec{\beta} \right) \left[ \vec{\beta} \times (\vec{E} \times \vec{p}) + \vec{\beta} \times (\vec{B} \times \vec{\mu}) \right] + \gamma^2 \left[ \vec{\beta} \cdot \frac{d}{dt} (\vec{E} \times \vec{p}) + \vec{\beta} \times (\vec{B} \times \vec{\mu}) \right].$$

### 3.2 Each Term With $u_\alpha u^\beta$

The terms that we now need to calculate are:

$$\frac{2}{3} q^2 u_\alpha u^\beta \frac{D}{d\tau} a_\beta$$  \hspace{1cm} (3.57a)

$$-\frac{1}{2} u_\alpha u^\beta Q^\gamma \nabla_\beta F_\gamma$$  \hspace{1cm} (3.57b)

$$u_\alpha u^\beta \frac{D}{d\tau} a^\gamma S_{\beta\gamma}$$  \hspace{1cm} (3.57c)

$$-2u_\alpha u^\beta \frac{D}{d\tau} u^\delta Q^\gamma [i F_\delta]_\gamma$$  \hspace{1cm} (3.57d)
Notice that we no longer have the necessity of $\alpha = \beta$ in any of these terms, as each component of the 4-velocity is non-zero.

### 3.2.1 Equation (3.57a): Temporal Component

As per usual, we start with the temporal component, which means $\alpha = 0$. However, notice that $\beta$ is not required to be the same as $\alpha$, and so we are now required to sum over $\beta$. Additionally, remember that we need to follow the rules of raising and lowering indexes. This will not really affect the spatial components of $u^\alpha$, but when lowering the index, the first component will gain a negative sign.

\[
\frac{2}{3}q^2 u_0 u^\beta \frac{D}{d\tau} a_{\beta} = \frac{2q^2}{3}(-\gamma) \left[ u_0 \frac{D}{d\tau} a_0 + u^1 \frac{D}{d\tau} a_1 + u^2 \frac{D}{d\tau} a_2 + u^3 \frac{D}{d\tau} a_3 \right]
\]

\[
= -\frac{2q^2}{3} \gamma \left( \gamma \frac{D}{d\tau} \left[ -\gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \right] + \gamma_\beta x \frac{D}{d\tau} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \dot{\beta}_x + \gamma^2 \ddot{\beta}_x \right] \right.
\]
\[
+ \gamma_\beta y \frac{D}{d\tau} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \dot{\beta}_y + \gamma^2 \ddot{\beta}_y \right] \left. + \gamma_\beta z \frac{D}{d\tau} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \dot{\beta}_z + \gamma^2 \ddot{\beta}_z \right] \right)
\]

In order to write this a little better, let’s take the derivatives of the first two terms, since the $y$ and $z$ components will simply follow the $x$ component, resulting in scalar products.

\[
\gamma \frac{d}{dt} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \right] = 4\gamma^7 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 + \gamma^5 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \dot{\vec{\beta}}^2 \right),
\]

\[
\gamma \frac{d}{dt} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\beta}_x + \gamma^2 \ddot{\beta}_x \right) \right] = 4\gamma^7 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \beta_x + \gamma^5 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \dot{\vec{\beta}}^2 \right) \beta_x + \gamma^5 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \dot{\beta}_x
\]
\[
+ 2\gamma^5 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_x + \gamma^3 \ddot{\beta}_x
\]
\[
= 4\gamma^7 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \beta_x + \gamma^5 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \dot{\vec{\beta}}^2 \right) \beta_x + 2\gamma^5 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_x
\]
\[
+ \gamma^3 \ddot{\beta}_x.
\]
Now, we multiply this second line by $\gamma \beta_z$ (and subsequently for the $y$ and $z$ components again) to get

$$u^\beta \frac{D}{d\tau} a_\beta = -4\gamma^8 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 - \gamma^6 (\vec{\beta} \cdot \vec{\ddot{\beta}} + \vec{\beta}^2) + 4\gamma^8 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 (\vec{\beta} \cdot \vec{\ddot{\beta}})
+ \gamma^6 (\vec{\beta} \cdot \vec{\beta} + \vec{\beta}^2) (\vec{\beta} \cdot \vec{\ddot{\beta}}) + 3\gamma^6 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 + \gamma^4 (\vec{\beta} \cdot \vec{\ddot{\beta}}).$$

Here, we see a couple of terms that seem to match well, so let’s play around with those. The first two terms that we can see match go as

$$-4\gamma^8 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 + 4\gamma^8 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 (\vec{\beta} \cdot \vec{\ddot{\beta}}) = 4\gamma^8 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 [\vec{\beta} \cdot \vec{\beta} - 1].$$

If we look at that quantity in the brackets, we see that this is actually a quantity known to us as

$$-\frac{1}{\gamma^2} = \vec{\beta} \cdot \vec{\beta} - 1,$$

and so we lose a factor of $\gamma^2$ in the coefficient. We can do the same with the two terms with a $\gamma^6$ in the coefficient, so that we get

$$u^\beta \frac{D}{d\tau} a_\beta = -\gamma^6 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 - \gamma^4 (\vec{\beta} \cdot \vec{\ddot{\beta}} + \vec{\beta}^2) + 3\gamma^6 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 + \gamma^4 (\vec{\beta} \cdot \vec{\ddot{\beta}})
= 2\gamma^6 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 - \gamma^4 (\vec{\beta} \cdot \vec{\ddot{\beta}})$$

Finally, we can put this back into the overall term to get

$$\frac{2}{3} q^2 u_0 u^\beta \frac{D}{d\tau} a_\beta = -\frac{2q^2 \gamma^5}{3} \left[ 2\gamma^2 (\vec{\beta} \cdot \vec{\dot{\beta}})^2 - (\vec{\beta} \cdot \vec{\ddot{\beta}}) \right].$$

It is rather interesting that the time derivative of the acceleration in fact ended up having only velocity and acceleration factors in the vector-form.
To simplify this even further, let's go back to Equation (3.28), which gave us

\[
\frac{2}{3} q^2 g_0 \frac{D}{d\tau} a_0 = - \frac{2q^2\gamma^5}{3} \left[ 4\gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 + (\vec{\beta} \cdot \dot{\vec{\beta}}) + (\vec{\beta} \cdot \ddot{\vec{\beta}}) \right].
\]

When we put these two equations together, we come out the temporal component of the overall ALD force terms

\[
\frac{2}{3} q^2 \left( g_0 \beta + u_\alpha u_\beta \right) \frac{D}{d\tau} a_\beta = - \frac{2q^2\gamma^5}{3} \left[ 4\gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 + (\vec{\beta} \cdot \dot{\vec{\beta}}) + (\vec{\beta} \cdot \ddot{\vec{\beta}}) \right].
\] (3.61)

3.2.2 Equation (3.57a): Spatial Components

Now that we are onto the spatial components, we will need to consider all components possible. To do this, we will specifically look at the $\alpha = 1$ component and summing over $\beta$ just as before. Additionally, we will ignore the coefficient of $\frac{2q^2}{3}$, as we know that it will not change, and we will just put it back at the end. When $\alpha = 1$, we get something very similar to the temporal component,

\[
\gamma \beta_x \left[ u_1 u_\beta \frac{D}{d\tau} a_\beta = \gamma \beta_x \left[ u_0 \frac{D}{d\tau} a_0 + u_1 \frac{D}{d\tau} a_1 + u_2 \frac{D}{d\tau} a_2 + u_3 \frac{D}{d\tau} a_3 \right] \right].
\] (3.62)

Notice that we have already calculated this quantity in brackets above as the result of Equation (3.59). We then put the factor of $\gamma \beta_x$ out in front to get

\[
\gamma \beta_x \left[ u_1 u_\beta \frac{D}{d\tau} a_\beta = 2\gamma^7 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 - \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x. \right.
\] (3.63)
If we were to then add up all of the spatial components, we would simply have the vector $\vec{\beta}$ out front, rather than the individual components. Putting in the coefficient as well, this gives us

$$\frac{2}{3} q^2 u_i u^\beta \frac{D}{d\tau} a_{\beta} = \frac{2 q^2 \gamma^5}{3} \vec{\beta} \left[ 2 \gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 - \left( \dot{\vec{\beta}} \cdot \dot{\vec{\beta}} \right) \right].$$  \hspace{1cm} (3.64)

Now, we can take the result from (3.35) and add it to the result that we have found here. In so doing, we find that only the first two terms combine nicely, leaving us with

$$\frac{2}{3} q^2 (g_i \beta + u_i u^\beta) \frac{D}{d\tau} a_{\beta} = \frac{2 q^2 \gamma^3}{3} \left[ 6 \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \vec{\beta} + \gamma^2 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} \right) \vec{\beta} + 3 \gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \ddot{\vec{\beta}} + \dddot{\vec{\beta}} \right].$$  \hspace{1cm} (3.65)

### 3.2.3 Equation (3.57b): Temporal Component

Now that we are onto our sixth term, we see that we have three indexes to sum over now, $\beta$, $\gamma$, and $\delta$. As per usual, we will start with $\beta = 0$ and sum over the remaining indexes. Within the specification of each $\beta$, we will also specify each $\gamma$, sum over $\delta$, add the results back up, and move down the line.

- $\beta = 0$:

$$-\frac{1}{2} u_0 u^0 Q^\gamma \nabla_0 F_{\gamma \delta} = \frac{1}{2} \gamma^2 Q^\gamma \nabla_0 F_{\gamma \delta}$$
$-\gamma = 0$

\[
\frac{1}{2} \gamma^2 Q^{00} \nabla_0 F_{00} = \frac{1}{2} \gamma^2 [Q^{00} \nabla_0 F_{00} + Q^{01} \nabla_0 F_{01} + Q^{02} \nabla_0 F_{02} + Q^{03} \nabla_0 F_{03}]
\]

\[
= \frac{1}{2} \gamma^2 \left[ p_x \frac{\partial E_x}{\partial t} + p_y \frac{\partial E_y}{\partial t} + p_z \frac{\partial E_z}{\partial t} \right]
\]

\[
= \frac{1}{2} \gamma^2 \left( \vec{p} \cdot \frac{\partial \vec{E}}{\partial t} \right)
\]

$-\gamma = 1$

\[
\frac{1}{2} \gamma^2 Q^{10} \nabla_0 F_{10} = \frac{1}{2} \gamma^2 (Q^{10} \nabla_0 F_{10} + Q^{11} \nabla_0 F_{11} + Q^{12} \nabla_0 F_{12} + Q^{13} \nabla_0 F_{13})
\]

\[
= \frac{1}{2} \gamma^2 \left( p_x \frac{\partial E_x}{\partial t} + (-\mu_z) \frac{\partial B_z}{\partial t} + \mu_z \frac{\partial (-B_z)}{\partial t} \right).
\]

$-\gamma = 2$

\[
\frac{1}{2} \gamma^2 Q^{20} \nabla_0 F_{20} = \frac{1}{2} \gamma^2 (Q^{20} \nabla_0 F_{20} + Q^{21} \nabla_0 F_{21} + Q^{22} \nabla_0 F_{22} + Q^{23} \nabla_0 F_{23})
\]

\[
= \frac{1}{2} \gamma^2 \left( p_y \frac{\partial E_y}{\partial t} - \mu_z \frac{\partial B_z}{\partial t} - \mu_x \frac{\partial B_x}{\partial t} \right)
\]

$-\gamma = 3$

\[
\frac{1}{2} \gamma^2 Q^{30} \nabla_0 F_{30} = \frac{1}{2} \gamma^2 (Q^{30} \nabla_0 F_{30} + Q^{31} \nabla_0 F_{31} + Q^{32} \nabla_0 F_{32} + Q^{33} \nabla_0 F_{33})
\]

\[
= \frac{1}{2} \gamma^2 \left( p_z \frac{\partial E_z}{\partial t} - \mu_y \frac{\partial B_y}{\partial t} - \mu_x \frac{\partial B_x}{\partial t} \right)
\]

As we can see when $\beta = 0$, once we add all of the components up, we get

\[
-\frac{1}{2} u_0 u^0 Q^{\gamma \delta} \nabla_0 F_{\gamma \delta} = \gamma^2 \left[ \vec{p} \cdot \frac{\partial \vec{E}}{\partial t} - \vec{\mu} \cdot \frac{\partial \vec{B}}{\partial t} \right].
\]

$\beta = 1$
– $\gamma = 0$:

$$\left( \frac{1}{2} u_0^2 \nabla_0 F_{05} \right)^\gamma = \frac{1}{2} \gamma^2 \beta_x \left[ \frac{Q_{00} \nabla_0 F_{00} + Q_{01} \nabla_1 F_{01} + Q_{02} \nabla_0 F_{02} + Q_{03} \nabla_0 F_{03}}{2} \right]$$

$$= \frac{1}{2} \gamma^2 \beta_x \left[ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial x} + \frac{\partial E_z}{\partial x} \right]$$

– $\gamma = 1$:

$$\left( \frac{1}{2} u_0^2 \nabla_1 F_{15} \right)^\gamma = \frac{1}{2} \gamma^2 \beta_x \left[ \frac{Q_{10} \nabla_0 F_{10} + Q_{11} \nabla_1 F_{11} + Q_{12} \nabla_0 F_{12} + Q_{13} \nabla_0 F_{13}}{2} \right]$$

$$= \frac{1}{2} \gamma^2 \beta_x \left[ \frac{\partial E_x}{\partial x} - \frac{\partial B_z}{\partial x} - \frac{\partial B_y}{\partial x} \right]$$

– $\gamma = 2$:

$$\left( \frac{1}{2} u_0^2 \nabla_2 F_{25} \right)^\gamma = \frac{1}{2} \gamma^2 \beta_x \left[ \frac{Q_{20} \nabla_1 F_{20} + Q_{21} \nabla_1 F_{21} + Q_{22} \nabla_1 F_{22} + Q_{23} \nabla_1 F_{23}}{2} \right]$$

$$= \frac{1}{2} \gamma^2 \beta_x \left[ \frac{\partial E_y}{\partial x} - \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial x} \right]$$

– $\gamma = 3$: Thanks to the help of the previous terms, it is clear that the final component comes out to:

$$\left( \frac{1}{2} u_0^2 \nabla_3 F_{35} \right)^\gamma = \frac{1}{2} \gamma^2 \beta_x \left( \frac{\partial E_z}{\partial x} - \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial x} \right)$$

From the $\beta = 1$ case, then, it is clear that the next two components are just going to result in a scalar product between the velocity and the spatial derivatives. When we add this to the $\beta = 0$ component, though, we get exactly the identity of a total time derivative, and so we write

$$\left( \frac{1}{2} u_0^2 \nabla_\beta F_{35} \right)^\gamma = \gamma^2 \left( \vec{p} \cdot \frac{d\vec{E}}{dt} \right) - \gamma^2 \left( \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right). \quad (3.67)$$
Remember that the lowering of the velocity’s index out front is what has given us the relative minus sign. So, if we were to compare this to the result of (3.36), it might seem that there was a mistake made, due to the change in the signs, but we can rest assured that there has not been.

### 3.2.4 Equation (3.57b): Spatial Components

We again start with $\alpha = 1$. Notice that the only component that gains a negative sign when lowering the index of the velocity is the temporal component, so none of these will cancel the negative sign out front automatically. If we look at the general form of the $u_\alpha = u_1$ component, we get

$$\frac{1}{2} u_1 u^\beta Q^{\gamma \delta} \nabla_\beta F_{\gamma \delta},$$

which will be basically the same as the temporal component that we calculated in the previous section, just with an extra factor of $\beta_0$ out front. And this should also hold true for the other two components as well, which means that we will get full time derivatives of the fields dotted with the dipole moments. However, with the factor of $-\frac{1}{2}$ out front, the signs will change relative to the result in (3.67). This means that we will get

$$-\frac{1}{2} u_i u^\beta Q^{\gamma \delta} \nabla_\beta F_{\gamma \delta} = \gamma^2 \beta^2 \left[ (\vec{\mu} \cdot \frac{d\vec{B}}{dt}) - (\vec{p} \cdot \frac{d\vec{E}}{dt}) \right],$$

(3.69)
We can then add this to the result in (3.37) to get

\[-\frac{1}{2}(g_i^\beta + u_i^\alpha)Q^{\gamma\delta}\nabla_\beta F_{\gamma\delta} = p_i \vec{\nabla} E_i + \mu_i \vec{\nabla} B_i\]

\[+ \gamma^2 \vec{\beta} \left[ \left( \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right) - \left( \vec{p} \cdot \frac{d\vec{E}}{dt} \right) \right]. \quad (3.70)\]

### 3.2.5 Equation (3.57c): Temporal Component

By this point, the process is clear. We will set \( \alpha = 0 \), start with \( \beta = 0 \) and sum over \( \gamma \).

\[u_0 u_3^\beta \frac{D}{d\tau} a^\gamma S_{\beta\gamma} = \gamma u_3^\beta \frac{D}{d\tau} a^\gamma S_{\beta\gamma}. \quad (3.71)\]

Conveniently, we actually do not have to concern ourselves with the \( \beta = 0 \) component, as that contributes nothing since all temporal components of the spin tensor are zero. We then move to the \( \beta = 1 \) component

1. \( \beta = 1 \):

\[
\gamma u_1^1 \frac{D}{d\tau} a^\gamma S_{1\gamma} = \gamma u_1^1 \frac{D}{d\tau} \left[ a_0^0 S_{10} + a_1^1 S_{11} + a_2^2 S_{12} + a_3^3 S_{13} \right]
= \gamma^3 \beta_x \frac{d}{dt} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_y S_z + \gamma^2 \beta_y S_z \right.
\left. + \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_z (-S_y) + \gamma^2 \beta_z (-S_y) \right]
= \gamma^3 \beta_x \frac{d}{dt} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) (\beta_y S_z - \beta_z S_y) + \gamma^2 (\beta_y S_z - \beta_z S_y) \right]
= \gamma^3 \beta_x \frac{d}{dt} \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left( \vec{\beta} \times \vec{S} \right)_x + \gamma^2 \left( \dot{\vec{\beta}} \times \vec{S} \right)_x \right]
\]
Doing the same thing for the $\beta = 2$ and $\beta = 3$ components, we will find

$$
\gamma u^2 \frac{D}{d\tau} a^2 S_{2\gamma} = \gamma^3 \beta u \frac{d}{dt} \left[ \gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \times \vec{S})_y + \gamma^2 (\vec{\beta} \times \vec{S})_y \right],
$$

$$
\gamma u^3 \frac{D}{d\tau} a^2 S_{3\gamma} = \gamma^3 \beta z \frac{d}{dt} \left[ \gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \times \vec{S})_z + \gamma^2 (\vec{\beta} \times \vec{S})_z \right].
$$

Adding these up will result in a scalar product between the velocity out front and the vector product in the time derivative, as expected.

Before continuing, let’s go ahead and again distribute the time derivative. By this point, we know what the values of each of these derivatives are, so we have

$$
u_0 u^3 \frac{D}{d\tau} a^2 S_{\beta\gamma} = \gamma^3 \left\{ 4 \gamma^6 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) + \gamma^4 (\vec{\beta} \cdot \ddot{\vec{\beta}} + \ddot{\vec{\beta}}^2) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) 
+ \gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}] + \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d\vec{S}}{dt} \right])
+ 2 \gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) + \gamma^2 (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}] + \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d\vec{S}}{dt} \right]) \right\}. 
$$

Combining the two like-terms, this becomes

$$
u_0 u^3 \frac{D}{d\tau} a^2 S_{\beta\gamma} = \gamma^5 \left\{ 4 \gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) + \gamma^2 (\vec{\beta} \cdot \ddot{\vec{\beta}} + \ddot{\vec{\beta}}^2) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}])
+ 3 \gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) + \gamma^2 (\vec{\beta} \cdot \ddot{\vec{\beta}}) (\vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d\vec{S}}{dt} \right])
+ \vec{\beta} \cdot [\vec{\beta} \times \vec{S}] + \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d\vec{S}}{dt} \right] \right\}. 
$$

(3.72)
3.2.6 Equation (3.57c): Spatial Components

Moving onto the spatial components, we again start at $u_{\alpha} = u_1$, setting the value for $\beta$, summing over $\gamma$, and then reconstruct the components. Again, when $\beta = 0$, all components will be zero, and so we can move onto $\beta = 1$ immediately. When we look to this, we have

$$u_1 u^1 \frac{D}{d\tau} a^\gamma S_{1\gamma} = \gamma^3 \beta_x \left[ \beta_x \frac{d}{dt} \left( a^{0} S_{10} + a^{1} S_{11} + a^2 S_{12} + a^3 S_{13} \right) \right],$$

this is exactly the same as the first line in the $\beta = 1$ for the temporal component, with just an extra factor of $\beta_x$ out front. So, by inspection, we would find that the result of the spatial components will be the same as the temporal component, with a $\vec{\beta}$ out front.

$$u_i u^i \frac{D}{d\tau} a^\gamma S_{\beta\gamma} = \gamma^5 \vec{\beta} \left\{ 4 \gamma^4 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} \right) \left( \vec{\beta} \cdot \left[ \vec{\beta} \times \vec{S} \right] \right) + \gamma^2 \left( \vec{\beta} \cdot \dddot{\vec{\beta}} + \ddot{\vec{\beta}}^2 \right) \left( \vec{\beta} \cdot \left[ \vec{\beta} \times \vec{S} \right] \right) \\
+ 3 \gamma^2 \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{\beta} \cdot \left[ \vec{\beta} \times \vec{S} \right] \right) + \gamma^2 \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d\vec{S}}{dt} \right] \right) \\
+ \vec{\beta} \cdot \left[ \vec{\beta} \times \vec{S} \right] + \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d\vec{S}}{dt} \right] \right\}$$

(3.73)
3.2.7 Equation (3.57d): Temporal Component

We are finally onto the last term that needs to be written in vector form. Again, we focus on the temporal component and then move onto the spatial components.

\[-2 u_0 u^\beta \frac{D}{d\tau} \left[ u^\delta Q^{\gamma}_{\beta \delta \gamma} F_{\delta \gamma} \right] = \gamma u^\beta \frac{D}{d\tau} \left[ u^\delta \left( Q^{\gamma}_{\beta \delta \gamma} F_{\delta \gamma} - Q^{\gamma}_{\beta \gamma} F_{\beta \gamma} \right) \right]. \tag{3.74}\]

Notice that we have lost the negative sign, which is due to the lower index of the velocity out front. When we start with \(\beta = 0\), we have already seen how the entire term goes to zero when \(\delta = 0\) as well, and so we start on \(\beta = 0, \delta = 1\).

1. \(\delta = 1\):

\[\gamma u^0 \frac{D}{d\tau} \left[ u^1 (Q^{\gamma}_{0 F_{1\gamma}} - Q^{\gamma}_{1 F_{0\gamma}}) \right] = \gamma^3 \frac{d}{dt} \left[ \gamma \beta_x (Q^{2}_{0 F_{12}} - Q^{2}_{1 F_{02}} + Q^{3}_{0 F_{13}} - Q^{3}_{1 F_{03}}) \right] \]
\[= \gamma^3 \frac{d}{dt} \left[ \gamma \beta_x (p_y B_z - \mu_z E_y + p_z (-B_y) - (-\mu_y) E_z) \right] \]
\[= \gamma^3 \frac{d}{dt} \left[ (\mathbf{p} \times \mathbf{B})_x + (\mathbf{\mu} \times \mathbf{E})_x \right] \]

2. \(\delta = 2\):

\[\gamma u^0 \frac{D}{d\tau} \left[ u^2 (Q^{\gamma}_{0 F_{2\gamma}} - Q^{\gamma}_{2 F_{0\gamma}}) \right] = \gamma^3 \frac{d}{dt} \left[ \gamma \beta_y (Q^{1}_{0 F_{21}} - Q^{1}_{2 F_{02}} + Q^{3}_{0 F_{23}} - Q^{3}_{2 F_{03}}) \right] \]
\[= \gamma^3 \frac{d}{dt} \left( \gamma \beta_y (p_x (-B_z) - (-\mu_z) E_y + p_z B_x - \mu_x E_z) \right) \]
\[= \gamma^3 \frac{d}{dt} \left( \gamma \beta_y \left[ (\mathbf{p} \times \mathbf{B})_y + (\mathbf{\mu} \times \mathbf{E})_y \right] \right). \]
3. $\delta = 3$:

$$
\gamma u^0 D_\tau [u^3 (Q^\gamma_0 F_{3\gamma} - Q^\gamma_0 F_{0\gamma})] = \gamma^3 \frac{d}{dt} \left[ \gamma \beta_x (Q^1_0 F_{31} - Q^1_0 F_{01} + Q^2_0 F_{32} - Q^2_0 F_{03}) \right] \\
= \gamma^3 \frac{d}{dt} \left( \gamma \beta_x \left[ (\vec{p} \times \vec{B})_z + (\vec{\mu} \times \vec{E})_z \right] \right)
$$

When we add all of this together, we get a nice result of

$$
\gamma u^0 D_\tau [u^3 (Q^\gamma_0 F_{3\gamma} - Q^\gamma_0 F_{0\gamma})] = \gamma^3 \frac{d}{dt} \left[ \gamma \beta \cdot \left( \vec{p} \times \vec{B} \right) + \gamma \beta \cdot \left( \vec{\mu} \times \vec{E} \right) \right]. \quad (3.75)
$$

However, we are not quite done with the temporal component, as we still have to find the components corresponding to $\beta = i$. Because of this, we will wait to take the time derivative until the very end of this part.

We will start the next components at the natural starting point, $\beta = 1$ again.

1. $\delta = 0$:

$$
\gamma u^1 D_\tau [u^0 (Q^\gamma_1 F_{0\gamma} - Q^\gamma_0 F_{1\gamma})] = \gamma^3 \beta_x \frac{d}{dt} \left[ \gamma \left( Q^2_1 F_{02} - Q^2_0 F_{12} + Q^3_1 F_{03} - Q^3_0 F_{13} \right) \right] \\
= \gamma^3 \beta_x \frac{d}{dt} \left( \gamma \left[ (\vec{E} \times \vec{\mu})_x + (\vec{B} \times \vec{p})_x \right] \right).
$$

Notice that this is extremely similar to the results that we had for $\beta = 0$, only that the velocity has been pulled out the time derivative, and the vector products have swapped order. This will play a nice part in the final stages, when we add everything together.
2. $\delta = 1$:

$$\gamma u^1 \frac{D}{d\tau} [u^1 (Q^\gamma_1 F_1 - Q^\gamma_1 F_1)] = 0.$$ 

3. $\delta = 2$:

$$\gamma u^1 \frac{D}{d\tau} [u^2 (Q^\gamma_1 F_2 - Q^\gamma_2 F_1)] = \gamma^3 \beta_x \frac{d}{dt} \left[ \gamma \beta_y \left( Q^0_1 F_{20} - Q^0_2 F_{10}+Q^3_1 F_{23} - Q^3_2 F_{13} \right) \right]$$

$$= \gamma^3 \beta_x \frac{d}{dt} \left( \gamma \beta_y \left[ p_x (-E_y) - p_y (-E_x) 
+ (-\mu_y) B_x - \mu_x (-B_y) \right] \right)$$

$$= \gamma^3 \beta_x \frac{d}{dt} \left( \gamma \beta_y \left[ \left( \vec{E} \times \vec{p} \right)_z + \left( \vec{\mu} \times \vec{B} \right)_z \right] \right)$$

4. $\delta = 3$:

$$\gamma u^1 \frac{D}{d\tau} [u^3 (Q^\gamma_1 F_3 - Q^\gamma_3 F_1)] = \gamma^3 \beta_x \frac{d}{dt} \left[ \gamma \beta_y \left( Q^0_1 F_{30} - Q^0_3 F_{10}+Q^2_1 F_{32} - Q^2_3 F_{12} \right) \right]$$

$$= \gamma^3 \beta_x \frac{d}{dt} \left( \gamma \beta_y \left[ p_x (-E_z) - p_z (-E_x) 
+ \mu_z (-B_x) - (-\mu_x) B_y \right] \right)$$

$$= \gamma^3 \beta_x \frac{d}{dt} \left( \gamma \beta_y \left[ -\left( \vec{E} \times \vec{p} \right)_y - \left( \vec{\mu} \times \vec{B} \right)_y \right] \right)$$
We add these components together and find this will end up in a scalar product with a triple cross product. The other $\beta = i$ components have exactly the same form, just with the $y$ and $z$ components, and so we can write

$$u_0 u^j \frac{D}{d\tau} [u^\delta (Q_{\gamma i} F_{\delta \gamma} - Q_{\delta i} F_{\gamma \delta})] = \gamma^3 \vec{\beta} \cdot \left[ \frac{d}{dt} (\gamma \vec{\beta} \times \vec{E}) \right] + \gamma^4 \vec{\beta} \cdot \left( \frac{d}{dt} [\gamma (\vec{B} \times \vec{p})] + \frac{d}{dt} [\gamma (\vec{E} \times \vec{p})] \right)$$

(3.76)

We then add (3.76) to the $\beta = 0$ component in (3.75). Before we do so, let’s just take the time derivative of (3.75), which will allow us to immediately cancel some things once we put it all together. We will leave out the factor of $\gamma^3$ for the time-being and put it back in at the end, since it is not under the time derivative.

$$\frac{d}{dt} [\gamma \vec{\beta} \cdot (\vec{p} \times \vec{B}) + \gamma \vec{\beta} \cdot (\vec{\mu} \times \vec{E})] = \vec{\beta} \cdot \left( \frac{d}{dt} [\gamma (\vec{p} \times \vec{B})] + \frac{d}{dt} [\gamma (\vec{\mu} \times \vec{E})] \right) + \left[ \gamma (\vec{p} \times \vec{B}) + \gamma (\vec{\mu} \times \vec{E}) \right] \cdot \dot{\vec{\beta}}.$$  

When applying the product rule, we have only kept it in two terms for the simple reason that when we put the factor of $\gamma^3$ back out front, we see that the first term in the equation above cancels completely with the second term in (3.76), thanks to the anti-symmetric property of vector products. With this in mind, we get

$$u_0 u^\delta \frac{D}{d\tau} [u^\gamma (Q_{\beta \gamma} F_{\delta \beta} - Q_{\delta \beta} F_{\gamma \beta})] = \gamma^3 \vec{\beta} \cdot \left[ \frac{d}{dt} (\gamma \vec{\beta} \times \vec{E}) \right] + \gamma^4 \vec{\beta} \cdot \left( \vec{p} \times \vec{B} \right) + \gamma^4 \vec{\beta} \cdot \left( \vec{\mu} \times \vec{E} \right).$$
The last task is to distribute the derivatives inside the brackets in the first term, which gives us

\[-2u_0 u^\beta \frac{D}{d\tau} (u^\delta Q^\gamma_{\beta\delta\gamma}) = \gamma^4 \tilde{\beta} \cdot \left( \tilde{\beta} \times \left[ \vec{E} \times \vec{p} \right] + \tilde{\beta} \times \left[ \vec{\mu} \times \vec{B} \right] \right) + \gamma^5 \left( \tilde{\beta} \cdot \tilde{\beta} \right) \tilde{\beta} \cdot \left( \tilde{\beta} \times \left[ \vec{E} \times \vec{p} \right] + \tilde{\beta} \times \left[ \vec{\mu} \times \vec{B} \right] \right) + \gamma^4 \tilde{\beta} \cdot \left( \tilde{p} \times \vec{B} \right) + \gamma^4 \tilde{\beta} \cdot \left( \tilde{\mu} \times \vec{E} \right).\]

(3.77)

### 3.2.8 Equation (3.57d): Spatial Components

Now onto the final piece of the puzzle before we put everything together. As usual, we start with \( \alpha = 1 \), but this time, we will immediately reverse the order of the subtraction, as to cancel out the negative sign out front.

\[2u_\alpha u^\beta \frac{D}{d\tau} \left[ u^\delta Q^\gamma_{\beta\delta\gamma} \right] = u_\alpha u^\beta \frac{D}{d\tau} \left[ u^\delta (Q^\gamma_{\beta\delta} - Q^\gamma_{\beta\delta}F^\gamma_{\beta\delta}) \right].\]

(3.78)

For \( u_1 = \gamma \beta_x \), we then set \( \beta = 0 \), which immediately sets \( \delta = 0 \) to zero, and so we have

1. \( \beta = 0 \):
(a) \( \delta = 1 \):

\[
\gamma^3 \beta_x \frac{d}{dt} \left[ u^1 (Q^\gamma_{1} F_{0\gamma} - Q^\gamma_{0} F_{1\gamma}) \right] = \gamma^3 \beta_x \frac{d}{dt} \left[ \gamma \beta_x \left( Q^2_{1} F_{02} - Q^2_{1} F_{12} + Q^3_{1} F_{03} - Q^3_{1} F_{13} \right) \right] \\
= \gamma^3 \beta_x \frac{d}{dt} \left( \gamma \beta_x \left[ \mu_z E_y - p_y B_z + (-\mu_y) E_z - p_z (-B_y) \right] \right) \\
= \gamma^3 \beta_x \frac{d}{dt} \left( \gamma \beta_x \left[ (\vec{E} \times \vec{\mu})_x + (\vec{B} \times \vec{p})_x \right] \right)
\]

We can do the same for \( \delta = 2 \) and \( \delta = 3 \) to get full scalar products on the inside of the brackets,

\[
-2 u_1 u^0 \frac{D}{dt} \left[ u^\delta Q^\gamma_{[i} F_{\delta \gamma]} \right] = \gamma^3 \beta_x \frac{d}{dt} \left[ \gamma \beta \cdot (\vec{E} \times \vec{\mu}) + \gamma \beta \cdot (\vec{B} \times \vec{p}) \right]. \quad (3.79)
\]

Before moving onto \( \beta = i \), let’s just sum over the remaining values of \( \alpha \). It is clear that when doing so will just give the total velocity vector out front, or

\[
-2 u_1 u^0 \frac{D}{dt} \left[ u^\delta Q^\gamma_{[\beta} F_{\delta \gamma]} \right] = \gamma^5 \beta \left( \frac{d}{dt} \left[ \gamma \beta \cdot (\vec{E} \times \vec{\mu}) + \gamma \beta \cdot (\vec{B} \times \vec{p}) \right] \right) \\
= \gamma^5 \beta \left( \beta \cdot \frac{d}{dt} \left[ \vec{E} \times \vec{\mu} \right] + \beta \cdot \frac{d}{dt} \left[ \vec{B} \times \vec{p} \right] \right) + \gamma^4 \beta \left[ \beta \cdot \frac{d}{dt} \left[ \vec{E} \times \vec{\mu} \right] + \beta \cdot \frac{d}{dt} \left[ \vec{B} \times \vec{p} \right] \right] \quad (3.80)
\]

Now that we have this, let’s go back to the \( \beta = i \) cases. Starting at \( \alpha = 1 \), we have

\[
-2 u_1 u^1 \frac{D}{dt} \left( u^\delta Q^\gamma_{[1} F_{\delta \gamma]} \right) = \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( u^\delta Q^\gamma_{[1} F_{\delta \gamma]} \right).
\]
The only value of $\delta$ that we can see immediately sets this equal to zero is when $\delta = 1$, so will jump it and go from $\delta = 0$ to $\delta = 2$.

1. $\delta = 0$:

$$-2u_1 u^1 \frac{D}{d\tau} \left( u^0 Q^\gamma \left|_{[0]} \right. \right) = \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( \gamma \left[ Q^0 \phi F_{12} - Q^2 \phi F_{02} \right. \right.$$  
$$\left. + Q^3 \phi F_{13} - Q^3 \phi F_{03} \right] \right)$$

$$= \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( \gamma \left[ p_y B_z - \mu_z E_y \right. \right.$$  
$$\left. + p_x (-B_y) - (-\mu_y) E_z \right] \right)$$

$$= \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( \gamma \left[ \left( \vec{p} \times \vec{B} \right)_x + \left( \vec{\mu} \times \vec{E} \right)_x \right] \right)$$

2. $\delta = 2$:

$$-2u_1 u^1 \frac{D}{d\tau} \left( u^2 Q^\gamma \left|_{[2]} \right. \right) = \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( \gamma \beta_y \left[ Q^0 \phi F_{10} - Q^0 \phi F_{20} \right. \right.$$  
$$\left. + Q^3 \phi F_{13} - Q^3 \phi F_{23} \right] \right)$$

$$= \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( \gamma \beta_y \left[ p_y (-E_x) - p_x (-E_y) \right. \right.$$  
$$\left. + \mu_x (-B_y) - (-\mu_y) B_x \right] \right)$$

$$= \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( \gamma \beta_y \left[ \left( \vec{p} \times \vec{E} \right)_z + \left( \vec{\mu} \times \vec{B} \right)_z \right] \right)$$

3. $\delta = 3$:

$$-2u_1 u^1 \frac{D}{d\tau} \left( u^3 Q^\gamma \left|_{[3]} \right. \right) = \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( \gamma \beta_z \left[ Q^0 \phi F_{10} - Q^0 \phi F_{20} \right. \right.$$  
$$\left. + Q^2 \phi F_{12} - Q^2 \phi F_{32} \right] \right)$$

$$= \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left( \gamma \beta_z \left[ p_z (-E_x) - p_x (-E_z) \right. \right.$$  
$$\left. + (-\mu_x) B_z - \mu_z (-B_x) \right] \right)$$

$$= \gamma^3 \beta_x (\beta_x) \frac{d}{dt} \left. \left( \gamma \beta_z \left[ \left( \vec{p} \times \vec{E} \right)_y - \left( \vec{\mu} \times \vec{B} \right)_y \right] \right. \right.$$
As per usual, we see that this will result in a vector product between the velocities and the two vector products already inside the time derivatives. We can immediately see that when we sum over the remaining components, we come out to the result of

\[-2u_iu^\beta \frac{D}{dt}\left(u^\gamma Q_{\beta F_\gamma}^\gamma\right) = \gamma^5 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) + \vec{\beta} \cdot \left( \vec{B} \times \vec{p} \right) \right) + \gamma^4 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) + \vec{\beta} \cdot \left( \vec{B} \times \vec{p} \right) \right) + \gamma^3 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) \right) + \gamma^4 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) \right).
\]

The first set of terms comes directly from the result in (3.80), while the remainder come from the calculation of $\beta = 1$ and extrapolation. Let’s take the time derivatives of the terms we have yet to “simplify”, so that we get the whole thing in as simple a form as possible.

\[
\gamma^3 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \gamma \left[ \vec{p} \times \vec{B} \right] \right) \right) = \gamma^5 \vec{\beta} \left( \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) \right) + \vec{\beta} \cdot \left( \vec{B} \times \vec{p} \right) \right) + \gamma^4 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) \right).
\]

\[
\gamma^3 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \gamma \left[ \vec{p} \times \vec{E} \right] \right) \right) = \gamma^5 \vec{\beta} \left( \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) \right) + \vec{\beta} \cdot \left( \vec{B} \times \vec{p} \right) \right) + \gamma^4 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) \right).
\]

\[
\gamma^3 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \gamma \left[ \vec{p} \times \vec{E} \right] \right) \right) = \gamma^5 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) \right) + \gamma^4 \vec{\beta} \left( \vec{\beta} \cdot \frac{d}{dt} \left( \vec{E} \times \vec{\mu} \right) \right).
\]
\[ \gamma^3 \vec{\beta} \left[ \vec{\beta} \cdot \frac{d}{dt} (\gamma \vec{\beta} \times \left[ \vec{B} \times \vec{\mu} \right]) \right] = \gamma^5 \vec{\beta} \left( \vec{\beta} \cdot \left( \vec{\beta} \times \left[ \vec{B} \times \vec{\mu} \right] \right) \right) \\
+ \gamma^4 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \left[ \vec{B} \times \vec{\mu} \right] \right) \right] \\
+ \gamma^4 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \frac{d}{dt} \left[ \vec{B} \times \vec{\mu} \right] \right) \right]. \]

Now that we have that out of the way, we finally put everything back together to get

\[ -2u_i u^\beta \frac{D}{d\tau} \left( u^\delta Q^\gamma [\beta F_{\beta \gamma}] \right) = \gamma^5 \vec{\beta} \left( \vec{\beta} \cdot \left( \vec{E} \times \vec{\mu} \right) + \vec{\beta} \cdot \left( \vec{B} \times \vec{p} \right) \right) + \gamma^4 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{E} \times \vec{\mu} \right) \right] \\
+ \gamma^4 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{B} \times \vec{p} \right) \right] + \gamma^5 \vec{\beta} \left[ \left( \vec{\beta} \cdot \vec{\beta} \right) \left( \vec{\beta} \cdot \left[ \vec{p} \times \vec{B} \right] \right) \right] \\
+ \gamma^5 \vec{\beta} \left[ \left( \vec{\beta} \cdot \vec{\beta} \right) \left( \vec{\beta} \cdot \left[ \vec{p} \times \vec{B} \right] \right) \right] + \gamma^5 \vec{\beta} \left( \vec{\beta} \cdot \left[ \vec{\beta} \times \vec{E} \right] \right) \\
+ \gamma^4 \vec{\beta} \left[ \vec{\beta} \cdot \frac{d}{dt} \left( \vec{\mu} \times \vec{E} \right) \right] + \gamma^5 \vec{\beta} \left( \vec{\beta} \cdot \vec{\beta} \right) \left( \vec{\beta} \cdot \left[ \vec{\mu} \times \vec{E} \right] \right) \\
+ \gamma^4 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{B} \times \vec{p} \right) \right] + \gamma^4 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{B} \times \vec{D} \right) \right] \\
+ \gamma^5 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{B} \times \vec{\mu} \right) \right] \left( \vec{\beta} \cdot \left[ \vec{\beta} \times \left( \vec{B} \times \vec{\mu} \right) \right] \right) \\
+ \gamma^5 \vec{\beta} \left( \vec{\beta} \cdot \vec{\beta} \right) \left( \vec{\beta} \cdot \left[ \vec{B} \times \vec{\mu} \right] \right) \right]. \]
However, this is not quite the end of the story, as the anti-symmetric property of vector products allows us to cancel eight of the sixteen terms above, which leaves us with

\[
-2u_i u^\alpha \frac{D}{d\tau} (u^\delta Q^\gamma \{_{\beta} F_{\delta\gamma}\}) = \gamma^4 \beta \cdot \left( \bar{E} \times \bar{p} \right) + \gamma^4 \bar{\beta} \cdot \left( \bar{B} \times \bar{p} \right)
+ \gamma^4 \bar{\beta} \cdot \left( \bar{B} \times \left[ \bar{B} \times \bar{p} \right] \right) + \gamma^4 \bar{\beta} \cdot \left( \bar{B} \times \frac{d}{dt} \left[ \bar{B} \times \bar{p} \right] \right)
+ \gamma^5 \bar{\beta} \cdot \left( \bar{B} \cdot \left( \bar{B} \times \bar{p} \right) \right)
+ \gamma^5 \bar{\beta} \cdot \left( \bar{B} \cdot \left( \bar{B} \times \bar{p} \right) \right).
\]

(3.82)

We have now written out all of the zeroth and spatial components of the equation of motion in a relativistically correct form. Before we add them all together, it will be helpful for us to keep them all separated as we go through a dimension analysis so that we can put the proper factors of the speed of light, \( c \) back in.

### 3.2.9 Dimensional Analysis

Before we go on to test the units in every single term, we should first discuss what the dimensions of the quantities in these equations are. Recall that we working in Gaussian units, which is a system in which the magnetic field shares the same units as the electric field. To find these, we can think of Coulomb’s law, which has a slightly more compact form that the typical SI unit form,

\[
|\vec{F}| = q|\vec{E}| = \frac{qQ}{r^2}.
\]

(3.83)
Clearly, the electric field (and thereby the magnetic field) has dimensions of

\[
\text{dim} (\vec{E}) = \text{dim} (\vec{B}) = \frac{[\text{Charge}]}{[\text{Distance}]^2}.
\]  (3.84)

The other quantities we have to concern ourselves with are shown below:

\[
\text{dim} (\vec{S}) = \frac{[\text{Mass}] \cdot [\text{Distance}]^2}{[\text{Time}]},
\]  (3.85a)

\[
\text{dim} (\vec{p}) = [\text{Charge}] \cdot [\text{Distance}],
\]  (3.85b)

\[
\text{dim} (\vec{\mu}) = \frac{[\text{Charge}] \cdot [\text{Distance}]^2}{[\text{Time}]},
\]  (3.85c)

\[
\text{dim} (\vec{\beta}) = 0,
\]  (3.85d)

\[
\text{dim} (\ddot{\vec{\beta}}) = \frac{1}{[\text{Time}]};
\]  (3.85e)

\[
\text{dim} (\dddot{\vec{\beta}}) = \frac{1}{[\text{Time}]^2}.
\]  (3.85f)

It may also be beneficial that the Lorentz factor, \( \gamma \), is a dimensionless quantity.

For the temporal component, we have 27 terms that we will have to look at, while the spatial component has a whopping 37 terms. We will do these in subsections, as to not overload too much.

### 3.3 Dimensional Analysis

#### 3.3.1 Temporal component

As mentioned above, there are indeed 27 terms in this equation, but there are quite a few that match dimensionally, and so the amount of work that needs to get
done is minimized by the recognizing of like-terms. Each term and its corrected
dimensions is seen below.

1. \( q\gamma^2 \left( \vec{\beta} \cdot \vec{E} \right) \): This one has exactly the same dimensions of force already, since
both \( \gamma \) and \( \vec{\beta} \) are dimensionless, so we do not need to multiply or divide this
by any factor of \( c \).

2. \( \vec{p} \cdot \frac{d\vec{E}}{dt} \): This has dimensions of

\[
\text{dim} \left[ \vec{p} \cdot \frac{d\vec{E}}{dt} \right] = \text{[Charge]} \cdot \text{[Distance]} \cdot \frac{\text{[Charge]}}{\text{[Distance]}^2 \cdot \text{[Time]}} = \frac{\text{[Charge]}^2}{\text{[Distance]} \cdot \text{[Time]}}
\]

so if we divide this quantity by \( c \), we will cancel out the factor of time and
bring in an extra dimension of distance in the denominator, giving us

\[
\vec{p} \cdot \frac{d\vec{E}}{dt} \rightarrow \vec{p} \cdot \frac{1}{c} \frac{d\vec{E}}{dt}.
\]

(3.86)

3. \( \vec{\mu} \cdot \frac{d\vec{B}}{dt} \): This has similar dimensions to the term above,

\[
\text{dim} \left[ \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right] = \frac{\text{[Charge]} \cdot \text{[Distance]}^2}{\text{[Time]}} \cdot \frac{\text{[Charge]}}{\text{[Distance]}^2 \cdot \text{[Time]}} = \frac{\text{[Charge]}^2}{\text{[Time]}^2}.
\]

This clearly shows that we now need to divide this whole factor by \( c^2 \), rather
than \( c \).

\[
\vec{\mu} \cdot \frac{d\vec{B}}{dt} \rightarrow \vec{\mu} \cdot \frac{1}{c^2} \frac{d\vec{B}}{dt}.
\]

(3.87)
4. $\gamma^2 \vec{p} \cdot \frac{d\vec{E}}{dt}$: This term has exactly the same dimensions as the equation seen in (3.85), and so we have

$$\gamma^2 \vec{p} \cdot \frac{d\vec{E}}{dt} \to \frac{\gamma^2 \vec{p}}{c} \cdot \frac{1}{c} \frac{d\vec{E}}{dt}.$$  
(3.88)

5. $-\gamma^2 \vec{\mu} \cdot \frac{d\vec{B}}{dt}$: This has the same dimensions as that seen in (3.86), so we have

$$-\gamma^2 \vec{\mu} \cdot \frac{d\vec{B}}{dt} \to \frac{\gamma^2 \vec{\mu}}{c^2} \cdot \frac{1}{c^2} \frac{d\vec{B}}{dt}.$$  
(3.89)

6. $\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) [\vec{\beta} \cdot (\vec{p} \times \vec{B})]$:

We now have slightly different dimensions yet,

$$\dim \left[ (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot (\vec{p} \times \vec{B})) \right] = \frac{1}{[\text{Time}]} \cdot [\text{Charge}] \cdot [\text{Distance}] \cdot \frac{[\text{Charge}]}{[\text{Distance}]^2}$$

$$= \frac{[\text{Charge}]^2}{[\text{Distance}] \cdot [\text{Time}]}.$$ 

so dividing this by a factor of $c$ will set this straight.

$$\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) [\vec{\beta} \cdot (\vec{p} \times \vec{B})] \to \frac{\gamma^4}{c} (\vec{\beta} \cdot \dot{\vec{\beta}}) [\vec{\beta} \cdot (\vec{p} \times \vec{B})].$$  
(3.90)

7. $\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) [\vec{\beta} \cdot (\vec{\mu} \times \vec{E})]$:

$$\dim (\vec{\beta} \cdot \dot{\vec{\beta}}) [\vec{\beta} \cdot (\vec{\mu} \times \vec{E})] = \frac{1}{[\text{Time}]} \cdot \frac{[\text{Charge}]}{[\text{Time}]} \cdot \frac{[\text{Distance}]}{[\text{Distance}]^2} \cdot \frac{[\text{Charge}]}{[\text{Distance}]^2}$$

$$= \frac{[\text{Charge}]^2}{[\text{Time}]^2}.$$
Following the example in (3.86), this needs to be divided by a factor of $c^2$.

$$\gamma^4 (\vec{\beta} \cdot \vec{\beta}) \left[ \vec{\beta} \cdot (\vec{\mu} \times \vec{E}) \right] \rightarrow \frac{\gamma^4}{c^2} (\vec{\beta} \cdot \vec{\beta}) \left[ \vec{\beta} \cdot (\vec{\mu} \times \vec{E}) \right]$$  \hspace{1cm} (3.91)

8. $\gamma^2 [\vec{\beta} \cdot (\vec{p} \times \vec{B})]$ : This is exactly the same in the units as (3.89), so we have

$$\gamma^2 \left[ \vec{\beta} \cdot (\vec{p} \times \vec{B}) \right] \rightarrow \frac{\gamma^2}{c} \left[ \vec{\beta} \cdot (\vec{p} \times \vec{B}) \right].$$  \hspace{1cm} (3.92)

9. $\gamma^2 [\vec{\beta} \cdot (\vec{\mu} \times \vec{E})]$ : This follows the same as (3.90), giving us

$$\gamma^2 \left[ \vec{\beta} \cdot (\vec{\mu} \times \vec{E}) \right] \rightarrow \frac{\gamma^2}{c^2} \left[ \vec{\beta} \cdot (\vec{\mu} \times \vec{E}) \right].$$  \hspace{1cm} (3.93)

10. $\gamma^2 \left[ \vec{\beta} \cdot \frac{d}{dt} (\vec{p} \times \vec{B}) \right]$ : As with the previous two, it is clear that this will become

$$\gamma^2 \left[ \vec{\beta} \cdot \frac{d}{dt} (\vec{p} \times \vec{B}) \right] \rightarrow \frac{\gamma^2}{c} \left[ \vec{\beta} \cdot \frac{d}{dt} (\vec{p} \times \vec{B}) \right].$$  \hspace{1cm} (3.94)

11. $\gamma^2 \left[ \vec{\beta} \cdot \frac{d}{dt} (\vec{\mu} \times \vec{E}) \right]$ : As with the previous two of this form, this needs to become

$$\gamma^2 \left[ \vec{\beta} \cdot \frac{d}{dt} (\vec{\mu} \times \vec{E}) \right] \rightarrow \frac{\gamma^2}{c^2} \left[ \vec{\beta} \cdot \frac{d}{dt} (\vec{\mu} \times \vec{E}) \right].$$

12. $\frac{8q^2 \gamma^7}{3} (\vec{\beta} \cdot \vec{\dot{\beta}})^2$ : These next two are easy to see that they both need to be divided by a factor of $c^2$.

$$\frac{8q^2 \gamma^7}{3} (\vec{\beta} \cdot \vec{\dot{\beta}})^2 \rightarrow \frac{8q^2 \gamma^7}{3c^2} (\vec{\beta} \cdot \vec{\dot{\beta}}).$$  \hspace{1cm} (3.95)
13. \( \frac{2q^2 \gamma^5}{3} (\vec{\beta} \cdot \vec{\beta}) \):

\[
\frac{2q^2 \gamma^5}{3} (\vec{\beta} \cdot \vec{\beta}) \rightarrow \frac{2q^2 \gamma^5}{3c^2} (\vec{\beta} \cdot \vec{\beta}).
\] (3.96)

14. \( 4\gamma^9 (\vec{\beta} \cdot \vec{\beta})^2 (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) \):

These next ones involving the spin vector are a little different, since there is no presence of charge in these terms. This means that we will need to have a classical mechanics form of the dimensions,

\[
\text{dim} \left[ \vec{F} \right] = \frac{[\text{Mass}] \cdot [\text{Distance}]}{[\text{Time}]^2}.
\]

So, we see the dimensions of these terms as

\[
\text{dim} \left( (\vec{\beta} \cdot \vec{\beta})^2 (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) \right) = \frac{1}{[\text{Time}]^2} \cdot \frac{[\text{Mass}] \cdot [\text{Distance}]^2}{[\text{Time}]} = \frac{[\text{Mass}] \cdot [\text{Distance}]^2}{[\text{Time}]^3}.
\]

From this, we can see that this term (and the following five terms) needs to be divided by \( c^2 \). With this, we have

\[
4\gamma^9 (\vec{\beta} \cdot \vec{\beta})^2 (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) \rightarrow \frac{4\gamma^9}{c^2} (\vec{\beta} \cdot \vec{\beta})^2 (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]). \] (3.97)

15. \( \gamma^7 (\vec{\beta} \cdot \vec{\beta} + \vec{\beta}^2) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}] \)

\[
\gamma^7 (\vec{\beta} \cdot \vec{\beta} + \vec{\beta}^2) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) \rightarrow \frac{\gamma^7}{c^2} (\vec{\beta} \cdot \vec{\beta} + \vec{\beta}^2) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]). \] (3.98)

16. \( 3\gamma^7 (\vec{\beta} \cdot \vec{\beta}) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}] \)

\[
3\gamma^7 (\vec{\beta} \cdot \vec{\beta}) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]) \rightarrow \frac{3\gamma^7}{c^2} (\vec{\beta} \cdot \vec{\beta}) (\vec{\beta} \cdot [\vec{\beta} \times \vec{S}]). \] (3.99)
17. $\gamma^7 (\vec{\beta} \cdot \dot{\vec{\beta}}) \left( \vec{\beta} \cdot \frac{d \vec{S}}{dt} \right)$

$$\gamma^7 (\vec{\beta} \cdot \dot{\vec{\beta}}) \left( \vec{\beta} \cdot \frac{d \vec{S}}{dt} \right) \rightarrow \frac{\gamma^7}{c^2} (\vec{\beta} \cdot \ddot{\vec{\beta}}) \left( \vec{\beta} \cdot \frac{d \vec{S}}{dt} \right) (3.100)$$

18. $\gamma^5 \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d \vec{S}}{dt} \right]$:

$$\gamma^5 \vec{\beta} \left[ \vec{\beta} \times \frac{d \vec{S}}{dt} \right] \rightarrow \frac{\gamma^5}{c^2} \dot{\vec{\beta}} \cdot \left[ \vec{\beta} \times \frac{d \vec{S}}{dt} \right] (3.101)$$

19. $\gamma^5 \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d \vec{S}}{dt} \right]$:

$$\gamma^5 \vec{\beta} \left[ \dot{\vec{\beta}} \times \frac{d \vec{S}}{dt} \right] \rightarrow \frac{\gamma^5}{c^2} \ddot{\vec{\beta}} \cdot \left[ \ddot{\vec{\beta}} \times \frac{d \vec{S}}{dt} \right] (3.102)$$

20. $\gamma^4 \vec{\beta} \cdot \left[ \dot{\vec{\beta}} \times \left( \vec{E} \times \vec{p} \right) \right]$:

These next six terms follow the examples of Equations (3.86) and (3.87).

$$\gamma^4 \vec{\beta} \cdot \left[ \dot{\vec{\beta}} \times \left( \vec{E} \times \vec{p} \right) \right] \rightarrow \frac{\gamma^4}{c^2} \dddot{\vec{\beta}} \cdot \left[ \dddot{\vec{\beta}} \times \left( \vec{E} \times \vec{p} \right) \right]. (3.103)$$

21. $\gamma^4 \vec{\beta} \cdot \left[ \dot{\vec{\beta}} \times \left( \vec{\mu} \times \vec{B} \right) \right]$:

$$\gamma^4 \vec{\beta} \cdot \left[ \dot{\vec{\beta}} \times \left( \vec{\mu} \times \vec{B} \right) \right] \rightarrow \frac{\gamma^4}{c^2} \dddot{\vec{\beta}} \cdot \left[ \dddot{\vec{\beta}} \times \left( \vec{\mu} \times \vec{B} \right) \right]. (3.104)$$

22. $\gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \vec{\beta} \cdot \left[ \vec{\beta} \times \left( \vec{E} \times \vec{p} \right) \right]$:

$$\gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \vec{\beta} \cdot \left[ \vec{\beta} \times \left( \vec{E} \times \vec{p} \right) \right] \rightarrow \frac{\gamma^5}{c} (\vec{\beta} \cdot \dot{\vec{\beta}}) \vec{\beta} \cdot \left[ \vec{\beta} \times \left( \vec{E} \times \vec{p} \right) \right]. (3.105)$$
23. \[ \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \vec{\beta} \cdot [\vec{\beta} \times (\vec{\mu} \times \vec{B})] : \]
\[ \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \vec{\beta} \cdot [\vec{\beta} \times (\vec{\mu} \times \vec{B})] \rightarrow \frac{\gamma^5}{c^2} (\vec{\beta} \cdot \dot{\vec{\beta}}) \vec{\beta} \cdot [\vec{\beta} \times (\vec{\mu} \times \vec{B})]. \]  \hspace{1cm} (3.106)

24. \[ \gamma^4 \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d}{dt} (\vec{E} \times \vec{p}) \right] : \]
\[ \gamma^4 \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d}{dt} (\vec{E} \times \vec{p}) \right] \rightarrow \frac{\gamma^4}{c} \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d}{dt} (\vec{E} \times \vec{p}) \right]. \]  \hspace{1cm} (3.107)

25. \[ \gamma^4 \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d}{dt} (\vec{\mu} \times \vec{B}) \right] : \]
\[ \gamma^4 \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d}{dt} (\vec{\mu} \times \vec{B}) \right] \rightarrow \frac{\gamma^4}{c^2} \vec{\beta} \cdot \left[ \vec{\beta} \times \frac{d}{dt} (\vec{\mu} \times \vec{B}) \right]. \]  \hspace{1cm} (3.108)

26. \[ \gamma^4 \dot{\vec{\beta}} \cdot (\vec{p} \times \vec{B}) : \] These last two now follow the example of Equations (3.91) and (3.90), respectively.
\[ \gamma^4 \dot{\vec{\beta}} \cdot (\vec{p} \times \vec{B}) \rightarrow \frac{\gamma^4}{c} \dot{\vec{\beta}} \cdot (\vec{p} \times \vec{B}). \]  \hspace{1cm} (3.109)

27. \[ \gamma^4 \dot{\vec{\beta}} \cdot (\vec{\mu} \times \vec{E}) : \]
\[ \gamma^4 \dot{\vec{\beta}} \cdot (\vec{\mu} \times \vec{E}) \rightarrow \frac{\gamma^4}{c^2} \dot{\vec{\beta}} \cdot (\vec{\mu} \times \vec{E}). \]  \hspace{1cm} (3.110)
Before moving onto the spatial components, let’s go ahead and finally put all of the terms for the temporal components together with the correct dimensions.

\[
ma_0 = \gamma^2 \left( \beta \cdot \vec{E} \right) + \vec{p} \cdot \frac{1}{c} \frac{d\vec{E}}{dt} (\gamma^2 + 1) + \vec{\mu} \cdot \frac{1}{c^2} \frac{d\vec{B}}{dt} (\gamma^2 - 1) + \frac{\gamma^2}{c} (\dot{\beta} \cdot \dot{\vec{B}}) [\dot{\beta} \cdot (\vec{p} \times \vec{B})] \\
+ \frac{\gamma^4}{c^3} (\beta \cdot \beta) [\beta \cdot (\vec{p} \times \vec{E})] + \frac{\gamma^2}{c} \left[ \beta \cdot (\vec{p} \times \vec{B}) \right] + \frac{\gamma^2}{c^2} [\dot{\beta} \cdot (\vec{p} \times \vec{E})] \\
+ \frac{\gamma^4}{c} [\beta - \frac{d}{dt}(\vec{p} \times \vec{B})] + \frac{\gamma^2}{c^2} [\beta \cdot \frac{d}{dt}(\vec{E} \times \vec{p})] + \frac{\gamma^4}{c} \ddot{\beta} \cdot \left[ \beta \times (\vec{E} \times \vec{p}) \right] \\
+ \frac{\gamma^4}{c^3} \beta \cdot \left[ \beta \times (\vec{p} \times \vec{B}) \right] + \frac{\gamma^4}{c^2} \beta \cdot [\beta \times \frac{d}{dt}(\vec{E} \times \vec{p})] + \frac{\gamma^4}{c^2} \ddot{\beta} \cdot \left[ \beta \times \frac{d}{dt}(\vec{p} \times \vec{B}) \right] \\
+ \frac{\gamma^5}{c^2} (\ddot{\beta} \cdot \beta) \beta \cdot \left[ \beta \times (\vec{p} \times \vec{E}) \right] + \frac{2q^2\gamma^5}{3e^2} [4\gamma^2 (\ddot{\beta} \cdot \dot{\beta}) + (\ddot{\beta} \cdot \dot{\beta})] \\
+ \frac{4\gamma^9}{c^2} (\ddot{\beta} \cdot \dot{\beta}) \beta \cdot \left[ \beta \times \vec{S} \right] + \frac{\gamma^7}{c^2} \left( \ddot{\beta} \cdot \dot{\beta} \right) \left( \beta \cdot \left[ \beta \times \vec{S} \right] \right) \\
+ \frac{3\gamma^7}{c^2} \left( \ddot{\beta} \cdot \dot{\beta} \right) \left( \beta \cdot \left[ \beta \times \vec{S} \right] \right) + \frac{\gamma^7}{c^2} \left( \ddot{\beta} \cdot \dot{\beta} \right) \left( \beta \cdot \left[ \beta \times \frac{d\vec{S}}{dt} \right] \right) \\
+ \frac{\gamma^5}{c^2} \ddot{\beta} \cdot \left[ \beta \times \vec{S} \right] + \frac{\gamma^5}{c^2} \ddot{\beta} \cdot \left[ \beta \times \frac{d\vec{S}}{dt} \right].
\]

(3.111)

### 3.3.2 Spatial Components

We now go through exactly the same process for the spatial components.

1. \( q\gamma (\vec{E} + \vec{\beta} \times \vec{B}) \): As can be seen, we have the proper dimensions already, and so we can leave this term as is.
2. $4\gamma^6 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 (\vec{S} \times \vec{\beta})$: As seen before with similar terms, this needs to be divided by a factor of $c^2$. It will go like this through the next five terms.

\[
4\gamma^6 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 (\vec{S} \times \vec{\beta}) \rightarrow \frac{4\gamma^6}{c^2} (\vec{\beta} \cdot \dot{\vec{\beta}})^2 (\vec{S} \times \vec{\beta}). \quad (3.112)
\]

3. $\gamma^4 (\vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2) (\vec{S} \times \vec{\beta})$:

\[
\gamma^4 (\vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2) (\vec{S} \times \vec{\beta}) \rightarrow \frac{\gamma^4}{c^2} (\vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2) (\vec{S} \times \vec{\beta}). \quad (3.113)
\]

4. $3\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{S} \times \dot{\vec{\beta}})$:

\[
3\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{S} \times \dot{\vec{\beta}}) \rightarrow \frac{3\gamma^4}{c^2} (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{S} \times \dot{\vec{\beta}}) \quad (3.114)
\]

5. $\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) \left( \frac{d\vec{S}}{dt} \times \dot{\vec{\beta}} \right)$:

\[
\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) \left( \frac{d\vec{S}}{dt} \times \dot{\vec{\beta}} \right) \rightarrow \frac{\gamma^4}{c^2} (\vec{\beta} \cdot \dot{\vec{\beta}}) \left( \frac{d\vec{S}}{dt} \times \dot{\vec{\beta}} \right). \quad (3.115)
\]

6. $\gamma^2 \left( \frac{d\vec{S}}{dt} \times \dddot{\vec{\beta}} \right)$:

\[
\gamma^2 \left( \frac{d\vec{S}}{dt} \times \dddot{\vec{\beta}} \right) \rightarrow \frac{\gamma^2}{c^2} \left( \frac{d\vec{S}}{dt} \times \dddot{\vec{\beta}} \right) \quad (3.116)
\]

7. $\gamma^2 (\vec{S} \times \dddot{\vec{\beta}})$:

\[
\gamma^2 (\vec{S} \times \dddot{\vec{\beta}}) \rightarrow \frac{\gamma^2}{c^2} (\vec{S} \times \dddot{\vec{\beta}}) \quad (3.117)
\]
8. $\gamma^4 (\vec{\beta} \cdot \vec{\beta}) (\vec{E} \times \vec{\mu})$: The next six terms follow the example of Equations (3.89) and (3.90), giving us

$$\gamma^4 (\vec{\beta} \cdot \vec{\beta}) (\vec{E} \times \vec{\mu}) \to \frac{\gamma^4}{c^2} (\vec{\beta} \cdot \vec{\beta}) (\vec{E} \times \vec{\mu}).$$

(3.118)

9. $\gamma^4 (\vec{\beta} \cdot \vec{\beta}) (\vec{B} \times \vec{p})$:

$$\gamma^4 (\vec{\beta} \cdot \vec{\beta}) (\vec{B} \times \vec{p}) \to \frac{\gamma^4}{c} (\vec{\beta} \cdot \vec{\beta}) (\vec{B} \times \vec{p}).$$

(3.119)

10. $\gamma^2 \frac{d}{dt} (\vec{E} \times \vec{\mu})$:

$$\gamma^2 \frac{d}{dt} (\vec{E} \times \vec{\mu}) \to \frac{\gamma^2}{c^2} \frac{d}{dt} (\vec{E} \times \vec{\mu}).$$

(3.120)

11. $\gamma^2 \frac{d}{dt} (\vec{B} \times \vec{p})$:

$$\gamma^2 \frac{d}{dt} (\vec{B} \times \vec{p}) \to \frac{\gamma^2}{c} \frac{d}{dt} (\vec{B} \times \vec{p}).$$

(3.121)

12. $\gamma^4 (\vec{\beta} \cdot \vec{\beta}) [\vec{\beta} \times (\vec{E} \times \vec{\mu})]$:

$$\gamma^4 (\vec{\beta} \cdot \vec{\beta}) [\vec{\beta} \times (\vec{E} \times \vec{\mu})] \to \frac{\gamma^4}{c^2} (\vec{\beta} \cdot \vec{\beta}) [\vec{\beta} \times (\vec{E} \times \vec{\mu})].$$

(3.122)

13. $\gamma^4 (\vec{\beta} \cdot \vec{\beta}) [\vec{\beta} \times (\vec{B} \times \vec{p})]$:

$$\gamma^4 (\vec{\beta} \cdot \vec{\beta}) [\vec{\beta} \times (\vec{B} \times \vec{p})] \to \frac{\gamma^4}{c} (\vec{\beta} \cdot \vec{\beta}) [\vec{\beta} \times (\vec{B} \times \vec{p})].$$

(3.123)
14. $\gamma^2 \left( \dot{\beta} \times [\tilde{E} \times \tilde{p}] \right)$: We are now onto the switching of dipoles and fields, so the factors of $c$ and $c^2$ will also be swapped.

\[
\gamma^2 \left( \dot{\beta} \times [\tilde{E} \times \tilde{p}] \right) \rightarrow \frac{\gamma^2}{c} \left( \dot{\beta} \times [\tilde{E} \times \tilde{p}] \right).
\] (3.124)

15. $\gamma^2 \left( \dot{\beta} \times [\tilde{B} \times \tilde{\mu}] \right)$:

\[
\gamma^2 \left( \dot{\beta} \times [\tilde{B} \times \tilde{\mu}] \right) \rightarrow \frac{\gamma^2}{c^2} \left( \dot{\beta} \times [\tilde{B} \times \tilde{\mu}] \right).
\] (3.125)

16. $\gamma^2 \left( \frac{d}{dt} \left( \dot{\beta} \times [\tilde{E} \times \tilde{p}] \right) \right)$:

\[
\gamma^2 \left( \frac{d}{dt} \left( \dot{\beta} \times [\tilde{E} \times \tilde{p}] \right) \right) \rightarrow \frac{\gamma^2}{c} \left( \frac{d}{dt} \left( \dot{\beta} \times [\tilde{E} \times \tilde{p}] \right) \right).
\] (3.126)

17. $\gamma^2 \left( \dot{\beta} \times \frac{d}{dt} [\tilde{B} \times \tilde{\mu}] \right)$:

\[
\gamma^2 \left( \dot{\beta} \times \frac{d}{dt} [\tilde{B} \times \tilde{\mu}] \right) \rightarrow \frac{\gamma^2}{c^2} \left( \dot{\beta} \times \frac{d}{dt} [\tilde{B} \times \tilde{\mu}] \right).
\] (3.127)

18. $\frac{2q^2\gamma^3}{3} \left[ 4\gamma^4 \left( \dot{\beta} \cdot \ddot{\beta} \right)^2 \dot{\beta} \right]$: As we saw in the other ALD term contributions, each one of these will be corrected by dividing by $c^2$.

\[
\frac{2q^2\gamma^3}{3} \left[ 4\gamma^4 \left( \dot{\beta} \cdot \ddot{\beta} \right)^2 \dot{\beta} \right] \rightarrow \frac{2q^2\gamma^3}{3c^2} \left[ 4\gamma^4 \left( \dot{\beta} \cdot \ddot{\beta} \right)^2 \dot{\beta} \right].
\] (3.128)

19. $\frac{2q^2\gamma^3}{3} \left[ \gamma^2 \left( \dot{\beta} \cdot \ddot{\beta} \right) \dot{\beta} \right]$:

\[
\frac{2q^2\gamma^3}{3} \left[ \gamma^2 \left( \dot{\beta} \cdot \ddot{\beta} \right) \dot{\beta} \right] \rightarrow \frac{2q^2\gamma^3}{3c^2} \left[ \gamma^2 \left( \dot{\beta} \cdot \ddot{\beta} \right) \dot{\beta} \right].
\] (3.129)
20. \( \frac{2q^2\gamma^3}{3} \left( 3\gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \dot{\vec{\beta}} \right) \):

\[
\frac{2q^2\gamma^3}{3} \left( 3\gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \dot{\vec{\beta}} \right) \rightarrow \frac{2q^2\gamma^3}{3c^2} \left( 3\gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \dot{\vec{\beta}} \right) \quad (3.130)
\]

21. \( \frac{2q^2\gamma^3}{3} \vec{\beta} \):

\[
\frac{2q^2\gamma^3}{3} \vec{\beta} \rightarrow \frac{2q^2\gamma^3}{3c^2} \vec{\beta} \quad (3.131)
\]

22. \( p_i \vec{\nabla} E_i \): When we consider this term, we see that the units of dimension in the definition of the electric dipole are canceled with the units of \( \frac{1}{[\text{Distance}]} \), which comes from the derivative, which means that we are immediately left with units of force, and so this term is okay as is.

23. \( \mu_i \vec{\nabla} B_i \): Unlike the electric dipole, the magnetic dipole now has units of time in the denominator, while the extra units of distance in the numerator reduces this to

\[
\text{dim} \left[ \mu_i \vec{\nabla} B_i \right] = \frac{[\text{Charge}]^2}{[\text{Distance}] \cdot [\text{Time}]},
\]

leaving us with the necessity to divide this by a factor of \( c \), and so we have

\[
\mu_i \vec{\nabla} B_i \rightarrow \frac{1}{c} \mu_i \vec{\nabla} B_i. \quad (3.132)
\]

24. \( \gamma^2 \vec{\beta} \left[ \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right] \):

\[
\gamma^2 \vec{\beta} \left[ \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right] \rightarrow \frac{\gamma^2}{c^2} \vec{\beta} \left[ \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right] \quad (3.133)
\]
25. \(-\gamma^2 \vec{\beta} \left[ \vec{p} \cdot \frac{d \vec{E}}{dt} \right] \):

\[-\gamma^2 \vec{\beta} \left[ \vec{p} \cdot \frac{d \vec{E}}{dt} \right] \rightarrow -\frac{\gamma^2}{c} \vec{\beta} \left[ \vec{p} \cdot \frac{d \vec{E}}{dt} \right] \quad (3.134)\]

26. \(4\gamma^9 \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \): These next six terms follow the same pattern as those from the temporal component with the spin vector.

\[4\gamma^9 \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \rightarrow \frac{4\gamma^9}{c^2} \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \quad (3.135)\]

27. \(\gamma^7 \vec{\beta} \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \dot{\vec{\beta}}^2 \right) \left[ \vec{\beta} \cdot \left[ \vec{\beta} \times \vec{S} \right] \right] \):

\[\gamma^7 \vec{\beta} \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \dot{\vec{\beta}}^2 \right) \left[ \vec{\beta} \cdot \left[ \vec{\beta} \times \vec{S} \right] \right] = \frac{\gamma^7}{c^2} \vec{\beta} \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \dot{\vec{\beta}}^2 \right) \left[ \vec{\beta} \cdot \left[ \vec{\beta} \times \vec{S} \right] \right] \quad (3.136)\]

28. \(3\gamma^7 \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \):

\[3\gamma^7 \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \rightarrow \frac{3\gamma^7}{c^2} \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \quad (3.137)\]

29. \(\gamma^7 \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \frac{d \vec{S}}{dt} \right) \right] \):

\[\gamma^7 \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \frac{d \vec{S}}{dt} \right) \right] \rightarrow \frac{\gamma^7}{c^2} \vec{\beta} \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \frac{d \vec{S}}{dt} \right) \right] \quad (3.138)\]

30. \(\gamma^5 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \):

\[\gamma^5 \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \rightarrow \frac{\gamma^5}{c^2} \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] \quad (3.139)\]
\[ 31. \gamma^5 \beta \left( \beta \cdot \left( \dot{\beta} \times \frac{d\vec{S}}{dt} \right) \right): \]
\[ \gamma^5 \beta \left( \beta \cdot \left( \dot{\beta} \times \frac{d\vec{S}}{dt} \right) \right) \rightarrow \gamma^5 c \beta \left( \beta \cdot \left( \dot{\beta} \times \frac{d\vec{S}}{dt} \right) \right). \quad (3.140) \]

\[ 32. \gamma^4 \beta \left( \beta \cdot [\vec{E} \times \vec{\mu}] \right): \]
\[ \gamma^4 \beta \left( \beta \cdot [\vec{E} \times \vec{\mu}] \right) \rightarrow \gamma^4 c \beta \left( \beta \cdot [\vec{E} \times \vec{\mu}] \right). \quad (3.141) \]

\[ 33. \gamma^4 \beta \left( \beta \cdot [\vec{B} \times \vec{p}] \right): \]
\[ \gamma^4 \beta \left( \beta \cdot [\vec{B} \times \vec{p}] \right) \rightarrow \gamma^4 c \beta \left( \beta \cdot [\vec{B} \times \vec{p}] \right). \quad (3.142) \]

\[ 34. \gamma^4 \beta \left( \beta \cdot \left( \beta \times \frac{d}{dt} [\vec{p} \times \vec{E}] \right) \right): \]
\[ \gamma^4 \beta \left( \beta \cdot \left( \beta \times \frac{d}{dt} [\vec{p} \times \vec{E}] \right) \right) \rightarrow \gamma^4 c \beta \left( \beta \cdot \left( \beta \times \frac{d}{dt} [\vec{p} \times \vec{E}] \right) \right). \quad (3.143) \]

\[ 35. \gamma^4 \beta \left( \beta \cdot \left( \beta \times \frac{d}{dt} [\vec{B} \times \vec{\mu}] \right) \right): \]
\[ \gamma^4 \beta \left( \beta \cdot \left( \beta \times \frac{d}{dt} [\vec{B} \times \vec{\mu}] \right) \right) \rightarrow \gamma^4 c \beta \left( \beta \cdot \left( \beta \times \frac{d}{dt} [\vec{B} \times \vec{\mu}] \right) \right). \quad (3.144) \]

\[ 36. \gamma^5 \beta \left( \beta \cdot \dot{\beta} \right) \left( \beta \cdot [\vec{p} \times \vec{E}] \right): \]
\[ \gamma^5 \beta \left( \beta \cdot \dot{\beta} \right) \left( \beta \cdot [\vec{p} \times \vec{E}] \right) \rightarrow \gamma^5 c \beta \left( \beta \cdot \dot{\beta} \right) \left( \beta \cdot [\vec{p} \times \vec{E}] \right). \quad (3.145) \]
37. $\gamma^5 \vec{\beta} (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot [\vec{\beta} \times (\vec{B} \times \vec{\mu})])$:

$$\gamma^5 \vec{\beta} (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot [\vec{\beta} \times (\vec{B} \times \vec{\mu})]) \rightarrow \frac{\gamma^5}{c^2} \vec{\beta} (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot [\vec{\beta} \times (\vec{B} \times \vec{\mu})]) .$$

(3.146)
We now have each term with the proper dimensions of force and have the opportunity to put it all back together.

\[
\begin{aligned}
\vec{F} &= q \left( \vec{E} + \vec{\beta} \times \vec{B} \right) + p_i \vec{\nabla}E_i + \frac{1}{c} \mu_i \vec{\nabla}B_i + \frac{2q^2 \gamma^3 \dot{\vec{\beta}}}{3c^2} \vec{\beta} + \frac{\gamma^4}{c^2} \left( \vec{\beta} \cdot \ddot{\vec{\beta}} \right) \left( \vec{E} \times \ddot{\vec{\beta}} \right) \\
&+ \frac{2q^2 \gamma^5}{3c^2} \left[ 4 \gamma^2 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} \right)^2 \vec{\beta} + \left( \vec{\beta} \cdot \ddot{\vec{\beta}} \right) \vec{\beta} + 3 \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \vec{\beta} \right] + \frac{\gamma^4}{c} \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{B} \times \dddot{\vec{\beta}} \right) \\
&+ \frac{\gamma^2}{c^2} \left( \vec{E} \times \dddot{\vec{\beta}} \right) \left( \vec{B} \times \dddot{\vec{\beta}} \right) + \frac{\gamma^2}{c^2} \left( \vec{B} \times \dddot{\vec{\beta}} \right) \left( \vec{E} \times \dddot{\vec{\beta}} \right) \\
&+ \frac{\gamma^4}{c} \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{\beta} \times \left( \vec{E} \times \dddot{\vec{\beta}} \right) \right) + \frac{\gamma^2}{c} \left( \vec{\beta} \times \left( \vec{E} \times \dddot{\vec{\beta}} \right) \right) \\
&+ \frac{\gamma^5}{c^2} \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{\beta} \times \left( \vec{B} \times \dddot{\vec{\beta}} \right) \right) + \frac{\gamma^7}{c^2} \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{\beta} \times \left( \vec{S} \times \dddot{\vec{\beta}} \right) \right) \\
&+ \frac{\gamma^6}{c^2} \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{\beta} \times \left( \vec{S} \times \dddot{\vec{\beta}} \right) \right) + \frac{\gamma^8}{c^2} \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{\beta} \times \left( \vec{S} \times \dddot{\vec{\beta}} \right) \right) \\
&+ \frac{131}{2} \gamma^7 \left( \vec{\beta} \cdot \dddot{\vec{\beta}} \right) \left( \vec{\beta} \times \left( \vec{S} \times \dddot{\vec{\beta}} \right) \right). \\
\end{aligned}
\]

We now have a relativistically and dimensionally correct form for the force on the particle in this perturbative theory. However, our work is still not done, as we will still need to go through the reduction of order process. Before we approach this
process, we will discuss the evolutions of spin and mass, so that we may go through the reduction of order for each of these quantities at once.

3.4 Evolutions Of Spin And Mass

For completeness, we should finally calculate the 3-vector forms of the evolutions of spin and mass, according to (2.124b) and (2.124c). We recall these two equations to be

\[
\begin{align*}
\frac{D}{d\tau} S_{\alpha\beta} &= 2(g^{\gamma}_{\alpha} + u_{\alpha} u^{\gamma})(g^{\delta}_{\beta} + u_{\beta} u^{\delta})Q_{\epsilon}[_{\gamma}F_{\delta}^{\epsilon} - 2a^\gamma S_{\gamma[\alpha} u_{\beta]}, \\
\frac{D}{d\tau} \hat{m} &= \frac{1}{2}Q^{\alpha\beta} D_{\alpha\beta} + 2Q^{\alpha}_{\delta} F_{\beta\gamma} a^{[\gamma} u^{\alpha]}.
\end{align*}
\]

We will start our added analysis with the evolution of the spin tensor. To do this, we first recall the form of the spin tensor, where all of the zeroth components are zero, and as we know, the derivative of 0 is 0. This means that we can focus all of our attention to the spin-vector, \( \vec{S} \). Since the vector itself can be written as

\[
\vec{S} = \hat{x}S_{x} + \hat{y}S_{y} + \hat{z}S_{z},
\]

we will find the evolution of the individual components of the spin tensor, so that we have

\[
\frac{D}{d\tau} \vec{S} = \frac{D}{d\tau} S_{23} + \frac{D}{d\tau} S_{31} + \frac{D}{d\tau} S_{12}. \quad (3.148)
\]
Additionally, rather than attempting to tackle this whole thing at once, we can break it up into separate terms, similar to how we did the computation of the force. In fact, we see that we have five terms that will come out of (2.120):

\[ 2g_\alpha \gamma g_\beta \delta F_{\gamma \delta} = g_\alpha \gamma g_\beta \delta \left( Q^\epsilon \gamma F_{\delta \epsilon} - Q^\epsilon \delta F_{\gamma \epsilon} \right), \]  \hspace{1cm} (3.149a)

\[ 2g_\alpha \gamma u_\beta u_\delta Q^\epsilon \gamma = g_\alpha \gamma u_\beta u_\delta \left( Q^\epsilon \gamma F_{\delta \epsilon} - Q^\epsilon \delta F_{\gamma \epsilon} \right), \]  \hspace{1cm} (3.149b)

\[ 2u_\alpha u_\delta g_\beta \delta Q^\epsilon \gamma = u_\alpha u_\gamma g_\beta \delta \left( Q^\epsilon \gamma F_{\delta \epsilon} - Q^\epsilon \delta F_{\gamma \epsilon} \right), \]  \hspace{1cm} (3.149c)

\[ 2u_\alpha u_\gamma u_\beta u_\delta Q^\epsilon \gamma = u_\alpha u_\gamma u_\beta u_\delta \left( Q^\epsilon \gamma F_{\delta \epsilon} - Q^\epsilon \delta F_{\gamma \epsilon} \right), \]  \hspace{1cm} (3.149d)

\[ a_\gamma S_{\gamma \alpha u_\beta} = a_\gamma S_{\gamma \alpha} u_\beta - a_\gamma S_{\gamma \beta} u_\alpha. \]  \hspace{1cm} (3.149e)

In order to find each of these quantities, we should be able to focus our attention to just one component of our choosing, from which we should be able to extrapolate the remainder of the results. This will be exactly the same process as we saw in the previous sections, where we write out the specific components of the tensors in order to find the typical vector form. Since we are free to choose the component that we will be looking at, let’s look at the \( S_{23} = S_x \) component for each of these.

We start with the very first term. To start, we know that \( \alpha = 2 \) and \( \beta = 3 \), which restricts the possible contributions that we could get from summing over \( \gamma \) and \( \delta \). This is again thanks to the diagonal from of the metric, which restricts us to \( \gamma = 2 \) and \( \delta = 2 \), giving us

\[ g_2^2 g_3^3 (Q^\epsilon \gamma F_{\delta \epsilon} - Q^\epsilon \delta F_{\gamma \epsilon}) = Q^0_2 F_{30} - Q^0_3 F_{20} + Q^1_2 F_{31} - Q^1_3 F_{21} \]

\[ = p_y (-E_z) - p_z (-E_y) + (-\mu_z) B_y - \mu_y (-B_z) \]  \hspace{1cm} (3.150)

\[ = \left( \vec{E} \times \vec{p} \right)_x + \left( \vec{\mu} \times \vec{B} \right)_x. \]
Next, our second term still restricts us to $\gamma = 2$, but we do not have any such restriction on $\delta$, since $\beta$ and $\delta$ are the labels for the 4-velocity components. Here, we will need to specify $\delta$ first, then sum over $\epsilon$, and add all of the results up at the end.

1. $\delta = 0$:

$$g_2^2 u_3 u^0 (Q^\epsilon_{\ 2} F_{0\epsilon}) = \gamma^2 \beta_x (Q^1_{\ 2} F_{01} - Q^1_{\ 0} F_{21} + Q^3_{\ 2} F_{03} - Q^3_{\ 0} F_{23})$$

$$= \gamma^2 \beta_x [(-\mu_z) E_x - p_x (-B_z) + \mu_x E_z - p_z B_x]$$

$$= \gamma^2 \beta_x \left[ \left( \vec{E} \times \vec{\mu} \right)_y + \left( \vec{B} \times \vec{p} \right)_y \right].$$

2. $\delta = 1$:

$$g_2^2 u_3 u^1 (Q^\epsilon_{\ 2} F_{1\epsilon} - Q^\epsilon_{\ 1} F_{2\epsilon}) = \gamma^2 \beta_x \beta_z (Q^0_{\ 2} F_{10} - Q^0_{\ 1} F_{20} + Q^3_{\ 2} F_{13} - Q^3_{\ 1} F_{23})$$

$$= \gamma^2 \beta_x \beta_z [p_y (-E_x) - p_x (-E_y) + \mu_x (-B_z) - (-\mu_y) B_x]$$

$$= \gamma^2 \beta_x \beta_z \left[ \left( \vec{p} \times \vec{E} \right)_z + \left( \vec{B} \times \vec{\mu} \right)_z \right].$$

3. $\delta = 2$:

$$g_2^2 u_3 u^2 (Q^\epsilon_{\ 2} F_{2\epsilon} - Q^\epsilon_{\ 2} F_{2\epsilon}) = 0.$$

4. $\delta = 3$:

$$g_2^2 u_3 u^3 (Q^\epsilon_{\ 2} F_{3\epsilon} - Q^\epsilon_{\ 3} F_{2\epsilon}) = \gamma^2 \beta_x \beta_z (Q^0_{\ 2} F_{30} - Q^0_{\ 3} F_{20} + Q^1_{\ 2} F_{31} - Q^1_{\ 3} F_{21})$$

$$= \gamma^2 \beta_x \beta_z [p_y (-E_z) - p_z (-E_y) + (-\mu_z) B_y - \mu_y (-B_z)]$$

$$= \gamma^2 \beta_x \beta_z \left[ \left( \vec{E} \times \vec{p} \right)_x + \left( \vec{\mu} \times \vec{B} \right)_x \right].$$
With all of these results, we can immediately see that for this term, we have the result of
\[
2g^2 u_3^\delta Q^e \left[ (\vec{E} \times \vec{\mu})_y + (\vec{B} \times \vec{p})_y \right] + \gamma^2 \beta_z \left[ (\vec{\beta} \times [\vec{E} \times \vec{p}])_y + (\vec{\beta} \times [\vec{\mu} \times \vec{B}])_y \right].
\]

Moving onto the third term, we actually immediately see that this is going to have the same form as the second term, only with a swap of indexes out front. We can then see that we get
\[
2u_2 u_3^\gamma g^3 Q^e \left[ (\vec{B} \times \vec{E})_z + (\vec{p} \times \vec{B}) \right] + \gamma^2 \beta_y \left[ (\vec{\beta} \times [\vec{p} \times \vec{E}])_z + (\vec{\beta} \times [\vec{B} \times \vec{\mu}])_z \right].
\]

Then, we compare these two results and see that we can write this a sum of triple and quadruple vector products (all of which are currently the $x$-component).
\[
2( g^2 u_3^\delta + u_2 u_3^\gamma g^3 ) Q^e \left[ \gamma F_{\delta} \right] = \gamma^2 \left[ (\vec{\beta} \times (\vec{\mu} \times \vec{E}))_x + (\vec{\beta} \times (\vec{p} \times \vec{B}))_x \right] + \gamma^2 \left[ (\vec{\beta} \times (\vec{B} \times \vec{\mu}))_x \right].
\]

Before moving onto the fourth term with all of the velocities out front, let’s move to the last term.
For the final term in (2.120), we need only sum over $\gamma$. The resulting form becomes even simpler when we recall that $S_{\alpha\beta}$ has non-contributing components for any $\alpha, \beta = 0$, as well as the diagonal components. This leaves us with

$$2a^3 S_{\gamma [2 u_3]} = a^1 S_{12} u_3 - a^1 S_{13} u_2 - a^2 S_{23} u_2 + a^3 S_{32} u_3$$

$$= \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_x + \gamma^2 \beta_x \right] (S_x)(\gamma/\beta_x) - \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_x + \gamma^2 \beta_x \right] (-S_y)(\gamma/\beta_y)$$

$$- \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_y + \gamma^2 \beta_y \right] (S_x)(\gamma/\beta_y) + \left[ \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_z + \gamma^2 \beta_z \right] (-S_x)(\gamma/\beta_z).$$

Here, it becomes possible to extrapolate what the overall vector will look like. First, we look to the leading two terms above, and notice that they have the same overall sign (thanks to the negative sign in front of the $S_y$) and the component of the spin matches the component of the velocity. If we were to add up each of the components, then, we would be able to find a dot product between $\vec{S}$ and $\vec{\beta}$. Meanwhile, the inside of the brackets in those first two terms share the same components, so we would expect to find that they lead to

$$2\gamma^3 \left( \vec{S} \cdot \vec{\beta} \right) \left[ \gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \vec{\beta} + \dot{\vec{\beta}} \right].$$

where the factor of 2 comes from having two terms here. If we were to do this out for the other two components, we would find an extra factor of each component, as well as two of the $x$-components. If we now look to the second two terms, they also share the same sign, but now we are left with a scalar product between the acceleration terms and the velocity outside, so we are left with a total result of

$$2a^3 S_{\gamma [\alpha u_3]} \implies 2\gamma^3 \left( \vec{S} \cdot \vec{\beta} \right) \left[ \gamma^2 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \vec{\beta} + \dot{\vec{\beta}} \right] + 2\gamma^3 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \left[ \gamma^2 \left( \vec{\beta} \cdot \vec{\beta} \right) + 1 \right] \vec{S}$$
We are not quite done yet, as we can take advantage of the definition of the Lorentz factor. The quantity in second set of brackets is actually just $\gamma^2$, as shown below.

$$
\gamma^2 (\vec{\beta} \cdot \vec{\beta}) + 1 = \frac{(\vec{\beta} \cdot \vec{\beta})}{1 - (\vec{\beta} \cdot \vec{\beta})} + 1
$$

$$
= \frac{(\vec{\beta} \cdot \vec{\beta})}{1 - (\vec{\beta} \cdot \vec{\beta})} + \frac{1 - (\vec{\beta} \cdot \vec{\beta})}{1 - (\vec{\beta} \cdot \vec{\beta})}
$$

$$
= \frac{1}{1 - (\vec{\beta} \cdot \vec{\beta})}
$$

$$
= \gamma^2,
$$

which reduces our result to

$$
2a^\gamma S_{\gamma[u_\beta]} \rightarrow 2\gamma^3 (\vec{S} \cdot \vec{\beta}) \left[ \gamma^2 (\vec{\beta} \cdot \vec{\beta}) \vec{\beta} + \vec{\dot{\beta}} \right] + 2\gamma^5 (\vec{\beta} \cdot \vec{\dot{\beta}}) \vec{S} \quad (3.152)
$$

In a similar style, we can extrapolate the 3-vector form of results from (3.150) and (3.151) as

$$
g_\alpha^\gamma g_\beta^\delta (Q^\epsilon \gamma F_{\delta \epsilon} - Q^\epsilon \delta F_{\gamma \epsilon}) \rightarrow \left( \vec{E} \times \vec{p} \right) + \left( \vec{\mu} \times \vec{B} \right), \quad (3.153)
$$

$$
2(g_\alpha^\gamma u_\beta u^\delta + u_\alpha u^\gamma g_\beta^\delta) = \gamma^2 \left[ \vec{\beta} \times \left( \vec{\mu} \times \vec{E} \right) \right] + \gamma^2 \left[ \vec{\beta} \times \left( \vec{p} \times \vec{B} \right) \right]
$$

$$
+ \gamma^2 \left[ \vec{\beta} \times \left( \vec{p} \times \vec{E} \right) \right] + \gamma^2 \left[ \vec{\beta} \times \left( \vec{B} \times \vec{\mu} \right) \right]. \quad (3.154)
$$

Now that we have those sorted, it is time to move onto the fourth term in (2.120). Again, we are still looking at the $\alpha = 2, \beta = 3$ component, but now we need to sum over $\gamma, \delta, \text{ and } \epsilon$. As usual, we start with specifying $\gamma$, then specifying $\delta$, and then summing over $\epsilon$. We can then add everything back up at the end.
1. $\gamma = 0$:

   (a) $\delta = 0$:
   
   $$u_2 u_0^0 u_3 u_0^0 (Q^e_{\alpha} F_{0\alpha} - Q^e_{\alpha} F_{0\alpha}) = 0.$$  

   (b) $\delta = 1$:
   
   $$u_2 u_0^0 u_3 u_1^1 (Q^e_{\alpha} F_{1\alpha} - Q^e_{\alpha} F_{0\alpha}) = \gamma^4 \beta_y \beta_z \beta_x (Q^2_{01} F_{12} - Q^2_{11} F_{02})$$
   $$+ Q^3_{01} F_{13} - Q^3_{11} F_{03})$$
   $$= \gamma^4 \beta_y \beta_z \beta_x [p_y B_z - \mu_z E_y + p_z (-B_y) - (-\mu_y) E_z]$$
   $$= \gamma^4 \beta_y \beta_z \beta_x \left[ (\vec{\mu} \times \vec{E})_x + (\vec{p} \times \vec{B})_x \right]$$

   (c) $\delta = 2$:
   
   $$u_2 u_0^0 u_3 u_2^2 (Q^e_{\alpha} F_{2\alpha} - Q^e_{\alpha} F_{0\alpha}) = \gamma^4 \beta_y \beta_z \beta_y (Q^1_{01} F_{21} - Q^1_{12} F_{01})$$
   $$+ Q^3_{01} F_{23} - Q^3_{12} F_{03})$$
   $$= \gamma^4 \beta_y \beta_z \beta_y [p_x (-B_z) - (\mu_z) E_x + p_z B_z - \mu_x E_x]$$
   $$= \gamma^4 \beta_y \beta_z \beta_y \left[ (\vec{\mu} \times \vec{E})_y + (\vec{p} \times \vec{B})_y \right]$$

   (d) $\delta = 3$:
   
   $$u_2 u_0^0 u_3 u_3^3 (Q^e_{\alpha} F_{3\alpha} - Q^e_{\alpha} F_{0\alpha}) = \gamma^4 \beta_y \beta_z \beta_z [Q^1_{10} F_{31} - Q^1_{31} F_{01})$$
   $$+ Q^2_{01} F_{32} - Q^2_{21} F_{02})$$
   $$= \gamma^4 \beta_y \beta_z \beta_z [p_x (-B_y) - \mu_y E_x + p_y B_x - (-\mu_z) E_y]$$
   $$= \gamma^4 \beta_y \beta_z \beta_z \left[ (\vec{\mu} \times \vec{E})_z + (\vec{p} \times \vec{B})_z \right]$$
We can now see that for the $\gamma = 0$ component, we have

$$u_2 u_0^0 u_3 u^\delta (Q^\epsilon_0 F_{0\epsilon} - Q^\epsilon_0 F_{0\epsilon}) = \gamma^4 \beta_y \beta_z \left[ \vec{\beta} \cdot (\vec{\mu} \times \vec{E}) + \vec{\beta} \cdot (\vec{p} \times \vec{B}) \right].$$

(3.155)

2. $\gamma = 1$:

(a) $\delta = 0$:

$$u_2 u^1 u_0^0 (Q^\epsilon_1 F_{0\epsilon} - Q^\epsilon_0 F_{1\epsilon}) = \gamma^4 \beta_y \beta_x \beta_z \left[ Q^2_{01} F_{02} - Q^2_{10} F_{12} + Q^3_{01} F_{13} - Q^3_{01} F_{03} \right]$$

$$= \gamma^4 \beta_y \beta_x \beta_z \left[ \mu_x E_y - p_y B_z + (-\mu_y) E_z - p_z (-B_y) \right]$$

$$= \gamma^4 \beta_y \beta_x \beta_z \left[ (\vec{\mu} \times \vec{E})_x + (\vec{B} \times \vec{p})_x \right].$$

(b) $\delta = 1$:

$$u_2 u^1 u_3 u^0 (Q^\epsilon_1 F_{1\epsilon} - Q^\epsilon_0 F_{1\epsilon}) = 0.$$

(c) $\delta = 2$:

$$u_2 u^1 u_3 u^2 (Q^\epsilon_1 F_{2\epsilon} - Q^\epsilon_2 F_{1\epsilon}) = \gamma^4 \beta_y \beta_x \beta_z \beta_y \left[ Q^0_{01} F_{20} - Q^0_{10} F_{10} + Q^3_{12} F_{23} - Q^3_{21} F_{13} \right]$$

$$= \gamma^4 \beta_y \beta_x \beta_z \beta_y \left[ p_x (-E_y) - p_y (-E_x) + (-\mu_y) B_x - \mu_x (-B_y) \right]$$

$$= \gamma^4 \beta_y \beta_x \beta_z \beta_y \left[ (\vec{E} \times \vec{p})_y + (\vec{\mu} \times \vec{B})_y \right].$$
(d) $\delta = 3$:

$$u_2u_1^1u_3u_3^3(Q^e_{1F_3e} - Q^e_{3F_1e}) = \gamma^4 \beta_y \beta_x \beta_z \beta_x (Q^0_{1F_30} - Q^0_{3F_10})$$

$$+ Q^2_{1F_32} - Q^2_{3F_12})$$

$$= \gamma^4 \beta_y \beta_x \beta_z \beta_x [p_x(-E_z) - p_z(-E_x)$$

$$+ \mu_z(-B_x) - (-\mu_x)B_z]$$

$$= \gamma^4 \beta_y \beta_x \beta_z \beta_x \left[ \left( \vec{\beta} \times \vec{E} \right)_y + \left( \vec{B} \times \vec{\mu} \right)_y \right]$$

When we compare results from (c) and (d), we see that this is of the same form as a triple vector product, and we get

$$u_2u_1^1u_3u_3^3(Q^e_{1F_3e} - Q^e_{3F_1e}) = \gamma^4 \beta_y \beta_x \beta_z \beta_x \left[ \left( \vec{\beta} \times \vec{E} \right)_y + \left( \vec{B} \times \vec{\mu} \right)_y \right]$$

$$+ \gamma^4 \beta_y \beta_x \beta_z \left[ \left( \vec{\beta} \times \left[ \vec{E} \times \vec{\beta} \right] \right)_x \right]$$

$$+ \gamma^4 \beta_y \beta_x \beta_z \left[ \left( \vec{\beta} \times \left[ \vec{\beta} \times \vec{B} \right] \right)_x \right].$$

We will take a tactical pause here to consider this result and what we would end up getting in the future components of $\gamma$. Above, we see that the second factor of velocity is the same as the component of the resulting triple vector products in brackets. When we go to $\gamma = 2$ and $\gamma = 3$, we will have the factor of $\beta_x$ become $\beta_y$ and $\beta_z$, respectively. Additionally, we can already see that
this triple vector product will be similar to the one above. We can therefore confidently write

\[
\begin{align*}
\mathbf{u}_2 \mathbf{u}_3 \mathbf{u}_4 \mathbf{u}_5 & (Q^\epsilon \gamma F_{\delta \epsilon} - Q^\epsilon \delta F_{\gamma \epsilon}) = \gamma^4 \beta_y \beta_z \left[ \bar{\beta} \cdot \left( \bar{\mu} \times \mathbf{E} \right) + \bar{\beta} \cdot \left( \bar{\mathbf{B}} \times \bar{p} \right) \right] \\
+ & \gamma^4 \beta_y \beta_z \left[ \bar{\beta} \cdot \left( \mathbf{E} \times \bar{\mu} \right) + \bar{\beta} \cdot \left( \bar{\mathbf{B}} \times \bar{p} \right) \right] \\
+ & \gamma^4 \beta_y \beta_z \left[ \bar{\beta} \cdot \left( \bar{\mu} \times \mathbf{E} \right) \right] \\
+ & \gamma^4 \beta_y \beta_z \left[ \bar{\beta} \cdot \left( \bar{\mathbf{B}} \times \bar{p} \right) \right].
\end{align*}
\]

However, we can immediately see that the first two lines cancel, thanks to the usual property of vector products, where we gain a negative sign by reversing the order of the vectors. This leaves us with

\[
\begin{align*}
\mathbf{u}_2 \mathbf{u}_3 \mathbf{u}_4 \mathbf{u}_5 & (Q^\epsilon \gamma F_{\delta \epsilon} - Q^\epsilon \delta F_{\gamma \epsilon}) = \gamma^4 \beta_y \beta_z \left[ \bar{\beta} \cdot \left( \mathbf{E} \times \bar{\mu} \right) \right] \\
+ & \gamma^4 \beta_y \beta_z \left[ \bar{\beta} \cdot \left( \bar{\mu} \times \mathbf{E} \right) \right].
\end{align*}
\]

Unfortunately, there is no clear answer about what to do regarding the factors of \( \beta_y \) and \( \beta_z \). It appears that perhaps we should take a gander at what the other two components will give us. If we look to the results seen above, though, we see that we can reduce the amount of work needed, as the only contributing terms are when \( \gamma = 1, 2, 3 \) and \( \delta = 1, 2, 3 \). This is evident from the fact that the temporal components all canceled out with each other. We start with \( \alpha = 3 \) and \( \beta = 1 \), which is the \( S_y \) component.

1. \( \gamma = 1 \):
   
   (a) \( \delta = 1 \):

   \[
   \mathbf{u}_3 \mathbf{u}_4 \mathbf{u}_5 \mathbf{u}_5 (Q^\epsilon \gamma F_{1 \epsilon} - Q^\epsilon \delta F_{1 \epsilon}) = 0.
   \]
(b) $\delta = 2$:

\[
u_3 u_1 u^2 (Q^e_1 F_{2e} - Q^e_2 F_{1e}) = \gamma^4 \beta_x \beta_x \beta_y (Q^0_1 F_{20} - Q^0_2 F_{10})
+ Q^3_1 F_{23} - Q^3_2 F_{13})
= \gamma^4 \beta_x \beta_x \beta_y [p_x (-E_y) - p_y (-E_x)
+ (-\mu_y) B_x - \mu_x (B_x)]
= \gamma^4 \beta_x \beta_x \beta_y \left[ (\vec{E} \times \vec{p})_z + (\vec{\mu} \times \vec{B})_z \right].
\]

(c) $\delta = 3$:

\[
u_3 u_1 u^3 (Q^e_1 F_{3e} - Q^e_3 F_{1e}) = \gamma^4 \beta_x \beta_x \beta_z (Q^0_1 F_{30} - Q^0_3 F_{10})
+ Q^2_1 F_{32} - Q^2_3 F_{12})
= \gamma^4 \beta_x \beta_x \beta_z [p_x (-E_z) - p_z (-E_x)
+ \mu_z (-B_x) - (-\mu_y) B_z]
= \gamma^4 \beta_x \beta_x \beta_z \left[ (\vec{\mu} \times \vec{B})_y \right] + \left[ (\vec{E} \times \vec{\mu})_y \right].
\]

As before, we see that the summation of these three results gives us a triple vector product,

\[
u_3 u_1 u^\delta (Q^e_1 F_{\delta e} - Q^e_\delta F_{1e}) = \gamma^4 \beta_x \beta_x \beta_x \left[ \left( \vec{\beta} \times \left[ \vec{E} \times \vec{p} \right] \right)_x + \left( \vec{\beta} \times \left[ \vec{\mu} \times \vec{B} \right] \right)_x \right].
\]

Again, we see above will result in a scalar product so that we get

\[
u_3 u^\gamma u^\delta = \gamma^4 \beta_x \beta_x \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \left[ \vec{E} \times \vec{p} \right] \right) + \vec{\beta} \cdot \left( \vec{\beta} \times \left[ \vec{\mu} \times \vec{B} \right] \right) \right].
\]
Perhaps the solution is becoming more obvious. Recall that we are currently looking at the \( y \)-component of the spin, and the current form is void of any \( y \)-component of velocity out front. It actually looks similar to one of the two resultant terms of a vector product. Then, we look back to (3.156), and see the same form there. How do we get around this? Well, we can do the same calculation for when \( \alpha = 1 \) and \( \beta = 3 \), which will give us a relative minus sign. This can still be considered a valid operation if we write

\[
2 \frac{D}{d\tau} \vec{S} = \frac{D}{d\tau} S_{31} - \frac{D}{d\tau} S_{13} + \frac{D}{d\tau} S_{12} - \frac{D}{d\tau} S_{21} + \frac{D}{d\tau} S_{23} - \frac{D}{d\tau} S_{32}.
\]

(3.159)

We now have the our result for the time evolution of the spin. Recall that we also need to use the \( \frac{d\vec{S}}{dt} \) form, rather than the covariant derivative, so we take out a factor of \( \gamma \) from each side to properly account for it. We then get

\[
\frac{d\vec{S}}{dt} = \frac{1}{\gamma} \left( \vec{E} \times \vec{p} \right) + \frac{1}{\gamma} \left( \vec{\mu} \times \vec{B} \right) + \gamma \left[ \vec{\beta} \times \left( \vec{\beta} \times \left( \vec{p} \times \vec{E} \right) \right) \right]
+ \gamma \left[ \vec{\beta} \times \left( \vec{\beta} \times \vec{B} \times \vec{\mu} \right) \right] + \gamma \left[ \vec{\beta} \times \left( \vec{\mu} \times \vec{E} \right) \right]
+ 2\gamma^2 \left( \vec{S} \cdot \vec{\beta} \right) \left[ \gamma^2 \left( \vec{\beta} \cdot \vec{\beta} \right) \vec{\beta} + \vec{\beta} \right]
+ 2\gamma^4 \left( \vec{\beta} \cdot \vec{\beta} \right) \vec{S}.
\]

(3.160)

We are now free to move onto the evolution of mass, or (2.121), which is (to reiterate)

\[
\frac{D}{d\tau} \dot{m} = \frac{1}{2} Q^{\alpha\beta} \frac{D}{d\tau} F_{\alpha\beta} + 2Q^{\beta}_{\alpha} F_{\beta\gamma} a^{[\gamma} u^{\alpha]},
\]

which we will calculate by breaking this into its two separate terms. Obviously, we need to sum over \( \alpha \) and \( \beta \) for the first term, as well as \( \gamma \) in the second term. It will
be easier if we start with the first term, particularly since we have already calculated this result at the very beginning of this paper, where we saw

$$\frac{1}{2} u^\beta Q^{\gamma \delta} \nabla_\beta F_{\gamma \delta} = \frac{1}{2} Q^{\gamma \delta} \frac{D}{d\tau} F_{\gamma \delta} = \gamma \left[ \left( \vec{p} \cdot \frac{d\vec{E}}{dt} \right) - \left( \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right) \right],$$

where we note that we have lost a factor of $\gamma$ out front, since we do not have the extra velocity term out front. Additionally, we see that we are taking the covariant derivative of the mass, and so we divide out the Lorentz factor altogether, so that we have our first term as

$$\frac{1}{2} Q^{\alpha \beta} \frac{D}{d\tau} F_{\alpha \beta} \rightarrow \left( \vec{p} \cdot \frac{d\vec{E}}{dt} \right) - \left( \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right). \quad (3.161)$$

Moving onto the second term, we first expand the anti-symmetric components, so that we get

$$2Q_{\alpha}^{\quad \beta} F_{\beta \gamma} a^{[\gamma u^\alpha]} = Q_{\alpha}^{\quad \beta} F_{\beta \gamma} (a^\gamma u^\alpha - a^\alpha u^\gamma). \quad (3.162)$$

Next, we need to consider the change in the location of indices on the dipole tensor. If we look back to (3.45), then we see that we lowered the second index so that we had $Q^{\alpha \beta}$, which meant that the first column was multiplied by a factor of -1, and all of the electric dipole components became positive. However, now that we are lowering the first index, the top row is multiplied by -1, and we get

$$Q_{\alpha}^{\quad \beta} = \begin{pmatrix} 0 & -p_x & -p_y & -p_z \\ -p_x & 0 & -\mu_z & \mu_y \\ -p_y & \mu_z & 0 & -\mu_x \\ -p_z & -\mu_y & \mu_x & 0 \end{pmatrix}. \quad (3.163)$$
This is pretty much the same as what we saw from $Q^{\alpha \beta}$, only that all of the electric dipole moments are negative, rather than positive. Now that we have the anti-symmetric components expanded out and the new form of the dipole tensor, we can start working on writing out the components of the second term. As per usual, we start by specifying each $\alpha$, then specifying $\beta$, summing over $\gamma$, and adding up each component at the end.

1. $\alpha = 0$: When $\alpha = 0$, we have

$$Q_0^\beta F_{\beta \gamma} (F_{\beta \gamma}) (a^\gamma u^0 - a^0 u^\gamma),$$

which tells us immediately that when $\beta = \alpha$, the whole thing will evaluate to zero, since we will have the diagonal component of $Q_{\alpha \beta}$. This allows us the freedom to start with $\beta = 1$.

(a) $\beta = 1$:

$$Q_0^{-1} F_{1 \gamma} (a^\gamma u^0 - a^0 u^\gamma) = (-p_x) \left[ F_{10}(a^0 u^0 - a^0 u^0) + F_{11}(a^1 u^0 - a^0 u^1) \right. + F_{12}(a^2 u^0 - a^0 u^2) + F_{13}(a^3 u^0 - a^0 u^3) \bigg]$$

Notice that when $\gamma = \alpha$ and $\gamma = \beta$, we have those cancel out because either the quantity in parentheses cancels out, or we have the diagonal component of the field strength tensor. In the future, we will simply
ignore those terms then, since we know they cancel. Getting back to it, we have

\[ Q_0^{-1} F_{1\gamma}(a^\gamma u^0 - a^0 u^\gamma) = (-p_x)[F_{12}(a^2 u^0 - a^0 u^2) + F_{13}(a^3 u^0 - a^0 u^3)] \]

\[ = (-p_x) \left( B_x \left[ \gamma^5 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_y + \gamma^3 \dot{\beta}_y \right] \right. \]

\[ - B_y \left[ \gamma^5 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) \beta_z + \gamma^3 \dot{\beta}_z \right] \]

\[ = (-p_x) \left( \gamma^3 \dot{\beta}_y B_z - \gamma^3 \dot{\beta}_z B_y \right) \]

\[ = \gamma^3 p_x (\vec{B} \times \dot{\vec{\beta}})_x \]

This is a nice feature of this term, as the majority of the result has canceled itself from the anti-symmetric property, and so we can immediately see that the next two components are

\[ Q_0^{-1} F_{2\gamma}(a^\gamma u^0 - a^0 u^\gamma) = \gamma^3 p_y (\vec{B} \times \dot{\vec{\beta}})_y, \]

\[ Q_0^{-1} F_{3\gamma}(a^\gamma u^0 - a^0 u^\gamma) = \gamma^3 p_z (\vec{B} \times \dot{\vec{\beta}})_z. \]

It is evident that the \( \alpha = 0 \) component gives us the nice result of

\[ Q_0^{-1} F_{\beta\gamma}(a^\gamma u^0 - a^0 u^\gamma) = \gamma^3 \left[ \vec{p} \cdot (\vec{B} \times \dot{\vec{\beta}}) \right]. \] (3.164)

Unfortunately, the next three components will prove to be slightly trickier.

2. \( \alpha = 1 \):
(a) $\beta = 0$:

\[
Q_1 \gamma F_0 (-a^1 u^1 - a^1 u^7) = -p_x \left[ F_{02} (a^2 u^1 - a^1 u^2) + F_{03} (a^3 u^1 - a^1 u^3) \right]
\]

\[
= (-p_x) \left( E_y \left[ \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_y \beta_x + \gamma^3 \dot{\beta}_y \beta_x \\
- \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_y \beta_y - \gamma^3 \dot{\beta}_y \beta_y \right] \\
- p_x \left( E_x \left[ \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x \beta_x + \gamma^3 \dot{\beta}_x \beta_x \\
- \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x \beta_x - \gamma^3 \dot{\beta}_x \beta_x \right] \right) \\
= (-p_x) \left( \gamma^3 E_y \left[ \vec{\beta} \times \dot{\vec{\beta}} \right]_x + \gamma^3 E_x \left[ \vec{\beta} \times \dot{\vec{\beta}} \right]_y \right) \\
= \gamma^3 p_x \left[ (\vec{\beta} \times \dot{\vec{\beta}}) \times \vec{E} \right]_x
\]

(b) $\beta = 2$: Recall that when $\alpha = \beta$, the whole thing goes to zero, which is why we skipped straight to $\beta = 2$ from $\beta = 0$.

\[
Q_1 \gamma F_2 (-a^1 u^1 - a^1 u^7) = (-\mu_z) \left[ F_{20} (a^0 u^1 - a^1 u^0) + F_{23} (a^3 u^1 - a^1 u^3) \right]
\]

\[
= (-\mu_z) \left[ (-E_y) \left[ \gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x - \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x - \gamma^3 \dot{\beta}_x \right] \\
+ B_x \left[ \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x \beta_x + \gamma^3 \dot{\beta}_x \beta_x \\
- \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x \beta_x - \gamma^3 \dot{\beta}_x \beta_x \right] \right) \\
= -\gamma^3 \mu_z \left[ E_y \dot{\beta}_x + B_x \dot{\beta} \times \vec{\beta} \right]_y
\]
(c) $\beta = 3$:

$$Q_1^3 F_{3\gamma}(a^\gamma u^1 - a^1 u^\gamma) = \mu_y [F_{30}(a^0 u^1 - a^1 u^0) + F_{32}(a^2 u^1 - a^1 u^2)]$$

$$= \mu_y \left[ (-E_z) \left[ \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x - \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x - \gamma^3 \beta_x \right] 
- B_x \left[ \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_y \beta_x + \gamma^3 \beta_y \beta_x 
- \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \beta_x \beta_y - \gamma^3 \beta_x \beta_y \right] 
\right]$$

$$= \gamma^3 \mu_y \left[ E_z \beta_x + B_x \left( \dot{\vec{\beta}} \times \vec{\beta} \right)_z \right]$$

We look at this result for $\alpha = 1$, and see some nice results. We start by looking at the $\beta = 0$ component, and we can see that for the $\alpha = 2$ and $\alpha = 3$ components, this will give us a result of a scalar product with that triple vector product. Next, the two spatial components give us two terms. The first is the $x$ component of the vector product between $\vec{p}$ and $\vec{E}$, which is then multiplied by the $x$ component of $\dot{\vec{\beta}}$. Meanwhile, we see the second term gives us a triple vector product between the magnetic dipole moment and the one already there.

With these results above, it becomes easy to see that the evolution of mass in time can be written as

$$\frac{d\dot{m}}{dt} = \vec{p} \cdot \frac{d\vec{E}}{dt} - \vec{p} \cdot \frac{d\vec{B}}{dt} + \gamma^2 \left[ \vec{p} \cdot \left( \vec{B} \times \dot{\vec{\beta}} \right) \right] + \gamma^2 \left[ \vec{p} \cdot \left( \left[ \dot{\vec{\beta}} \times \vec{\beta} \right] \times \vec{E} \right) \right]$$

$$+ \gamma^2 \left[ \vec{E} \cdot \left( \vec{\beta} \times \vec{\mu} \right) \right] + \gamma^2 \left[ \vec{B} \cdot \left( \vec{\mu} \times \left[ \dot{\vec{\beta}} \times \vec{\beta} \right] \right) \right].$$

(3.165)

It is now time to move onto the reduction of order process so that we may tackle the numerical calculations.
3.5 Reduction Of Order

We look to replace each factor of acceleration and time derivative of acceleration within our results with this “reduction of order” process in order keep our equations to being second order in time.

We start with Equation (2.122), focusing first on the temporal component as usual.

\[ \dot{a}_0 = a_0 = \frac{q}{m} F_{0\beta} u^{\beta} \]

\[ = \frac{q}{m} [E_{0\sigma} u^{\sigma} + F_{01} u^{1} + F_{02} u^{2} + F_{03} u^{3}] \]

\[ = \frac{q}{m} \gamma (E_x \beta_x + E_y \beta_y + E_z \beta_z) . \]

When we add these components up, we see yet another scalar product (as expected), which gives us

\[ a_0 = \frac{q}{m} \gamma (\vec{\beta} \cdot \vec{E}) . \quad (3.166) \]

This is in fact a very positive result, as the temporal component of the 4-force is related to the change of energy over the change in time, or rather, the work done in a quantity of time. Notice that this quantity is void of any magnetic field component, thereby holding to the standard that magnetic fields do no work.

Next, when we move onto the spatial components, this becomes a well-known form of the traditional Lorentz force, but with an important note. When we consider this result, we will see an \( \vec{a} \) on the left-hand side. However, this is not describing the mechanical acceleration vector, which we have been using as \( \vec{\beta} \). Instead, it is describing the vector form of the spatial components of the acceleration 4-vector,
which we have defined in (3.18). Additionally, it is also important to note that this
form is the derivative of the 4-velocity with respect to the proper time.

\[
\frac{d\vec{u}}{d\tau} = \ddot{\vec{a}} = \frac{q}{m} \left( \vec{E} + \vec{\beta} \times \vec{B} \right). \tag{3.167}
\]

Now that we have this first part, we are free to move onto the second equation that
must be considered in the reduction of order process.

We now move our attention to the second equation that we use for the reduction
process, (2.123), in which we can immediately see that there is no possibility of
gaining any more values of acceleration or higher orders in time.

\[
\dot{a}_\beta = u^\sigma u^\gamma \nabla_\gamma \left( \frac{q}{m} F_{\beta\sigma} \right) + \left( \frac{q}{m} F_{\beta\sigma} \right) \frac{q}{m} F^{\sigma\delta} u_\delta.
\]

We should probably attack these two terms separately. Recall from Equation (3.19)
that the covariant derivative can be written as

\[
\frac{D}{d\tau} = u^\gamma \nabla_\gamma = \gamma \frac{d}{dt}. \tag{3.168}
\]

This will make dealing with the first term a little easier,

\[
u^\sigma u^\gamma \nabla_\gamma \left( \frac{q}{m} F_{\beta\sigma} \right) = \gamma u^\sigma \frac{d}{dt} \left( \frac{q}{m} F_{\beta\sigma} \right). \tag{3.169}
\]
This leaves us with only having to sum over the index $\sigma$, so for the temporal component, we have

$$\frac{\gamma q}{m} u^\sigma \frac{d}{dt} F_{0\sigma} = \frac{\gamma q}{m} \left( u^0 \frac{d}{dt} F_{00} + u^1 \frac{d}{dt} F_{01} + u^2 \frac{d}{dt} F_{02} + u^3 \frac{d}{dt} F_{03} \right)$$

$$= \frac{\gamma^2 q}{m} \left( \beta_x \frac{dE_x}{dt} + \beta_y \frac{dE_y}{dt} + \beta_z \frac{dE_z}{dt} \right)$$

$$= \frac{\gamma^2 q}{m} \left( \vec{\beta} \cdot \frac{d\vec{E}}{dt} \right).$$

Before we move onto the calculation for the second term, we will deal with the spatial components. As we can see from the temporal components, this will result in the time derivative of the fields within the Lorentz force, and so we should have (remembering to add in the extra factor of $\gamma$ from the covariant derivative)

$$\gamma u^\sigma \frac{d}{dt} \left( \frac{q}{m} F_{i\sigma} \right) = \frac{\gamma^2 q}{m} \left( \frac{d\vec{E}}{dt} + \vec{\beta} \times \frac{d\vec{B}}{dt} \right). \quad (3.171)$$

Moving onto the second term, we see the second factor of the electromagnetic tensor has one raised index and one lowered index. To calculate this, we take $F^{\alpha\beta}$ and multiply it by the metric,

$$F^{\alpha}_\beta = F^{\alpha\gamma} g_{\gamma\beta} = F^{\alpha0} g_{00} + F^{\alpha1} g_{11} + F^{\alpha2} g_{22} + F^{\alpha3} g_{33},$$

where we have set $\gamma = \beta$, since the metric is a diagonal matrix. Each of the spatial components of the field-strength tensor will be multiplied by 1, and thereby staying
the same. However, all of the components in the 0th column (or the electric field components) will be multiplied by -1, and so we have

$$F_{\alpha \beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \quad (3.172)$$

We can now move onto calculating this second term, starting with the $\beta = 0$ component. We then have two components to sum over, $\sigma$ and $\delta$.

$$\frac{q^2}{m^2} F_{0\sigma} F^{\sigma \delta} u^\delta = \frac{q^2}{m^2} \left( E_{0\sigma} E^{0 \sigma} u^\sigma + F_{01} F^{1 \delta} u^\delta + F_{02} F^{2 \delta} u^\delta + F_{03} F^{3 \delta} u^\delta \right)$$

With these three remaining terms, we can sum over $\delta$, recognizing that when $\delta$ is the same as the raised index on the field strength tensor, those terms go to zero. We can focus on the first term, giving us

$$F_{01} F^{1 \delta} u^\delta = F_{01} F^{1 0} u^0 + E_{01} E^{1 \sigma} u^\sigma + F_{01} F^{1 2} u^2 + F_{01} F^{1 3} u^3$$

$$= E_x E_x \gamma + E_x B_z \gamma \beta_y + E_x (-B_y) \gamma \beta_z$$

$$= \gamma E_x \left[ E_x + (\vec{\beta} \times \vec{B}) \right],$$

and as per usual, this gives us a nice scalar product result when we add everything up.

$$\frac{q^2}{m^2} F_{0\sigma} F^{\sigma \delta} u^\delta = \gamma \left[ E \cdot \vec{E} + \vec{E} \cdot (\vec{\beta} \times \vec{B}) \right]. \quad (3.173)$$

We now move onto the spatial components, with

$$\frac{q^2}{m^2} F_{1\sigma} F^{\sigma \delta} y^\delta,$$
and we should focus on the $\sigma = 0$ component for $\beta = 1$.

$$F_{10} F^0_\delta u^\delta = F_{10} F^0_0 u^0 + F_{10} F^0_1 u^1 + F_{10} F^0_2 u^2 + F_{10} F^0_3 u^3$$

$$= \gamma E_x (E_x \beta_x + E_y \beta_y + E_z \beta_z),$$

and so summing over the remaining $\sigma = 0$ terms for $\beta = 2, 3$ will give us

$$F_{10} F^0_\delta u^\delta = \gamma \vec{E} \left( \vec{\beta} \cdot \vec{E} \right). \quad (3.174)$$

If we then focus on the $\sigma = 1, 2, 3$ components, we should find the magnetic field contributions.

$$F_{12} F^2_\delta u^\delta = F_{12} F^2_0 u^0 + F_{12} F^2_1 u^1 + F_{12} F^2_2 u^2 + F_{12} F^2_3 u^3$$

$$= \gamma (B_z E_y + B_z (-B_z) \beta_x + B_z B_x \beta_z)$$

$$= \gamma B_z \left[ E_y + (\vec{\beta} \times \vec{B} \big)_y \right]$$

$$F_{13} F^3_\delta u^\delta = -B_y \left( F^3_0 u^0 + F^3_1 u^1 + F^3_2 u^2 + F^3_3 u^3 \right)$$

$$= -\gamma B_y (E_z + B_y \beta_x + (-B_x) \beta_y)$$

$$= -\gamma B_y \left[ E_z + (\vec{\beta} \times \vec{B} \big)_z \right]$$

If we add these two results together, we find that the singular magnetic field will go to the right-hand side of the resultant vector products,

$$\frac{q^2}{m^2} F_{i\sigma} F^\sigma_\delta u^\delta = \gamma \vec{E} \left( \vec{\beta} \cdot \vec{E} \right) + \gamma (\vec{E} \times \vec{B}) + \gamma \left[ (\vec{\beta} \times \vec{B}) \times \vec{B} \right]. \quad (3.175)$$
Lastly, we need to put all of the results from (3.171) through (3.174) together, to produce

\[
\dot{a}_0 = \frac{\gamma^2 q}{m} \left( \bar{\beta} \cdot \frac{d\bar{E}}{dt} \right) + \frac{\gamma q^2}{m^2} \left[ \bar{E} \cdot \bar{E} \right] + \frac{\gamma q^2}{m^2} \bar{E} \cdot (\bar{\beta} \times \bar{B})
\]

(3.176a)

\[
\dot{a} = \frac{\gamma^2 q}{m} \left( \frac{d\bar{E}}{dt} + \bar{\beta} \times \frac{d\bar{B}}{dt} \right) + \frac{\gamma q^2}{m^2} \left[ (\bar{\beta} \cdot \bar{B}) \bar{E} + (\bar{E} \times \bar{B}) \right]
\]

(3.176b) + \frac{\gamma q^2}{m^2} \left[ (\bar{\beta} \times \bar{B}) \times \bar{B} \right].

One thing that we need to mention is that these derived forms of (3.166), (3.167), and (3.176) are not the mechanical forms of acceleration and time derivative of acceleration, but rather the components of the 4-acceleration. The notation here is any factor that we write as an “\(a\)” will be the component of the 4-vector, while \(\beta\), \(\dot{\beta}\), and \(\ddot{\beta}\) are the specific mechanical quantities that are tied directly to the motion of the body.

In each of our results, whether it be the equations of motion or evolutions of spin, we need to find replacements for \(\dot{\beta}\) and \(\ddot{\beta}\), and so we need to find equations for these quantities. Additionally, in (3.176), we have on the left side our 4-vector components, so we start by writing \(\dot{a}^\mu\) in terms of the mechanical quantities by taking the time derivative of \(u^\mu\). Recall that we found \(a^\mu\) by taking the time derivative of \(u^\mu\), and so when we follow the same process, we find

\[
\dot{a}_0 = 4\gamma^7 \left( \bar{\beta} \cdot \dot{\bar{\beta}} \right)^2 + \gamma^5 \left( \bar{\beta} \cdot \ddot{\bar{\beta}} + \dddot{\bar{\beta}} \right)
\]

(3.177)

\[
\dot{a} = 4\gamma^7 \left( \bar{\beta} \cdot \dot{\bar{\beta}} \right)^2 \bar{\beta} + \gamma^5 \left( \bar{\beta} \cdot \ddot{\bar{\beta}} + \dddot{\bar{\beta}} \right) \bar{\beta} + \gamma^5 (\bar{\beta} \cdot \dot{\bar{\beta}}) \bar{\beta}
\]

+ 2\gamma^5 (\bar{\beta} \cdot \dot{\bar{\beta}}) \dot{\bar{\beta}} + \gamma^3 \dddot{\bar{\beta}}
\]

(3.178)
Now, we can set the appropriate quantities equal to each other, so that we get

\[
\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) = \frac{q \gamma}{m} (\vec{\beta} \cdot \vec{E}),
\]

(3.179)

\[
\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}}) \vec{\beta} + \gamma^2 \ddot{\vec{\beta}} = \frac{q}{m} (\vec{E} + \vec{\beta} \times \vec{B}),
\]

(3.180)

\[
4\gamma^7 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 + \gamma^5 (\vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2) = \frac{\gamma^2 q}{m} (\vec{\beta} \cdot \frac{d\vec{E}}{dt}) + \frac{\gamma q^2}{m^2} (\vec{E} \cdot \vec{E})
\]

(3.181)

\[
\frac{\gamma^2 q}{m} \left( \frac{d\vec{E}}{dt} + \vec{\beta} \times \frac{d\vec{B}}{dt} \right) + \frac{\gamma q^2}{m^2} \left[ (\vec{\beta} \cdot \vec{B}) \vec{E} + (\vec{E} \times \vec{B}) + (\vec{\beta} \times \vec{B}) \times \vec{B} \right]
\]

\[
= 4\gamma^7 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 \vec{\beta} + \gamma^5 (\vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2) \vec{\beta} + \gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \ddot{\vec{\beta}} + 2\gamma^5 (\vec{\beta} \cdot \dot{\vec{\beta}}) \ddot{\vec{\beta}} + \gamma^3 \dddot{\vec{\beta}}
\]

(3.182)

These results may seem intimidating, but they do provide quite a bit of help for us. First, recall that the reduction of order is designed for us to replace any form of acceleration or jerk, so that we keep the initial conditions to that of needing to specify only the positions and velocities. So, we look to (3.186), and see that we can immediately rewrite the ALD term using this last relationship.

\[
\frac{2q^2 \gamma^3}{3c^2} \left[ 4\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 \vec{\beta} + \gamma^2 (\vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2) \vec{\beta} + 3\gamma^2 (\vec{\beta} \cdot \dot{\vec{\beta}}) \ddot{\vec{\beta}} + \dddot{\vec{\beta}} \right]
\]

\[
\downarrow
\]

\[
\frac{2q^3 \gamma^2}{3mc^2} \left( \frac{d\vec{E}}{dt} + \vec{\beta} \times \frac{d\vec{B}}{dt} \right) + \frac{2q^4 \gamma}{3m^2 c^2} \left[ (\vec{\beta} \cdot \vec{B}) \vec{E} + (\vec{E} \times \vec{B}) + (\vec{\beta} \times \vec{B}) \times \vec{B} \right].
\]
Similarly, there is the same relationship in the first set of terms with the spin-vector. To see this, let’s write out every term with the spin vector together. Additionally, we will utilize a property of a scalar triple product,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}).$$

in the final set of terms (starting with the factor of \(\gamma^9\)). The purpose for this identity will soon become clear for us.

What do we see here? There are in fact four replacements that we can make. First, we see that the first four terms above are exactly the same form as the right side of (3.182), only in a vector product with \(\vec{S}\). So, we can take those four terms, and replace them with the left side of (3.182), only with an extra factor of \(1/\gamma c^2\).

The second replacement is again, the same as the right side of (3.182), with an extra factor thrown in, and it applies to (essentially) the third line. However, this replacement is a little more interesting, as it results in the left side of (3.182) being dotted into another vector product! However, this is still a legal operation, as we
will still end up with a scalar in the brackets multiplying a vector on the right-hand side.

The last two that we see come out of this are both of the same form as (3.180). The only difference here between these two are that they also involve a vector product or a scalar product with what remains in the terms. However, notice that in every single term that is above, we have the ability to make these substitutions, as they all have exactly the same common factors. We can then apply all of these changes so that (3.183) becomes:

\[
\vec{F}_{\text{spin}} = \frac{\gamma q}{mc^2} \left[ \left( \vec{S} \times \frac{d\vec{E}}{dt} \right) + \left( \vec{S} \times \left[ \vec{\beta} \times \frac{d\vec{B}}{dt} \right] \right) \right] + \frac{q^2}{m^2 c^2} \left[ (\vec{\beta} \cdot \vec{B}) \left( \vec{S} \times \vec{E} \right) \right] + \frac{q^2}{m^2 c^2} \left[ \left( \vec{S} \times \vec{E} \right) \right] + \frac{q^2}{m^2 c^2} \left[ \left( \vec{S} \times \vec{B} \right) \times \vec{B} \right] + \frac{q}{mc^2} \left[ \vec{S} \cdot \left( \frac{d\vec{S}}{dt} \times \vec{\beta} \right) \right] + \frac{q}{mc^2} \left[ \left( \vec{\beta} \times \frac{d\vec{B}}{dt} \right) \cdot \left( \vec{S} \times \vec{\beta} \right) \right] + \gamma \frac{q}{mc^2} \left[ \vec{S} \cdot \left( \frac{d\vec{S}}{dt} \times \vec{\beta} \right) \right] + \gamma \frac{q}{mc^2} \left[ \left( \vec{\beta} \times \frac{d\vec{B}}{dt} \right) \cdot \left( \vec{S} \times \vec{\beta} \right) \right] + \gamma \frac{q}{mc^2} \left[ \left( \vec{S} \times \vec{\beta} \right) \cdot \left( \vec{E} \times \vec{\beta} \right) \right] + \gamma \frac{q}{mc^2} \left[ \left( \vec{S} \times \vec{\beta} \right) \cdot \left( \vec{E} \times \vec{\beta} \right) \right] + \gamma \frac{q}{mc^2} \left[ \left( \vec{S} \times \vec{\beta} \right) \cdot \left( \vec{E} \times \vec{\beta} \right) \right] + \gamma \frac{q}{mc^2} \left[ \left( \vec{S} \times \vec{\beta} \right) \cdot \left( \vec{E} \times \vec{\beta} \right) \right] + \gamma \frac{q}{mc^2} \left[ \left( \vec{S} \times \vec{\beta} \right) \cdot \left( \vec{E} \times \vec{\beta} \right) \right]
\]

Before we completely reduce the order of our result in (3.147), we wish to physically reduce it by introducing the form of the dipole tensor that we will utilize in the numerical calculations. Earlier we mentioned that we cannot write a general form of the time evolution of the dipole tensor because it depends on the specific charged particle. In our numerical calculations, we wish to simulate the results for
elementary particles, such as the electron, which have a negligible electric dipole moment, so that \( \vec{p} = 0 \). For this, we find that the dipole tensor takes the form of

\[
Q^{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -\mu_z & \mu_y \\
0 & \mu_z & 0 & -\mu_x \\
0 & -\mu_y & \mu_x & 0
\end{pmatrix},
\]

which allows us to now write the force on the particle as

\[
\vec{F} = q(\vec{E} + \vec{\beta} \times \vec{B}) + \frac{1}{c} \mu_i \vec{v} B_i + \frac{2q^2 \gamma^3}{3c^2} \left[ 4\gamma^4 (\vec{\beta} \cdot \dot{\vec{\beta}})^2 \vec{\beta} + 2\gamma^2 (\vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2) \vec{\beta} \right] - \frac{2q^2 \gamma^3}{3c^2} \left[ 3\gamma^2 (\vec{\beta} \cdot \ddot{\vec{\beta}}) \vec{\beta} + \vec{\beta} \right] + \frac{\gamma^4}{c^2} (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{E} \times \vec{\mu}) + \frac{\gamma^2 d}{c^2 dt} (\vec{E} \times \vec{\mu}) + \frac{\gamma^2}{c^2} \vec{\beta} \times \left( \vec{\beta} \times \frac{d}{dt} \left[ \vec{B} \times \vec{\mu} \right] \right) + \frac{4\gamma^6}{c^4} (\vec{\beta} \cdot \dot{\vec{\beta}})^2 (\vec{S} \times \vec{\beta}) \]

\[
+ \frac{\gamma^2}{c^2} \vec{\beta} \left( \vec{\beta} \cdot \left( \vec{\beta} \times \frac{d}{dt} \left[ \vec{B} \times \vec{\mu} \right] \right) \right) + \frac{4\gamma^6}{c^4} (\vec{\beta} \cdot \dot{\vec{\beta}})^2 (\vec{S} \times \vec{\beta}) 
\]

\[
+ \frac{\gamma^4}{c^2} (\vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2) (\vec{S} \times \vec{\beta}) + \frac{3\gamma^4}{c^2} (\vec{\beta} \cdot \ddot{\vec{\beta}}) (\vec{S} \times \vec{\beta}) + \frac{\gamma^2}{c^2} (\vec{\beta} \cdot \ddot{\vec{\beta}}) (\vec{S} \times \vec{\beta}) \]

\[
+ \frac{\gamma^7}{c^2} \vec{\beta} \left( \vec{\beta} \cdot \ddot{\vec{\beta}} + \vec{\beta}^2 \right) \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right] + \frac{\gamma^2}{c^2} \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \frac{d}{dt} \vec{S} \right) \right] 
\]

\[
+ \frac{3\gamma^7}{c^2} \vec{\beta} (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{S} \times \vec{\beta}) + \frac{\gamma^7}{c^2} \vec{\beta} (\vec{\beta} \cdot \dot{\vec{\beta}}) \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \frac{d}{dt} \vec{S} \right) \right] 
\]

\[
+ \frac{\gamma^5}{c^2} \vec{\beta} \left[ \vec{\beta} \cdot \left( \vec{\beta} \times \vec{S} \right) \right].
\]

(3.186)
Using the same process as we did for the force due to the spin, we reduce the order of the terms involving the electric and magnetic fields, and find:

\[
\vec{F} = q\left(\vec{E} + \vec{\beta} \times \vec{B}\right) + \frac{2q^3\gamma^3}{3mc^2}\left(\frac{d\vec{E}}{dt} + \vec{\beta} \times \frac{d\vec{B}}{dt}\right) + \frac{2q^4\gamma}{3m^2c^2}\left(\left(\vec{\beta} \cdot \vec{B}\right)\vec{E} + \left(\vec{E} \times \vec{B}\right)\right) \\
+ \frac{2q^4\gamma}{3m^2c^2}\left(\left[\left(\vec{\beta} \times \vec{B}\right) \times \vec{B}\right]\right) + \frac{1}{c}\vec{\mu}_i\nabla \vec{B}_i + \frac{q\gamma}{mc^2}\left(\vec{\beta} \cdot \vec{E}\right)\left(\vec{E} \times \vec{\mu}\right) \\
+ \frac{\gamma^2}{c^2}\frac{d}{dt}\left(\vec{E} \times \vec{\mu}\right) + \frac{q\gamma}{mc^2}\left(\vec{\beta} \cdot \vec{E}\right)\left[\vec{\beta} \times \left(\vec{E} \times \vec{\mu}\right)\right] + \frac{\gamma^2}{c^2}\left[\vec{\beta} \times \frac{d}{dt}\left(\vec{E} \times \vec{\mu}\right)\right] \\
+ \frac{\gamma^2}{c^2}\left[\vec{\beta} \times \left(\vec{E} \times \vec{\mu}\right)\right] + \frac{q^2}{mc^2}\left[\left(\vec{S} \times \frac{d\vec{E}}{dt}\right)\right] + \frac{q^2}{mc^2}\left[\left(\vec{S} \times \frac{d\vec{B}}{dt}\right)\right] \\
+ \frac{q^2}{m^2c^2}\left[\left(\vec{\beta} \cdot \vec{B}\right)\left(\vec{S} \times \vec{E}\right) + \left(\vec{S} \times \left[\vec{E} \times \vec{B}\right]\right)\right] + \frac{q^2}{m^2c^2}\left(\vec{S} \times \left[\left(\vec{\beta} \times \vec{B}\right) \times \vec{B}\right]\right) \\
+ \frac{q}{mc^2}\left[\left(\frac{d\vec{S}}{dt} \times \vec{E}\right)\right] + \frac{q}{mc^2}\left[\left(\frac{d\vec{S}}{dt} \times \left[\vec{\beta} \times \vec{B}\right]\right)\right] + \frac{q}{mc^2}\left[\left(\vec{E} \cdot \left(\frac{d\vec{S}}{dt} \times \vec{\beta}\right)\right)\right] \\
+ \frac{\gamma^2q^2}{mc^2}\left[\left[\vec{\beta} \times \frac{d\vec{B}}{dt}\right] \cdot \left(\vec{S} \times \vec{\beta}\right)\right] + \frac{\gamma^2q^2}{m^2c^2}\left[\left(\vec{\beta} \cdot \vec{B}\right)\left(\vec{E} \cdot \left[\vec{S} \times \vec{\beta}\right]\right)\right] \\
+ \frac{\gamma^2q^2}{m^2c^2}\left[\left[\left(\vec{E} \times \vec{B}\right) \cdot \left(\vec{S} \times \vec{\beta}\right)\right] + \left[\left(\vec{\beta} \times \vec{B}\right) \times \vec{B}\right] \cdot \left(\vec{S} \times \vec{\beta}\right)\right]\vec{\beta}.
\]

Now, paying close attention to the fourth line, it would only be natural to shake one's head and think that there is still a \(\vec{\beta}\) so we cannot possibly be done with the reduced form. However, if we look up to (3.179) through (3.182), there is no explicit equation that we have that gives us just \(\vec{\beta}\) by itself. However, we can find one by doing something rather clever.
To find a reduced form of $\dot{\vec{\beta}}$, we start by looking to (3.180), and solving it just for $\dot{\vec{\beta}}$, which gives us

$$
\dot{\vec{\beta}} = \frac{q}{\gamma^2 m} \left( \vec{E} + \vec{\beta} \times \vec{B} \right) - \gamma^2 (\vec{\beta} \cdot \vec{\beta}) \vec{\beta}.
$$

Where does this get us? There is still a factor of $\dot{\vec{\beta}}$ on the right-hand side. However, if we look now to (3.179), this gives us a relationship for exactly that scalar product!

Here, we are short two factors of $\gamma$, but this works just fine, as we can account for them on the right-hand side, and we are left with

$$
\dot{\vec{\beta}} = \frac{q}{\gamma^2 m} \left( \vec{E} + \vec{\beta} \times \vec{B} \right) - \frac{q}{\gamma m} (\vec{\beta} \cdot \vec{E}) \vec{\beta}.
$$

(3.187)

With this, we have the ability to have the right-side completely reduced. Before typing out the whole thing, let’s just rewrite these two terms that need to corrected out:

$$
\frac{\gamma^4}{c^2} \left[ \dot{\vec{\beta}} \cdot (\vec{E} \times \vec{\mu}) \right] \vec{\beta} = \frac{\gamma^2 q}{mc^2} \left[ \vec{E} \cdot (\vec{E} \times \vec{\mu}) + \left( [\vec{\beta} \times \vec{B}] \cdot [\vec{E} \times \vec{\mu}] \right) \right] \vec{\beta}
$$

$$
- \frac{\gamma^2 q}{mc^2} (\vec{\beta} \cdot \vec{E}) \left( \vec{\beta} \cdot [\vec{E} \times \vec{\mu}] \right) \vec{\beta}.
$$

$$
\frac{\gamma^2}{c^2} \left[ \dot{\vec{\beta}} \times (\vec{B} \times \vec{\mu}) \right] = \frac{q}{mc^2} \left[ (\vec{E} \times (\vec{B} \times \vec{\mu})) \right] + \left( [\vec{\beta} \times \vec{B}] \times (\vec{B} \times \vec{\mu}) \right) \vec{\beta}
$$

$$
- \frac{\gamma q}{mc^2} (\vec{\beta} \cdot \vec{E}) \left( \vec{\beta} \times (\vec{B} \times \vec{\mu}) \right).
$$
We can finally put everything back so that we can write out the completely reduced form of this perturbed force, which we will use in our numerical calculations.

\[
\vec{F} = q\left(\vec{E} + \vec{\beta} \times \vec{B}\right) + \frac{2q^3\gamma^3}{3mc^2} \left(\frac{d\vec{E}}{dt} + \vec{\beta} \times \frac{d\vec{B}}{dt}\right) + \frac{2q^4\gamma}{3m^2c^2} \left[(\vec{\beta} \cdot \vec{B})\vec{E} + (\vec{E} \times \vec{B})\right]
\]

\[
+ \left(\frac{2q^4\gamma}{3m^2c^2} \left[(\vec{\beta} \times \vec{B}) \times \vec{B}\right] + \frac{1}{c^2} \vec{\nabla} B_i + \frac{q\gamma}{mc^2} \left(\vec{\beta} \cdot \vec{E}\right) \left(\vec{E} \times \vec{\mu}\right) + \frac{\gamma^2}{c^2} \frac{d}{dt} \left(\vec{E} \times \vec{\mu}\right)\right)
\]

\[
+ \frac{q\gamma}{mc^2} \left(\vec{\beta} \cdot \vec{E}\right) \left(\vec{\beta} \times \left(\vec{E} \times \vec{\mu}\right)\right) + \frac{\gamma^2}{c^2} \left(\vec{\beta} \times \frac{d}{dt} \left(\vec{B} \times \vec{\mu}\right)\right) + \frac{\gamma^2}{c^2} \left(\vec{\mu} \cdot \frac{d\vec{B}}{dt}\right)
\]

\[
+ \frac{\gamma^2}{mc^2} \left[\vec{E} \cdot \left(\vec{E} \times \vec{\mu}\right)\right] + \left[(\vec{\beta} \times \vec{B}) \cdot \left(\vec{E} \times \vec{\mu}\right)\right]
\]

\[
+ \frac{\gamma}{mc^2} \left(\vec{S} \times \vec{E}\right) + \left(\vec{\beta} \times \vec{E}\right) \left(\vec{E} \times \vec{B}\right)\right]
\]

\[
+ \left(\vec{S} \times \left[(\vec{\beta} \times \vec{B}) \times \vec{B}\right]\right) + \frac{q^2}{m^2c^2} \left[\vec{E} \cdot \left(\vec{d\vec{S}}/dt \times \vec{\beta}\right)\right]
\]

\[
+ \frac{q}{mc^2} \left[\left(\vec{\beta} \times \vec{B}\right) \cdot \left(\vec{d\vec{S}}/dt \times \vec{\beta}\right)\right] + \frac{\gamma^4q}{mc^2} \left(\vec{d\vec{E}}/dt \cdot \left(\vec{S} \times \vec{\beta}\right)\right)
\]

\[
+ \frac{\gamma^4q}{m^2c^2} \left(\vec{S} \times \left[(\vec{\beta} \times \vec{B}) \cdot \left(\vec{E} \cdot \left(\vec{d}\vec{S}/dt \times \vec{\beta}\right)\right)\right]\right)
\]

\[
+ \frac{\gamma^3q^2}{m^2c^2} \left[(\vec{E} \times \vec{B}) \cdot \left(\vec{S} \times \vec{\beta}\right)\right] + \left[(\vec{\beta} \times \vec{B}) \cdot \left(\vec{E} \cdot \vec{S} \times \vec{\beta}\right)\right]
\]

(3.188)

While this may seem to look like a step backwards and may not appear very “reduced,” we have successfully written this to not have any factor of \(\dot{\vec{\beta}}\) or any other factor of higher order in time.
In a similar manner, we find the reduced forms of the evolutions of spin and mass (eliminating the terms with any factor of the electric dipole moment as well). Additionally, we also add in each factor of $c$ in order have these terms be dimensionally correct by following the same procedure for force.

\[
\frac{d\vec{S}}{dt} = \frac{1}{\gamma c} \left( \vec{\mu} \times \vec{B} \right) + \frac{\gamma}{c} \left[ \vec{\beta} \times \left( \vec{\beta} \times \left[ \vec{B} \times \vec{\mu} \right] \right) \right] + \frac{\gamma}{c} \left[ \vec{\beta} \times \left( \vec{\mu} \times \vec{E} \right) \right]
\]
\[
+ \frac{2\gamma q}{mc} \left( \vec{\beta} \cdot \vec{E} \right) \vec{S} + \frac{2\gamma q}{mc} \left( \vec{\beta} \cdot \vec{S} \right) \left( \vec{\beta} \cdot \vec{E} \right) \vec{\beta} - \frac{2\gamma q}{mc} \left( \vec{\beta} \cdot \vec{S} \right) \left( \vec{\beta} \cdot \vec{E} \right) \vec{\beta}
\]
\[
+ \frac{2q}{mc} \left( \vec{\beta} \cdot \vec{S} \right) \left( \vec{E} + \vec{\beta} \times \vec{B} \right).
\]

\[
\frac{d\vec{m}}{dt} = -\frac{1}{c^2} \left( \vec{\mu} \cdot \frac{d\vec{B}}{dt} \right) + \frac{q}{mc^3} \left[ \vec{E} \cdot \left( \vec{E} \times \vec{\mu} \right) \right] + \frac{q}{mc^3} \left[ \vec{E} \cdot \left( \left[ \vec{\beta} \times \vec{B} \right] \times \vec{\mu} \right) \right]
\]
\[
- \frac{\gamma q}{mc^3} \left[ \vec{E} \cdot \left( \vec{B} \times \vec{\mu} \right) \right] \left( \vec{\beta} \cdot \vec{E} \right) + \frac{q}{mc^3} \left[ \vec{B} \cdot \left( \vec{\beta} \times \vec{E} \right) \right] \left( \vec{\mu} \times \left[ \vec{E} \times \vec{\beta} \right] \right)
\]
\[
+ \frac{q}{mc^3} \left[ \vec{B} \cdot \left( \vec{\mu} \times \left[ \left( \vec{\beta} \times \vec{B} \right) \times \vec{\beta} \right] \right) \right]
\]
\[
- \frac{\gamma q}{mc^3} \left( \vec{\beta} \cdot \vec{E} \right) \left[ \vec{B} \cdot \left( \vec{\mu} \times \left[ \vec{B} \times \vec{\beta} \right] \right) \right] \right).
\]

(3.189)

(3.190)

It is now time to take our results and consider the numerical results of what these corrections will provide.


3.6 Short Detour

3.6.1 Magnetic Fields Still Do No Work

Before discussing the process of the numerical calculations and their results, we wish to draw attention back to the evolution of mass in (3.165). As we had discussed during the reduction of order process, the electric dipole of elementary particles is zero and the dipole tensor reduces to the form of (3.185), and in the case of an absence of an electric field, (3.165) reduces to

\[
\frac{dm}{dt} = -\vec{\mu} \cdot \frac{d\vec{B}}{dt} + \gamma^2 \left[ \vec{B} \cdot \left( \vec{\mu} \times [\vec{\beta} \times \vec{\beta}] \right) \right].
\]

We can now address the age old contradiction that we introduced at the start of this paper: we know that magnetic fields do no work. However, we are also correctly taught that if we put a particle at rest with some magnetic dipole moment into a magnetic field, that it starts to move. If the particle is indeed at rest, then the above equation reduces even further to

\[
\frac{d}{dt} \left( m + \vec{\mu} \cdot \vec{B} \right) = 0,
\]

and so we can immediately see that this quantity in parentheses is what must be conserved in time. This gives us an interesting result, as it helps us answer the contradiction that we had introduced right at the beginning of the paper. Recall that magnetic fields cannot do work, but yet we also learn that a stationary particle with magnetic moment \( \vec{\mu} \) will start to move when put into a non-uniform magnetic
field. What we can see from (3.191) is that the interaction energy between the magnetic dipole moment and the non-uniform magnetic field is taking away energy from the rest mass of the particle!

We can now take solace in the fact that both concepts that we are taught in elementary physics courses are indeed correct. It is true that magnetic fields cannot do work and add energy to a particle in its own right. It is also true that a stationary particle with a dipole moment will start to move when placed in a non-uniform magnetic field. We can now see that what is truly happening is that the magnetic field is not actually doing work, but rather taking the energy of the particle and converting it into kinetic energy.

### 3.6.2 Moving To Higher Order

Up to this point, we have only been working with first order of \( \lambda \), our single parameter, in perturbation theory. As with other uses of perturbation theory, as say quantum mechanics, there is always the potential to move to higher orders if necessary. The process to do so in the context of this problem is outlined below.

To start, each of the assumptions will remain the same and we take Maxwell’s equations and the conservation of stress-energy as axioms. When we then move to the near-zone limit, the same general approach applies by scaling each of the quantities appropriately such that the body, mass, and charge each approach zero in the same asymptotic limit, while the zeroth order in the near-zone limit corresponds to the first order in the far-zone limit. It is with this in mind that we recall that the
time-time component of the metric when we used it remained flat, and our metric had the form

\[ \tilde{g}_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

However, if we were to move to higher orders, the metric could become extremely complicated extremely quickly and take the form of

\[ \tilde{g}_{00} = -1 - 2\lambda a_i(t_0)x^i, \]

or even (in higher order still)

\[ \tilde{g}_{00} = -1 - 2\lambda a_i(t_0) - \lambda^2 \left[ 2\dot{a}_i(t_0)x^i + (a_i(t_0)x^i)^2 + 2(\delta a_i(t_0)x^i) \right]. \]

What this means is that at some level, not only the coordinates that the body is in, but also its acceleration, perturbed acceleration from previous calculations of lower orders, and an explicit time dependence can arise when defining the center of mass and carrying out the calculations. It is not necessarily clear as to how one would have to redefine the center of mass (if at all necessary) and the same calculations done in first order could prove to be even more complicated and rigorous.

However, as we shall see from the results below, it is possible that second order perturbation theory may not necessary be needed in certain cases. Of course it is always important to be as thorough as possible and check, but the time needed for the calculation may not be worth the size of the corrections due to the perturbations.
CHAPTER 4
NUMERICAL CALCULATIONS AND SIMULATIONS

For the numerical simulations, we will consider only elementary particles with no electric dipole moment, so that the force on the particle is given by equation (3.188). Before discussing the coding and notebooks utilized in these calculations, we should first discuss the relationship between the spin vector and the magnetic dipole moment.

To find the spin tensor, from which we can find the spin vector, one would have to know what the energy of the stress-energy tensor of the particle is in general. This is not a trivial task. Instead, we know that for elementary particles, the spin tensor and the dipole tensor are proportional to each other through some constant. This constant relating the two vectors is written in terms of the dimensionless $g$-factor, the charge of the particle, and the mass of the particle.

\[ \vec{\mu} = g \frac{e}{2m} \vec{S}. \]  

(4.1)

As we know the value of the magnetic moments for elementary particles through the collection of data, we can turn this equation around and find the spin vector based on the measured values of the magnetic dipole moment,

\[ \vec{S} = \frac{2m}{ge} \vec{\mu}. \]  

(4.2)
Table 4.1: Table of particles to use with conversion between the SI MKS system of units and the Gaussian CGS system of units.

<table>
<thead>
<tr>
<th>Particle Name</th>
<th>Symbol</th>
<th>Magnetic Moment (J · T⁻¹)</th>
<th>Magnetic Moment erg · G⁻¹</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron</td>
<td>e</td>
<td>$-9.284 \times 10^{-24}$</td>
<td>$-9.284 \times 10^{-13}$</td>
</tr>
<tr>
<td>Proton</td>
<td>$p^+$</td>
<td>$1.410 \times 10^{-26}$</td>
<td>$1.410 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

We will be focusing on two particles in which we assume that the electric dipole moment is zero, or at least negligibly small. The values of the magnetic moment are seen in the table above.

## 4.1 Equations Used For Numerical Analysis

For the numerical calculations in Mathematica, we wish to mostly consider the Integrable Options Test Accelerator (IOTA) ring at Fermi National Accelerator Lab, which has time-independent fields. However, this time independence does not necessarily mean that the fields themselves do not change, and so we must go back to our discussion about the chain rule to write

$$\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + \left( \vec{v} \cdot \nabla \right) \vec{B}.$$  

Notice that here, the explicit dependence on time goes away, but this still leaves room for variations in the spatial components of the fields. Additionally, in the application we consider, the electric fields used in the acceleration of the particle
are turned off, leaving us with just the terms involving the spin of the particle and
the magnetic fields. (3.188) becomes

\[
\frac{d\vec{p}}{d\tau} = q\left(\vec{\beta} \times \vec{B}\right) + \frac{2q^3\gamma^3}{3mc^2}\left(\vec{\beta} \times \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\right) + \frac{2q^4\gamma}{3m^2c^2}\left(\vec{B} \times \vec{B}\right) \\
+ \frac{1}{c}\mu_i\vec{\nabla}B_i + \frac{\gamma^2}{c^2}\left[\vec{\beta} \times \left(\vec{v} \times \vec{\nabla}\right)\left(\vec{B} \times \vec{\mu}\right)\right] + \frac{\gamma^2}{c^2}\left[\vec{\mu} \cdot \left(\vec{v} \times \vec{\nabla}\right)\vec{B}\right] \vec{\beta} \\
+ \frac{\gamma^2}{c^2}\left[\vec{\beta} \cdot \left(\vec{\beta} \times \left[\vec{v} \times \vec{\nabla}\right]\vec{B}\right)\right] \vec{\beta} + \frac{q}{mc^2}\left[\left(\vec{B} \times \vec{B}\right) \times \left(\vec{B} \times \vec{\mu}\right)\right] \\
+ \frac{\gamma q}{mc^2}\left[\vec{S} \times \left(\vec{B} \times \left[\vec{v} \times \vec{\nabla}\right]\vec{B}\right)\right] + \frac{q^2}{m^2c^2}\left[\vec{S} \times \left[\left(\vec{B} \times \vec{B}\right) \times \vec{B}\right]\right] \\
+ \frac{q}{mc^2}\left(\frac{d\vec{S}}{dt} \times \left[\vec{\beta} \times \vec{B}\right]\right) + \frac{q}{mc}\left(\vec{B} \times \left[\left(\vec{B} \times \vec{B}\right) \times \vec{B}\right]\right) \vec{\beta} \\
+ \frac{q^4}{mc}\left(\left[\vec{S} \times \left(\vec{B} \times \left[\vec{v} \times \vec{\nabla}\right]\vec{B}\right)\right] \cdot \left[\vec{\beta} \times \vec{B}\right]\right) \vec{\beta} \\
+ \frac{q^3}{m^2c^2}\left(\left[\left(\vec{B} \times \vec{B}\right) \times \vec{B}\right] \cdot \left[\vec{S} \times \vec{\beta}\right]\right) \vec{\beta}.
\]

On the left-hand side of this equation, we have replaced the variable \( \vec{F} \) with the
definition of force, since force is given as the time derivative of momentum. In
relativistic systems, we know \( \vec{p} = \gamma m\vec{v} \), and so we will also need to consider that
we are taking the total time derivative of this whole left-hand side, or

\[
\frac{d\vec{p}}{d\tau} = \gamma^2 \frac{dm}{dt} \vec{v} + \gamma^2 m \frac{d\vec{v}}{dt} + \gamma m \frac{d\gamma}{dt} \vec{v}.
\]

What is interesting is that since we will now have more than one term that is second
order in time, we will need to go through the reduction of order process when we
evaluate the time derivative of the Lorentz factor. We have already seen, though,
that the time derivative of the Lorentz factor is

\[
\frac{d\gamma}{dt} = \gamma^3 \left(\vec{\beta} \cdot \dot{\vec{\beta}}\right).
\]
and since we have the extra factor of $\gamma$ out front, we see that this gives us exactly what we had in (3.179). Better yet, since we are considering a system with no electric field, this term will go to zero, and we write the entire force equation as

\[
\frac{dm}{dt} \gamma^2 \vec{v} + \gamma^2 m \frac{d\vec{v}}{dt} = q \left( \vec{\beta} \times \vec{B} \right) + \frac{2q^3 \gamma^3}{3mc^2} \left( \vec{\beta} \times \left( \vec{\nu} \cdot \vec{\nabla} \right) \vec{B} \right) + \frac{2q^4 \gamma}{3m^2 c^2} \left[ \left( \vec{\beta} \times \vec{B} \right) \times \vec{B} \right] \\
+ \frac{1}{c} \mu_i \vec{\nu} B_i + \frac{\gamma^2}{c^2} \left[ \vec{\beta} \times \left( \vec{\nu} \cdot \vec{\nabla} \right) \left( \vec{B} \times \vec{\mu} \right) \right] + \frac{\gamma^2}{c^2} \left[ \vec{\mu} \cdot \left( \vec{\nu} \cdot \vec{\nabla} \right) \vec{B} \right] \vec{\beta} \\
+ \frac{\gamma^2}{c^2} \left[ \vec{\beta} \cdot \left( \vec{\nu} \cdot \vec{\nabla} \right) \left( \vec{B} \times \vec{\mu} \right) \right] + \frac{q}{mc^2} \left[ \left( \vec{\beta} \times \vec{B} \right) \times \left( \vec{B} \times \vec{\mu} \right) \right] \\
+ \frac{\gamma q}{mc^2} \left[ \vec{S} \times \left( \vec{\beta} \times \left[ \vec{\nu} \cdot \vec{\nabla} \right] \vec{B} \right) \right] + \frac{q^2}{m^2 c^2} \left[ \vec{S} \times \left[ \left( \vec{\beta} \times \vec{B} \right) \times \vec{B} \right] \right] \\
+ \frac{q}{mc^2} \left( \frac{d\vec{S}}{dt} \times \left[ \vec{\beta} \times \vec{B} \right] \right) + \frac{q}{mc^2} \left[ \left( \vec{\beta} \times \vec{B} \right) \cdot \left( \frac{d\vec{S}}{dt} \times \vec{\beta} \right) \right] \vec{\beta} \\
+ \frac{\gamma^4 q}{mc^2} \left[ \left( \vec{\beta} \times \left[ \vec{\nu} \cdot \vec{\nabla} \right] \vec{B} \right) \cdot \left[ \vec{S} \times \vec{\beta} \right] \right] \vec{\beta} \\
+ \frac{\gamma^3 q^2}{m^2 c^2} \left[ \left( \vec{\beta} \times \vec{B} \right) \times \vec{B} \right] \cdot \left[ \vec{S} \times \vec{\beta} \right] \vec{\beta} + qmc^2 \left[ \vec{\beta} \right. \\
\times \left( \vec{\beta} \right. \\
\times \vec{B} \left. \right) \left. \right] + \left. \frac{q}{mc^2} \left( \vec{\beta} \cdot \vec{S} \right) \left( \vec{\beta} \times \vec{B} \right) \right].
\]

(4.4)

We also still need to consider the evolutions of spin and mass as well, and so we must also include the equations given by

\[
\frac{d\vec{S}}{dt} = \frac{1}{\gamma c} \left( \vec{\mu} \times \vec{B} \right) + \frac{\gamma}{c} \left[ \vec{\beta} \times \left( \vec{\beta} \times \left[ \vec{B} \times \vec{\mu} \right] \right) \right] + \frac{2q}{mc} \left( \vec{\beta} \cdot \vec{S} \right) \left( \vec{\beta} \times \vec{B} \right),
\]

(4.5)

and

\[
\frac{dm}{dt} = -\frac{1}{c^2} \left( \vec{\mu} \cdot \left[ \vec{\nu} \cdot \vec{\nabla} \right] \vec{B} \right) + \frac{q}{mc^2} \left[ \vec{B} \cdot \left( \vec{\mu} \times \left( \vec{\beta} \times \vec{B} \right) \times \vec{\beta} \right) \right].
\]

(4.6)

We have the time-independent fields, but we should quickly discuss the reason that we wish to use IOTA as our main consideration. As of mid-2020, the IOTA experiment is expected to be running single-electron experiments within the ring [15]. In fact, there have been observation points established already at the ends of
Table 4.2: Electron Conversion Between CGS and Working Units.

<table>
<thead>
<tr>
<th></th>
<th>CGS</th>
<th>Working</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>$9.11 \times 10^{-28}$ g</td>
<td>1</td>
</tr>
<tr>
<td>Charge</td>
<td>$-4.803 \times 10^{-10}$ statC</td>
<td>$-300$</td>
</tr>
<tr>
<td>Magnetic Moment</td>
<td>$-9.284 \times 10^{-15}$ erg·G$^{-1}$</td>
<td>$-0.6$</td>
</tr>
</tbody>
</table>

Table 4.3: Proton Conversion Between CGS and Working Units.

<table>
<thead>
<tr>
<th></th>
<th>CGS</th>
<th>Working</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>$1.67 \times 10^{-24}$ g</td>
<td>1835</td>
</tr>
<tr>
<td>Charge</td>
<td>$4.803 \times 10^{-10}$ statC</td>
<td>300</td>
</tr>
<tr>
<td>Magnetic Moment</td>
<td>$1.410 \times 10^{-15}$ erg·G$^{-1}$</td>
<td>$9.11 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

dipole magnets for when the experiments do begin [16]. This is the first experiment that is capable of handling single-particle dynamics within the context of accelerator physics, and so we wish to shed as much light as possible on this topic for future considerations.

4.2 Mathematica Process

In order to evaluate these numerical calculations, we utilize the program Mathematica 12’s NDSolve [17], which solves for numerical differential equations for specified times and evaluates them in terms of interpolating functions. These interpolating functions are what we base our results upon.

Due to the restrictions within the working precision and accuracy of the program, we make use of our own “unit system,” which we use to scale each of the quantities that we need. The following two tables provide the relative size of the values in the CGS unit system and our working system. We must also note that in our system, we keep the speed of light to a value of $c = 3 \times 10^8 \frac{m}{s}$.
With the results that we produce, it is important to understand that the numerical values are going to be relatively large, but it is not the value itself with which we concern ourselves. Rather, we simply wish to see what the magnitude of the possible corrections to the trajectory of the particle would be due to these perturbative terms.\footnote{The notebook file used to calculate the single electron in a dipole field is attached in Appendix E.}

### 4.2.1 Electron In A Dipole Field

The goal of this simulation is to observe the size of the perturbation over extended periods of time for single electrons of differing energies within a dipole field. To do this, we start by considering an electron with non-relativistic speeds, such that in our unit system $|\vec{v}| = 200$, giving us a Lorentz factor of

$$\gamma_{e,\text{non-rel}} = 1.$$  \hspace{1cm} (4.7)

The strength of the field used is equivalent to 1 T while pointing in the vertical direction, and the particle is set in an initial trajectory along the z-axis. This ensures that we will produce a system where the particle remains in the $x-z$ plane.

We begin the numerical calculations by evaluating the trajectory of the particle and the forces acting upon it for the Lorentz force only, as in (1.6). Once we calculate the interpolating functions, we then evaluate a second set of equations, this time involving the additional terms from (4.4)-(4.6). Lastly, we take the difference between the two results to observe the exact magnitude of the perturbations. For each of these calculations, we generate individual plots for the position, force, and
angular momentum of the body in each direction. The non-relativistic results are seen below in Figures (4.1)-(4.6).²

With the non-relativistic results, we see start by seeing the expected behavior for the Lorentz force in Figure 4.1. However, once we add in the perturbations, we can see that the motion stays similar, but begins to experience a damping that grows larger in time, which we can particularly see in Figure 4.3. In the $y$-direction, we can account for the deviation from 0 as numerical noise, as it is 15 orders of magnitude less than the other two directions, and the resulting helix-shaped trajectory would not be noticeable. On the other hand, we can see deviations that are reducing the amplitude of oscillations by nearly 10% in just 10 turns. This is certainly a noticeable effect, and one that should be considered.

In a similar manner to the $y$-component of the position and force vectors, the angular momentum appears to oscillate in the $x$ and $z$ components. However, when observing the magnitude of these oscillations, we can account for them as numerical noise. Meanwhile, the $y$-component of the spin of the body does deviate from zero, indicating a slight decrease in angular momentum as the particle experiences these perturbations.

While these results are certainly interesting, it is more illuminating to observe these results for high-energy electrons, since single-particle dynamics experiments of this type would most likely have to be done at relativistic energies. Just like the low-energy case, the Lorentz force produces the exact expected behavior, and so we will focus our attention to only the results of the Lorentz force and perturbation, as well as the resulting perturbation magnitude itself. We conduct these numerical

²For each of the $z$-component plots, we needed to adjust the initial value in order to avoid dividing by zero, as can be seen in Appendix E.
Figure 4.1: Lorentz Trajectory of a Single, Non-relativistic Electron in a Dipole Magnetic Field. The left column consists of graphs for position, while the right column consists of acceleration plots. Each descending row gives $x$, $y$, and $z$ components, respectively. Here, we see exactly what one would expect for the Lorentz trajectory of an electron: oscillatory motion in the $x - z$ plane, and no motion or acceleration in the $y$-direction.
Figure 4.2: A single electron in a static and uniform dipole magnetic field will experience a damping in its oscillations in the $x - z$ plane. The $y$-direction is accounted for as numerical noise.
Figure 4.3: We plot the magnitude of the perturbation as a percentage of the main Lorentz force contribution and can see that even in these few oscillations, the scale of the perturbation contributes a damping of up to 8% of the original trajectory of a single electron.
Figure 4.4: Electron’s Lorentz Trajectory Spin. Again, we come out to an expected result of the electron having oscillating angular momentum in the $y$-direction due to the change in directions from the usual motion of the particle, while the angular momentum that comes from the magnetic moment of the particle stays constant in the $z$-direction.
Figure 4.5: Total evolution of spin for $x$, $y$, and $z$ components, respectively. The $x$-component is accounted for as numerical noise, while there is no change to the $y$ or $z$ components from the spin seen when calculated with the Lorentz force.
Figure 4.6: Perturbation-Only Evolution Of Spin for the electron. Again, each of these are shown as percentages relative to the main contribution from the Lorentz force. Both the $x$ and $z$ components can be contributed as numerical noise, while the spin of the body along the $y$-direction slowly decreases. Over the course of these oscillations, the angular momentum reduces by less than 0.5%.
simulations for four cases: \( \beta = 0.8, \beta = 0.9, \beta = 0.99, \) and \( \beta = 0.9999. \) Respectively, their equivalent Lorentz factors are

\[
\begin{align*}
\gamma_{0.8} &= 1.66667, \\
\gamma_{0.9} &= 2.29416, \\
\gamma_{0.99} &= 7.08881, \\
\gamma_{0.9999} &= 70.71245.
\end{align*}
\] (4.8a-d)

While these simulations do not reach ultra-relativistic energy regimes, the resulting trend on the trajectories of the particles should be applicable to particles with higher energies.

Starting with the \( \beta = 0.8 \) case, we observe exactly the same behavior as the non-relativistic case, with the damping of the motion in the \( x-z \) plane, while producing what appears to be numerical noise in the \( y \)-direction. When plotting the relative magnitude of the perturbation to the major contribution from the Lorentz force, we take the ratio of the total perturbation and the Lorentz force, as can be seen in Appendix E.

In addition to the similar behavior for the position and force experienced by the electron, we notice more sporadic changes within the spin of the particle while still providing the same general trend.

One thing that we can immediately notice from the results of Figures 4.7-4.16 is that as the energy of the particle increases, the damping of the oscillations becomes larger. The damping of these oscillations, in the consideration of single particle dynamics in accelerators, could potentially cause the electron to be lost very quickly. It is therefore imperative that when designing these types of experiments that these effects be under consideration.
Figure 4.7: The expected trajectory for an electron at $\beta = 0.8$. Unlike the Lorentz force trajectory, where the particle’s oscillation amplitudes are fixed, we immediately see a damping effect within the $x$ and $z$ components. The $y$ components can be attributed to numerical noise, as it is still about 16 orders of magnitude smaller than the other two directions.
Figure 4.8: Relative magnitude of the perturbation. It is interesting to note that the perturbation grows and shrinks in time, similar to the of a wave packet.
Figure 4.9: Electron with velocity $\beta = 0.8$ evolutions of spin. The $x$ component is attributed to numerical noise. The $y$ component shares a similar behavior to that when the particle experiences only the Lorentz force, but the transitions are becoming more violent at higher velocities.
Figure 4.10: The effect of the perturbation terms on the spin are shown to be significant for the $y$-component and we attribute that to the violent behavior in the previous figure. The $x$ and $z$ components, while still oscillating, are insignificant due to their magnitude.
Figure 4.11: For an electron at $\beta = 0.9$, the damping effect becomes stronger in the $x$ and $y$ components. We can start see that the behavior of the particle is to eventually come to some equilibrium position.
Figure 4.12: The magnitude of the perturbation on the electron appears to be growing in importance for the trajectory and forces as the energy of the particle increases.
Figure 4.13: When comparing to the evolution of spin from the Lorentz force, as in 4.4, the violent changes in the angular momentum are becoming more dramatic at higher energies. It is also of note that the frequency begins to noticeably increase.
Figure 4.14: We can see that the oscillatory nature for the $y$-component of the spin becomes more extreme as the energy of the electron has increased.
Figure 4.15: The expected trajectory for an electron at $\beta = 0.99$ experiences extreme damping. The maximum amplitude by the end of the simulation is only 17% the initial amplitude the particle experiences. Meanwhile, the force in each direction becomes stronger in order to keep the damping higher.
Figure 4.16: Here, we can see that the magnitude of the total perturbation is extremely small to that expected from the Lorentz force. We can see the general shape of the Lorentz force trajectory and the magnitude of the expected trajectory vs the magnitude of a Lorentz force trajectory.
Figure 4.17: At $\beta = 0.9999$, we note that the trajectory in the $x$ and $z$ directions are mostly flat over this time scale.

One more illustration of this increase in damping is shown in Figures 4.17-4.19. To truly show the size of the effect, we include the initial Lorentz trajectory calculation before showing the results involving the perturbation.

While these results are certainly interesting, they are not exactly unexpected. If we take the $\beta = 0.9999$ case, and test the order of magnitude of each term in (4.4), the second term alone is roughly just two orders of magnitude smaller than that of
Figure 4.18: Comparing these plots to the previous Lorentz force trajectory, we can see just how dramatic the damping of the particles motion becomes at high energies.
Figure 4.19: At $\beta = 0.99$, we see that the magnitude of the overall expected trajectory remains extremely small when compared to the Lorentz force trajectory.
the Lorentz force term, which can lead to very large corrections in a short amount of time. This is exactly the behavior that we are seeing exhibited here.

It is also of note that when running the simulations, there were no changes to the mass of the electron at any of the energies. This is accountable to two factors: the first being that the first term in (4.6) does not produce a non-zero contribution from the nature of our field being a static and constant magnetic dipole field. The second term, however, appears that it could have some contribution. We are assuming that the dipole moment of the electron is pointed in the $z$-direction, and so when we take the vector product of this quantity with the triple product of $\left( \vec{\beta} \times \vec{B} \right) \times \vec{B}$, when the velocity vector is pointing in the $z$-direction, it is clear that this would go to zero. Additionally, since the velocity vector points in both a positive and negative direction for its $x$-component, any contribution from the $x$ will be immediately canceled. We would then see that this will apply to any particle where the magnetic moment is pointed purely in the $z$-direction.

### 4.2.2 Proton in a Dipole Field

For the proton, we follow exactly the same procedure as that with the electron, but we can immediately see that the results we produce give much smaller corrections. Starting with the same initial velocity for the proton in the non-relativistic regime, we see that the corrections due to the perturbation are about six orders of magnitude smaller than the trajectory itself, as seen below. Note that the time scale has been increased in order to properly show the relative effect over a few oscillations.
Figure 4.20: A proton experiencing a Lorentz force trajectory. As with the electron, the left column is position while the right is the acceleration.
Figure 4.21: For a proton at non-relativistic energies, the resulting trajectory and acceleration are exactly the same as that from the Lorentz force. We still account the small deviation $y$ to numerical noise.
Figure 4.22: While the perturbation and deviation from the Lorentz force trajectory does indeed seem to grow in time, the overall effect appears to be extremely small and would more than likely be unnoticeable.
Moving onto the relativistic cases, we again start at $\beta = 0.8$ and work our way up to $\beta = 0.99$. With each of these cases, the perturbation is extremely small, and ranges between five to six orders of magnitude smaller than the result from the Lorentz force trajectory. This could be more problematic than having a noticeable damping, as the magnitude difference could eventually lead to a loss of the proton in the beam pipe since the magnitude of the perturbation is so small, and could build over time without notice from the observer. The results are shown below.

Now that we can see that the scale of the perturbations is extremely small, we isolate just the $z$-component of the trajectory in order to observe the damping in just one direction. Additionally, we have increased the length of time for the simulation, so as to gain a better understanding of the damping magnitude. What we are seeing is that in each of these cases, no matter the time scale, the damping seems to have an extremely small effect and is not even noticeable in most cases. When this is all scaled down to the width of a beam pipe, it is possible that these effects could go unnoticed and lead to a loss of the proton once the effect builds over many revolutions.
Figure 4.23: The overall trajectory again does not have any noticeable damping for a proton at $\beta = 0.8$. The line extending vertically in the $x$ acceleration can be attributed to a bug in the resulting interpolating function from Mathematica.
Figure 4.24: The resulting magnitude of the perturbation that the single proton experiences is so small that it would go unnoticed.
Figure 4.25: As with the $\beta = 0.8$ case, a proton at $\beta = 0.9$ experiences a total trajectory of a form that appears to be identical to that of a Lorentz force trajectory.
Figure 4.26: We again see that the magnitude of the perturbation is growing in time. However, when comparing the scales of these plots to those of the overall trajectory, we can say that this is effectively zero due a difference of 20 orders of magnitude.
Figure 4.27: Again, at $\beta = 0.9999$ this damping effect that the proton experiences is extremely small. While it does build over time, it appears that it would take many turns around a beam pipe in order to become something to note.
CHAPTER 5
SUMMARY

The object of this thesis has been to shed light upon the history and a novel approach to the electromagnetic self-interaction force for point-like charged particles and expand upon the results previously obtained recently. We have discussed the process of rewriting the covariant formulation of the force, evolution of spin, and evolution of mass equations, resulting in Equations (3.111), (3.147), (3.160), and (3.165). These are the most general, relativistic forms of the first-order perturbation evaluation of the self-interaction force in vector notation.

We also showed that an apparent contradiction in elementary physics courses is not a contradiction at all. We saw that magnetic fields still do no work and do not add kinetic energy to a particle. Instead, particles that have some magnetic moment begin to move in non-uniform magnetic fields by converting some of the particle’s rest energy from its mass into kinetic energy.

After completing the analytical work, we took the results from equations (3.188)-(3.190) and applied them to numerical simulations. These simulations were specified to placing an electron and a proton into a constant magnetic dipole field in order to gauge the magnitude of the perturbative effect. These simulations resulted in the perturbations contributing large corrections to motion of the particle through time, especially as the initial energy of the particle increased. As expected, the magnitude of these corrections are reduced in significance as the particle’s mass increases. We have been able to show through simulations in Mathematica that
these damping effects to the oscillations might need to be considered when designing future experiments that probe single particle dynamics.


APPENDIX A

MATHEMATICAL BACKGROUND
We should first describe the three main elements that can be used to describe a physical object within a mathematical theory: scalars, vectors, and tensors.

1. A *scalar* can be used to describe the scaling, or the size, of physical quantities. For example, the mass of a particle is considered a scalar, as it tells us how much matter makes up that particle, which is a scaling property. Typically, they are written in the form $a$, and will be the notation that we follow in this paper.

2. A *vector* describes directed motion of a physical quantity. Typically, the three fundamental vectors that we think of the most are position, velocity, and acceleration. However, the notion of these three quantities should always be in reference to an observation’s point, which is usually described to be at the origin of some coordinate system. The typical notation, which we will also follow, is of the form $\vec{a}$, noting the bold-face of the letter and the arrow above.

3. A *tensor* can be used to describe the motion (or deformation) on a surface or a volume. Tensors are a more general form of vectors, which in turn give direction to scalars. The notation that will be used during the course of this paper is: $F^{\alpha\beta}$ (or depending on the context, $F_{\alpha\beta}$). This is the form of a rank-2 tensor, which can be represented by a matrix. The rank is determined by the number of indexes, and so a rank-1 tensor, $F^\alpha$, can be written as a column (or row) vector. Tensors of higher rank instead would have be represented as a multi-dimensional array of matrices. However, we will only need to concern ourselves with tensors up to rank-2.

Before moving onto vector analysis and fields, we should quickly touch upon the notion of a coordinate system. The world in which we live in can be described by
three spatial dimensions, which we can usually write in one of three different ways: Cartesian coordinates, cylindrical coordinates, and spherical coordinates. Each of these systems are equally valid and produce the same results, but they can provide a deeper insight into problems, depending on the system we choose.

Since we can define these coordinate systems using three directions, we have a number of ways in which we can write them. Starting with Cartesian coordinates, we can think of them as describing a flat space, so that each direction is pointed in a straight line. We then use the unit vector, \( \hat{a} \), to describe which direction we are pointed. It is common to define the coordinates in the form

$$ \vec{x} = \sum_{i=1}^{3} x_i \hat{x}_i = x_1 \hat{x} + x_2 \hat{y} + x_3 \hat{z} = (x_1, x_2, x_3), \quad (A.1) $$

where each \( x_i \) is simply the scaling to tell us how much the overall vector is pointed in each direction. However, in a more general form, we can write these unit vectors as

$$ (\hat{x}_1, \hat{x}_2, \hat{x}_3). $$

This becomes convenient when we would like to switch between coordinate systems, or even consider problems in a system not constrained by a singular coordinate system until the final steps.

We can then take this idea of a coordinate system, and expand our concept of space infinitely far away in all directions from some central location, or origin, and assign certain values at each point. This is what constitutes a field. Vector fields are exactly as one might expect, then: a vector field has a vector assigned to every point in space. On the other hand, we also have scalar fields, which have a scalar (or a value but no direction) assigned to every point in space. These fields are central
to the study of electricity and magnetism, as it has really provided the model of a field theory to describe physical phenomena.

We can now move onto vector algebra and vector analysis. As we know, the typical vector that we use to describe physical quantities in our world has three spatial components. When we have multiple vectors to consider, we then have certain mathematical operations that govern the resulting vector or scalar. To start, we consider vector addition: when adding two vectors together, the result will be a third vector of the form

\[ \vec{C} = \vec{A} + \vec{B} \]

\[ = (a_1 \hat{x}_1 + a_2 \hat{x}_2 + a_3 \hat{x}_3) + (b_1 \hat{x}_1 + b_2 \hat{x}_2 + b_3 \hat{x}_3) \]

\[ = (a_1 + b_1) \hat{x}_1 + (a_2 + b_2) \hat{x}_2 + (a_3 + b_3) \hat{x}_3 \]

\[ = c_1 \hat{x}_1 + c_2 \hat{x}_2 + c_3 \hat{x}_3. \]

Notice that each unit vector determines the result of the addition of the components for each vector. We can then do exactly the same thing for vector subtraction, which would simply result

\[ \vec{C} = \vec{A} - \vec{B} = \sum_{i=1}^{3} (a_i - b_i) \hat{x}_i. \]

We can also multiply vectors with scalars, which results in the multiplication of each component of the vector by the scalar.

\[ c\vec{A} = ca_1 \hat{x}_1 + ca_2 \hat{x}_2 + ca_3 \hat{x}_3. \]

Since we can add and subtract vectors together, as well as multiply vectors by scalars, it is then a natural question to ask if we can multiply and divide vectors
together. While there is in fact no defined operation dividing vectors, there are two specific ways to multiply vectors. The first, which we will refer to as the *scalar product*, results in a scalar. We can write this operation in the following way:

\[
\vec{A} \cdot \vec{B} = (a_1\hat{x}_1 + a_2\hat{x}_2 + a_3\hat{x}_3) \cdot (b_1\hat{x}_1 + b_2\hat{x}_2 + b_3\hat{x}_3) = (a_1b_1) + (a_2b_2) + (a_3b_3). \tag{A.2}
\]

Notice that in this operation, only the values for each corresponding component multiply each other. We then add up each of those products, which results in a single scalar value. This notation of the \( \cdot \) in between the two vectors is why this operation can also be called the “dot product.” However, the notion of “scalar product” is typically more illuminating. There are a number of properties of the scalar product, which we define below:

\[
\begin{align*}
\vec{A} \cdot \vec{B} &= \left| \vec{A} \right| \left| \vec{B} \right| \cos \theta, \\
\vec{A} \cdot \vec{B} &= \vec{B} \cdot \vec{A}, \\
\vec{A} \cdot (\vec{B} + \vec{C}) &= \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}, \\
\vec{A} \cdot \vec{A} &= \left| \vec{A} \right|^2 \cos(0) = A^2.
\end{align*}
\]

Here, \( \theta \) is the angle between the two vectors, so when we take the scalar product of \( \vec{A} \) with itself, there is no angle between the two vectors, and so \( \cos(0) = 1 \). However, if we take the scalar product \( \vec{A} \cdot \vec{B} \) when \( \vec{A} \) and \( \vec{B} \) are perpendicular to each other, we will get a null result.

\[
\vec{A} \cdot \vec{B} = 0 \quad \text{if} \quad \vec{A} \perp \vec{B}.
\]

The second operation through which we can multiply vectors is (rather unsurprisingly) called the *vector product*, which results in a vector. When taking the
vector product between two vectors, we can use the determinant form in order find
the result:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{x}_1(a_2b_3 - a_3b_2) + \hat{x}_2(a_3b_1 - a_1b_3) + \hat{x}_3(a_1b_2 - a_2b_1). \quad (A.3)$$

This form will provide us with great insight later on. This notation of the $\times$
sometimes leads to the notion of the “cross product,” but we shall stick with “vector
product” as it illuminates the result of the operation better (much like scalar versus
dot product). We can immediately notice that if $\vec{A} = \vec{B}$, then each component will
be the same, and so

$$\vec{A} \times \vec{A} = 0. \quad (A.4)$$

As with the scalar product, the operation of the vector product can be written in a
number of ways and has certain properties:

$$\vec{A} \times \vec{B} = \vec{A} \vec{B} \sin \theta \hat{n},$$

$$\vec{A} \times \vec{B} = -\left( \vec{B} \times \vec{A} \right)$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

We should then ask ourselves: can we pile these operations onto each other, so that
we have triple and quad products between numerous vectors? The answer is of
course, yes we can. The process to do so is simply apply the rules above numerous
times until something simpler comes out.
There are two examples of triple products, then, that we should briefly mention: the scalar triple product and the vector triple product. As one can guess from the names, they result in a scalar and vector, respectively. The scalar product can be written as a scalar product between a vector and cross product, and it can go through equivalent permutations.

\[
\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}). \tag{A.5}
\]

However, if we reverse the order of any of these vectors in the expressions above, we would need to apply the rule that \( \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \). On the other hand, we have the vector triple product, which is sometimes referred to as the “BAC-CAB” rule, since it can be written as

\[
\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}). \tag{A.6}
\]

Here, we see two quantities in parentheses that will give us scalars through the vector multiplication, and so we will end up with a scalar multiplying a vector in each term, which we have already seen results in a vector.

These operations that have been seen above are particularly useful when we apply them to the vector operator, \( \vec{\nabla} \), which is defined by each component acting as a derivative with respect to that specific component:

\[
\vec{\nabla} = \frac{\partial}{\partial x_1} \hat{x}_1 + \frac{\partial}{\partial x_2} \hat{x}_2 + \frac{\partial}{\partial x_3} \hat{x}_3 = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right). \tag{A.7}
\]
This operator can be used in exactly the same ways as seen above with vector multiplication. When we apply this to a scalar, we find the gradient of the scalar, which tells us how much that value changes through space.

\[ \nabla a = \sum_{i=1}^{3} \frac{\partial a}{\partial x_i} \hat{x}_i = \frac{\partial a}{\partial x_1} \hat{x}_1 + \frac{\partial a}{\partial x_2} \hat{x}_2 + \frac{\partial a}{\partial x_3} \hat{x}_3. \]  

(A.8)

The del operator plays a central role in Classical Electrodynamics with the next two operations: the divergence and curl. The divergence is the scalar product equivalent of vector multiplication, and it acts in exactly the same manner as scalar products:

\[ \nabla \cdot \vec{A} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \sum_i \partial_i a_i(x_j). \]  

(A.9)

This operation tells us how much a certain quantity either moves away from (hence divergence) or into a central point. This is seen in Gauss’ law and unnamed magnetic corollary describing the lack of magnetic monopoles. Before discussing the curl, the divergence gives us a very important identity, especially for this paper. We consider the chain rule, which we can define as

\[ \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}. \]

Here, we see that we can find the total derivative of some function with respect to a variable by instead taking the derivative of the same function with respect to a secondary function and then multiplying to the derivative of that secondary function with respect to the variable. This can then be applied to the time derivative of any quantity with the use of

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\vec{r}} \cdot \nabla. \]  

(A.10)
Let’s break this relationship down very quickly. First, we have separated out the temporal and spatial components of the total time derivative into separate terms. In the second term, we have the velocity, or time derivative of position and we take the scalar product of the velocity with the del operator. To see that this is exactly chain rule, we can look at just one component of the vectors,

\[ v_x \frac{\partial}{\partial x} = \frac{\partial x}{\partial t} \frac{\partial}{\partial x}. \]

We can then put the physical quantity that we desire to learn about to the right.

The curl gives a vector field that curls around (in three dimensions, of course) that central location, and can be found again using the determinant method.

\[ \vec{\nabla} \times \vec{A} = \begin{vmatrix}
\hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\
\partial_1 & \partial_2 & \partial_3 \\
a_1 & a_2 & a_3
\end{vmatrix} 
= \hat{x}_1 \left( \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right) + \hat{x}_2 \left( \frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3} \right) + \hat{x}_3 \left( \frac{\partial a_1}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right). \]  

This operation is what is seen in Faraday’s Law and Ampere’s Law and is how we can relate electric fields to magnetic fields in time-dependent cases.

We next need to discuss some tensor operations. The first concept to address, then, would be Einstein summation notation, which simply removes the summation symbol, \( \sum \), and implies that any index that is repeated in a term must be summed over each of the corresponding components. For example, if we go back to Equation (A.1), the summation can simply be written as

\[ \vec{x} = x_i \hat{x}_i. \]
However, things become a little more stringent when discussing tensor notation. Recall that rank-2 tensors have two indexes, such as with the electromagnetic field strength tensor,

\[
F^{\alpha\beta} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & B_z & -B_y \\
E_y & -B_z & 0 & B_x \\
E_z & B_y & -B_x & 0 \\
\end{pmatrix}
\]  

(A.13)

This tensor itself has some interesting properties that we will discuss later. However, the placement of the indexes matters because in order to raise or lower them, we need to multiply the tensor by the metric, which is what is used to describe the space-time geometry of the system. For a flat metric with time-reversal symmetry, we specifically choose a metric with signature \((-1, 1, 1, 1)\), so that the metric takes the form

\[
g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]  

(A.14)

We can then transform \(F^{\alpha\beta}\) into \(F_{\alpha\beta}\) with the operation

\[
F_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}F^{\mu\nu}
= g_{\beta\nu}F^{\nu}_{\alpha}
= \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & -B_y \\
-E_y & -B_z & 0 & B_x \\
-E_z & B_y & -B_x & 0 \\
\end{pmatrix}
\]  

(A.15)
Here, we can see that we first did a sum over the index $\mu$, which we can call a contraction, and the first index on $F$ became $\alpha$. We then contract the $\nu$ index, and the second index on $F$ becomes $\beta$, and looking to the matrix representation, we see that the signs on the electric field components have swapped. This is due to the nature of the flat metric and the fact that the electric field components are only along the first components of the field-strength tensor.

This contraction of tensor indexes can start to become rather complicated, but the idea stays the same throughout. One way to help oneself through the process is to specify numerically each index. A simple example, then, would be to find $F_{\alpha\beta}^\alpha$ from $F_{\alpha\beta}^{\alpha\beta}$ using the metric defined in (A.14). If we start by indexing our numbers from 0 and going up to 3, we have

$$F_{\alpha\beta}^\alpha = g_{\beta\mu} F^{\alpha\mu}$$

$$= g_{\beta 0} F^{\alpha 0} + g_{\beta 1} F^{\alpha 1} + g_{\beta 2} F^{\alpha 2} + g_{\beta 3} F^{\alpha 3},$$

and now we will need to consider the values in $\beta$. Since it is a diagonal matrix, we are lucky in that we know $\beta = \mu$, and so we can write

$$F_{\alpha\beta}^\alpha = (-1) F^{\alpha 0} + F^{\alpha 1} + F^{\alpha 2} + F^{\alpha 3}.$$
Now, we can specify each value of $\alpha$ from 0 to 3 in order to find the resulting tensor and matrix representation. In so doing, we would find that all electric field components would be zero, while the magnetic field components remain the same.

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}.$$  

This idea of specifying the desired indexes, as well as contracting and summing over them will be crucial later in the thesis.

Beyond the height of the indexes, the order is also important. Notice that when we were finding $F_{\alpha\beta}$, we had the first term as $F^{\alpha 0}$, which gave us that the first column in $F_{\alpha\beta}$ gained a negative sign. However, if we had been rather lazy and accidentally swapped the indexes of $\alpha$ and 0, instead of the first column being multiplied by -1, we would have found only positive values of the electric field.

$$F_{\alpha\beta} \neq \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$  

We can immediately see the importance, then, of the order of the indexes. Now that we have discussed the required mathematical background in order to find our way throughout the duration of the thesis, we can move onto our discussion of Classical Electrodynamics and how our problem arises.
APPENDIX B

DERIVATION OF THE LORENTZ FORCE LAW
We start by considering the conservation of linear momentum. We start by taking the full time derivative of the vector product between the electric and magnetic fields.

\[
\frac{\partial}{\partial t} \left( \vec{E} \times \vec{B} \right) = \vec{E} \times \frac{\partial \vec{B}}{\partial t} - \vec{B} \times \frac{\partial \vec{E}}{\partial t}.
\] (B.1)

Notice that the product rule produced a negative sign in front of the second term, due to the reversal of order within the cross product. We can then recall Maxwell’s equations, which relates the time derivatives of the fields to each other. As a reminder, the two equations that we are looking for are (using SI units)

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},
\]

\[
\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}
\]

Rearranging the second equation, and recognizing that these partial derivatives are equivalent to full time derivatives if the fields are only dependent upon time, we can write

\[
\frac{d}{dt} \left( \vec{E} \times \vec{B} \right) = -\vec{E} \times \left( \nabla \times \vec{E} \right) - c^2 \vec{B} \times \left( \nabla \times \vec{B} \right) - \frac{1}{\epsilon_0} \vec{j} \times \vec{B}
\] (B.2)

We can start by writing the first term for the electric field (and subsequently for the magnetic field) by using the following definition for a triple cross product:

\[
-\vec{E} \times \left( \nabla \times \vec{E} \right) = \frac{1}{2} \nabla \left( \vec{E} \cdot \vec{E} \right) - \nabla \cdot \left( \vec{E} \vec{E} \right) + \left( \nabla \cdot \vec{E} \right) \vec{E},
\] (B.3)
where the quantity $\vec{E}\vec{E}$ is known as the **dyadic product**, which produces a rank-2 tensor.

\[
A = \vec{a}\vec{b} = a_1\hat{x}_1b_1\hat{x}_2 + a_1\hat{x}_1b_2\hat{x}_2 + \cdots + a_3\hat{x}_3b_3\hat{x}_3
\]

\[
= a_i\hat{x}_ib_j\hat{x}_j
\]

\[
= \hat{x}_ia_ib_j\hat{x}_j,
\]

where a summation over the two indexes $i$ and $j$ is implied by the repetition in the term, known as Einstein summation notation. We can think of this in terms of vectors and matrices, where we have

\[
A = \left(\hat{x}_1, \hat{x}_2, \hat{x}_3\right) \begin{pmatrix}
  a_1b_1 & a_1b_2 & a_1b_3 \\
  a_2b_1 & a_2b_2 & a_2b_3 \\
  a_3b_1 & a_3b_2 & a_3b_3
\end{pmatrix} \begin{pmatrix}
  \hat{x}_1 \\
  \hat{x}_2 \\
  \hat{x}_3
\end{pmatrix}, \tag{B.4}
\]

and we can calculate this central matrix with the use of

\[
\begin{pmatrix}
  a_1b_1 & a_1b_2 & a_1b_3 \\
  a_2b_1 & a_2b_2 & a_2b_3 \\
  a_3b_1 & a_3b_2 & a_3b_3
\end{pmatrix} = \begin{pmatrix}
  a_1, & a_2, & a_3
\end{pmatrix} \begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{pmatrix}. \tag{B.5}
\]

Looking back to (A.3), we see that the last term has Gauss’ Law from Maxwell’s equations, which we can use to write this term as $\frac{\rho}{\epsilon_0}\vec{E}$. Using this procedure for
the magnetic field, as there are no magnetic monopoles, the third term will go away
and we can then write these two as

\[ \vec{E} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \cdot \left[ \frac{1}{2} (\vec{E} \cdot \vec{E}) I_3 - \vec{E} \vec{E} \right] \frac{\rho}{\epsilon_0} \vec{E}, \]  
(B.6)

\[ \vec{B} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \cdot \left[ \frac{1}{2} (\vec{B} \cdot \vec{B}) I_3 - \vec{B} \vec{B} \right], \]  
(B.7)

where we note that \( I_3 \) stands for the identity matrix,

\[ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

We can now define the electromagnetic linear momentum flux tensor, also known as
the Maxwell Stress Tensor as

\[ T = \epsilon_0 \left[ \frac{1}{2} (\vec{E} \cdot \vec{E} + c^2 \vec{B} \cdot \vec{B}) I_3 - (\vec{E} \vec{E} + \vec{B} \vec{B}) \right]. \]  
(B.8)

Armed with this stress tensor and our identities for the triple cross products, we
add these two triple products together, and so the time derivative can be written as
(remembering that we still have the term \( \vec{j} \times \vec{B} \) in (A.2))

\[ \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = -\frac{1}{\epsilon_0} \left( \vec{\nabla} \cdot T + \rho \vec{E} + \vec{j} \times \vec{B} \right). \]  
(B.9)

These last two terms are what is known as the force density,

\[ \vec{f} = \rho \vec{E} + \vec{j} \times \vec{B}. \]  
(B.10)
We can then integrate this force density over all space to find the total force on a test point charge:

\[ \mathbf{\bar{F}} = \int d^3x \mathbf{\bar{f}} \]

\[ = \int d^3x \left( \rho \mathbf{\bar{E}} + \mathbf{\bar{j}} \times \mathbf{\bar{B}} \right) \]

\[ = q \left( \mathbf{\bar{E}} + \mathbf{\bar{v}} \times \mathbf{\bar{B}} \right). \]  

(B.11)

Alas, we have arrived at the Lorentz force law. Notice that this came about by using the mathematical definitions of triple cross products and Maxwell’s equations. This means that the Lorentz force law is a consequence of the postulates of Classical Electrodynamics, and does not have to be postulated separately.
APPENDIX C

WHY IS THE MULTIPLICATION OF DISTRIBUTIONS NOT WELL-POSED?
We should start by defining the mathematical meaning of what a distribution is, and will be using [18] for the entirety of this discussion. A distribution is a linear functional which can map a function to a set of real numbers. In other words, a distribution maps a vector space to a field of scalar values. It can also be considered as a generalized function. If we take some test function, say \( \varphi \), and \( T \) as a distribution, we can write the value of \( T \) acting on the test function in one dimension as

\[
\langle T, \varphi \rangle = \int T(x)\varphi(x) \, dx.
\]  

(C.1)

However, we immediately run into a problem: there is no general meaning of a distribution having a value at some given point.

We consider three specific cases of multiplying distributions together:

1. A distribution multiplied to a smooth function.

2. Two distributions with singular supports disjointed.

3. Multiplying two distributions together using a Fourier transform.

We will look at each of these cases separately.

\section*{C.1 Smooth Functions}

We define our distribution \( u \) to be in the domain of all real numbers, and some smooth function \( \varphi \) to be a complex function of real numbers, or

\[
u \in \mathcal{D}(\mathbb{R}),
\]

(C.2a)

\[
\varphi \in \mathcal{C}^\infty(\mathbb{R}).
\]

(C.2b)
This tells us that the smooth function can be real or in an infinite complex plane.
The multiplication of these two on a test function is well defined, since we can write

$$\langle u\varphi, f \rangle = \langle u, \varphi f \rangle.$$  
(C.3)

So, we do not have a problem here. [18]

### C.2 Distributions with Disjoint Singular Supports

Let us start with a couple of definitions.

1. **Support of a Function**: A set of points where a function is non-zero. For example, the support of the Heaviside step-function, defined as

$$H(x) = \begin{cases} 
1 & x \geq 0 \\
0 & x < 0 
\end{cases},$$

is

$$\text{Supp}[H(x)] = [0, +\infty].$$

Meanwhile, we can define the support of a Dirac delta as

$$\text{Supp}[\delta(x)] = \{0\}.$$
2. **Singular Support**: the points at which a distribution fails to be a smooth function. For the two examples of the Heaviside step-function and the Dirac Delta, we have

\[ \text{SingSupp}[\delta(x)] = \text{SingSupp}[H(x)] = \{0\}. \]

3. **Disjoint**: when the intersection between two objects is null.

\[ A \cap B = \emptyset \]

**Theorem C.2.1** If \( u \) and \( v \) are two distributions and \( \text{SingSupp}(u) \cap \text{SingSupp}(v) = \emptyset \), then the product between the two is well defined. [19]

To prove this, we start with some test function \( f \), which is non-zero outside of the singular support of \( v \). We can then multiply the two such that \( vf \) is a smooth function. Since \( vf \) is a smooth function, we can define the product of \( u \) and \( v \) through the product of \( u \) and \( vf \) since

\[ \langle u, vf \rangle = \langle uf, v \rangle, \]

as we saw earlier. This works similarly for \( f \) being non-zero outside the singular support of \( u \). So far so good. Notice that this theorem only holds for the case of the singular supports of \( u \) and \( v \) not intersecting. This is not the case with the same Dirac delta, as we saw in the Introduction, so we move onto the last case.
C.3 Using a Fourier Transform

The Fourier transform of a product of distributions is the convolution of the Fourier transform of each individual distribution, where a convolution is the integral product of two functions after one is reversed and shifted.

\[(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) \, d\tau.\]  \hfill (C.4)

We can define our Fourier transform as

\[\mathcal{F}[u(x)] = \hat{u}(k) = \int \! dx \, u(x)e^{ikx}.\]  \hfill (C.5)

With these two definitions, we can write the Fourier transform of the product of distributions as

\[\mathcal{F}(uv) = \mathcal{F}(u) * \mathcal{F}(v).\]  \hfill (C.6)

Now, if \(u\) and \(v\) are distributions, we can define their product as \(w\) if and only if for each point \(x\), there exists some function \(f\) where \(f = 1\) near that point. So, for each value \(k\), the following integral must be convergent.

\[\hat{f}^2w = \left(\hat{f}u * \hat{f}v\right) = \int \hat{f}u(q)\hat{f}v(k - q) \, dq.\]  \hfill (C.7)

It is important to note that in order for an integral to be absolutely convergent the integral of the absolute value of some function must evaluate to some finite, real number, such that

\[\int_{0}^{\infty} |f(x)| \, dx = L,\]  \hfill (C.8)
where $L$ is the finite, real number.

To highlight the issue at hand, we apply this to the Dirac delta. First, we note that the Fourier transform of the Dirac delta, $\hat{\delta}$ evaluates as

$$\hat{\delta} = \int_{-\infty}^{\infty} \delta(x - x_0)e^{ikx} \, dx = e^{ikx_0},$$  \hspace{1cm} (C.9)

and a reasonable assumption for us to make is that the particle we consider is at the origin, and so $x_0 = 0$, which forces the exponential to evaluate to 1. \text{[13]}

$$e^{ikx_0} = e^0 = 1.$$  \hspace{1cm} (C.10)

Then, for any test function fulling the previous conditions, it follows that

$$f \delta(x) = f(0)\delta(x),$$  \hspace{1cm} (C.11)

such that the Fourier transform of the function multiplied with the Dirac delta also evaluates to 1.

$$\hat{f} \delta = \hat{\delta} = 1.$$  \hspace{1cm} (C.12)

We look back to C.7), and putting this definition back into that integral, when we multiply $(f\delta)(f\delta)$, we will have

$$\hat{f^2\delta^2} = \int \hat{f}\delta(x)\hat{f}\delta(q - x) \, dx = \int (1)(1) \, dx.$$  \hspace{1cm} (C.13)
For reasons of being thorough, we check to see if this is absolutely convergent.

\[ \hat{f}^2 \delta^2 = \int_{0}^{\infty} |1| \, dx = \infty - 0 = \infty. \]  \hspace{1cm} (C.14)

Since infinity is not a real, finite number by definition, this integral is not absolutely convergent and the product of two Dirac deltas is not defined.

This is the essence and heart of our problem at hand. If this integral converged, we could easily take the Fourier transform of the Lorentz force of a particle interacting with the fields that it emitted, solve the differential equation, and take the inverse Fourier transform. However, this is not possible, and so other methods must be followed.
APPENDIX D

DERIVATION OF THE LARMOR FORMULA
To see why accelerating particles radiate, we look to a derivation of the Larmor formula and then generalize it to a relativistic form. This derivation will be following that done in Chapter 14 of Jackson, starting with an accepted form of the electric field from an accelerating point charge [2]:

\[
\vec{E}(\vec{x}, t) = e \left[ \frac{\hat{n} - \vec{\beta}}{\gamma^2 (1 - \vec{\beta} \cdot \hat{n})^3 R^2} \right]_{ret} + \frac{e}{c} \left[ \frac{\hat{n} \times \left[ \left( \hat{n} - \vec{\beta} \right) \times \vec{\dot{\beta}} \right]}{(1 - \vec{\beta} \cdot \hat{n})^3 R} \right]_{ret},
\]

where the subscript “ret” means that the quantities inside the brackets are evaluated at the retarded time. Now, if the charge is in a frame where its velocity is small compared to the speed of light, \(\vec{\beta}\) becomes small, and so the first term drops out and we are left with

\[
\vec{E} = \frac{e}{c} \left( \frac{\hat{n} \times \left( \hat{n} \times \dot{\vec{\beta}} \right)}{R} \right)_{ret}.
\]

We also already know that we can write the magnetic field as

\[
\vec{B} = \left[ \hat{n} \times \vec{E} \right]_{ret},
\]

which means that when we find the Poynting vector (recall the definition back in Chapter 2), this will become

\[
\vec{S} = \frac{c}{4\pi} \left| \vec{E} \right|^2 \hat{n}.
\]

As we have already seen, the power radiated is the Poynting vector integrated over some surface area. Applying that operation here, we notice that the factors of \(e\) and \(c\) become squared, the acceleration, \(\ddot{\vec{\beta}}\) will stay in the absolute value, and the
4\pi factor will cancel from the angular part of the integral. This leaves us with the total power radiated as

\[ P = \frac{2e}{3c} |\vec{\beta}|^2 = \frac{2e^2}{3c^3} |\vec{\dot{v}}|^2. \]  

(D.4)

Immediately, we see that the power radiated by a charge goes to zero if there is no acceleration. If a particle is in a static system, it will not radiate.

We can now look to generalize (D.4) to a relativistic form, in order to ensure that even at relativistic speeds and energies that a particle will not radiate if it is static. Since we have the acceleration, \( \vec{\dot{v}} = \vec{a} \), squared in (D.4), we can write it in terms of a scalar product,

\[ P = \frac{2e^2}{3c^3} (\frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt}). \]  

(D.5)

In order to write this in relativistic form, we go to covariant and contravariant notation, where the 4-velocity is defined as

\[ u^\mu = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}, \]

and the derivative (or 4-acceleration) will have to be with respect to the proper time, giving us

\[ P = -\frac{2e^2}{3c^3} \left( \frac{du_\mu}{d\tau} \frac{d\mu}{d\tau} \right). \]  

(D.6)

However, from relativistic kinematics, we know that the proper time can be written in terms of \( t \) (the time that the observer experiences), using the notion of time dilation, such that

\[ \gamma d\tau = dt, \]
and so when we take the derivative of $u_\mu$ with respect to $\tau$, we can actually use $t$ in the usual way. Starting with the $0^{th}$ component, we see that we will only have to take the derivative of the Lorentz factor, since $c$ is a constant. This process is shown below.

\[
\frac{d\gamma}{dt} = c\frac{d\gamma}{dt} = c\frac{d}{dt} \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = c\left(-\frac{1}{2}\right) \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \left(-2\frac{v}{c^2}\right) = \gamma^3 \frac{v}{c}. \tag{D.7}
\]

Next, we move onto the spatial component, $\gamma\vec{v}$ of the four-velocity, and see that we will have to use the product rule. However, rather than making it harder on ourselves, we already see that we have taken the derivative of the Lorentz factor. The second term will just be a derivative of the velocity, which we already know to be the acceleration. This means that we can immediately write

\[
\frac{d\gamma\vec{v}}{dt} = \left(\gamma^3 \frac{v}{c^2}\right) \vec{v} + \gamma \dot{\vec{v}}. \tag{D.8}
\]
Putting this all together, we can write
\[
\frac{du^\mu}{d\tau} = \gamma \frac{du^\mu}{dt}
\]
\[
= \gamma \begin{pmatrix}
\gamma^3 \frac{v \dot{v}}{c} \\
\left( \gamma^3 \frac{v \dot{v}}{c^2} \right) \vec{v} + \gamma \dot{\vec{v}} \\
\gamma^4 \frac{v \dot{v}}{c} \\
\left( \gamma^4 \frac{v \dot{v}}{c^2} \right) \vec{v} + \gamma^2 \dot{\vec{v}}
\end{pmatrix}
\] (D.9)

Instead of the metric used throughout the majority of this paper, with a signature of (-1,1,1,1), we will be reversing the components of the metric so that the spatial components are now negative: (1,-1,-1,-1). Due to this, when we lower the index \(\mu\), we will get
\[
\frac{du_\mu}{d\tau} = \left( \gamma^4 \frac{v \dot{v}}{c} \vec{v} - \left( \gamma^4 \frac{v \dot{v}}{c^2} \right) \vec{v} - \gamma^2 \dot{\vec{v}} \right)
\] (D.10)

Notice that both (D.9) and (D.10) both have factors of \(v\) and \(\dot{v}\) multiplied together, so we can rewrite them as the scalar product between the two vectors. For convenience, we will write both of these 4-vectors as column vectors, and we have
\[
\frac{du^\mu}{d\tau} = \begin{pmatrix}
\gamma^4 \vec{v} \cdot \dot{\vec{v}} \\
\gamma^4 \frac{v \dot{v}}{c} \vec{v} + \gamma^2 \dot{\vec{v}}
\end{pmatrix}
\] (D.11a)
\[
\frac{du_\mu}{d\tau} = \begin{pmatrix}
\gamma^4 \vec{v} \cdot \dot{\vec{v}} \\
\gamma^4 \frac{v \dot{v}}{c} \vec{v} - \gamma^2 \dot{\vec{v}}
\end{pmatrix}
\] (D.11b)
Now, if we take the inner product between the two accelerations in (D.11), we will get

$$
\frac{du_\mu}{d\tau} \frac{du^\mu}{d\tau} = \gamma^8 \left( \frac{\vec{v} \cdot \dot{\vec{v}}}{c^2} \right)^2 - \gamma^8 \left( \frac{\vec{v} \cdot \dot{\vec{v}}}{c^2} \right) \left( \vec{v} \cdot \ddot{\vec{v}} \right) - \gamma^4 \left( \vec{v} \cdot \ddot{\vec{v}} \right) - \gamma^6 \frac{\vec{v} \cdot \dot{\vec{v}}}{c^2} \left( \vec{v} \cdot \ddot{\vec{v}} \right) 
$$

$$
= \gamma^8 \frac{\vec{v} \cdot \dot{\vec{v}}}{c^4} \left( c^2 - \vec{v} \cdot \ddot{\vec{v}} \right) - \gamma^4 \left( \vec{v} \cdot \ddot{\vec{v}} \right) - \gamma^6 \frac{\vec{v} \cdot \dot{\vec{v}}}{c^2} .
$$

If we take this result and put it back into (D.6), we can then absorb the factor of $1/c^2$ into this result, which will allow us to write everything in terms of $\beta$ and $\dot{\beta}$.

$$
P = -\frac{2}{3} \frac{e^2}{c^3} \left[ \gamma^8 \left( \frac{\vec{v} \cdot \dot{\vec{v}}}{c^4} \right) \left( c^2 - \vec{v} \cdot \ddot{\vec{v}} \right) - \gamma^4 \left( \vec{v} \cdot \ddot{\vec{v}} \right) - \gamma^6 \frac{\vec{v} \cdot \dot{\vec{v}}}{c^2} \right] 
$$

$$
= -\frac{2}{3} \frac{e^2}{c} \left[ \gamma^8 \frac{\vec{v} \cdot \dot{\vec{v}}}{c^4} \left( 1 - \frac{\vec{v} \cdot \ddot{\vec{v}}}{c^2} \right) - \gamma^4 \frac{\vec{v} \cdot \dot{\vec{v}}}{c^2} - \gamma^6 \frac{\vec{v} \cdot \ddot{\vec{v}}}{c^2} \right] 
$$

$$
= -\frac{2}{3} \frac{e^2}{c^2} \left[ \gamma^8 \frac{\vec{\beta} \cdot \dot{\vec{\beta}}}{c^2} \left( 1 - \vec{\beta} \cdot \ddot{\vec{\beta}} \right) - \gamma^4 \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right) - \gamma^6 \left( \vec{\beta} \cdot \ddot{\vec{\beta}} \right)^2 \right].
$$

Here, the first term will go away because the factor of $c^2$ that is still in the denominator will always dominate over the scalar product, leaving us with the other two terms. If we use the property of the vector product squared, we can rewrite the scalar product between $\vec{\beta}$ and $\dot{\vec{\beta}}$, bringing us to the form that we will need for the relativistic Larmor formula. Recall that this vector product identity is

$$
\left( \vec{\beta} \times \dot{\vec{\beta}} \right)^2 = \left| \vec{\beta} \right|^2 \left| \dot{\vec{\beta}} \right|^2 - \left( \vec{\beta} \cdot \dot{\vec{\beta}} \right)^2 ,
$$
where the absolute square is to be taken as a scalar product between the vector and itself. From this, we can write (D.12) such that

\[
P = \frac{2}{3} e^2 \left[ \gamma^4 (\ddot{\vec{\beta}} \cdot \vec{\beta}) + \gamma^6 (\vec{\beta} \cdot \dot{\vec{\beta}}) (\vec{\beta} \cdot \dot{\vec{\beta}}) - \gamma^6 (\vec{\beta} \times \dot{\vec{\beta}})^2 \right]
\]

\[
= \frac{2}{3} e^2 \left[ \gamma^6 (\ddot{\vec{\beta}} \cdot \vec{\beta}) \left( \frac{1}{\gamma^2} + \vec{\beta} \cdot \vec{\beta} \right) - \gamma^6 (\vec{\beta} \times \dot{\vec{\beta}})^2 \right].
\]

(D.13)

Here, we recall that we can use the definition of the Lorentz factor to write \( \frac{1}{\gamma^2} = 1 - \vec{\beta} \cdot \vec{\beta} \). We put this into the first term, which reduces the expression in parentheses to 1, and we recover the relativistic Larmor formula,

\[
P = \frac{2}{3} e^2 \gamma^6 \left[ (\ddot{\vec{\beta}} \cdot \vec{\beta}) - (\vec{\beta} \times \dot{\vec{\beta}})^2 \right].
\]

(D.14)
APPENDIX E

MATHEMATICA 12 NOTEBOOK SAMPLE
(*Relativistic form, numerically stable (own units).*

(*Often the mass has been the biggest problem, so we will use that the mass
of the electron is going to be \(e\text{Mass}=1\). This will lead to simpler conversions
between these units and SI or CGS units.*

(*The charge of the electron needs to be bigger than the mass, so we set
\(e\text{Charge}=-300\), allowing us to keep it sort manageable, while still keeping the
relationship of having a large enough difference between the two.*

(*In CGS units, \(\frac{e\text{Charge}}{e\text{MagMom}} = 517\) numerically, so we choose \(e\text{MagMom}\) to be
about 500 times smaller than the charge of the electron, or \(e\text{MagMom}=0.6\).*

(*Speed of Light needs to be high, so let’s just set it to the usual \(s\text{Light} = 3 \times 10^8\).*

\begin{verbatim}
eMass = 1;
eCharge = -300;
eMagMom = -0.6;
gFactor = 2;

eSpinValInit = \frac{2e\text{Mass}}{gFactor*e\text{Charge}} * e\text{MagMom};
sLight = 30000000;
posVec = \{x[t], y[t], z[t]\};
velVec = D[posVec, t];
accVec = D[velVec, t];

lorFactor = \frac{1}{\sqrt{1 - \frac{\text{Dot}[\text{velVec}, \text{velVec}]}{s\text{Light}^2}}};

momentumVec = lorFactor * e\text{Mass} * \text{velVec};
sVec = \{0, 0, e\text{SpinValInit}\} + \text{Cross}[\text{posVec}, \text{momentumVec}];

\text{betaVec} = \frac{\text{velVec}}{s\text{Light}};
tMin = 0;
tMax = 500;
\end{verbatim}
(*Fields*)
fElec = {0, 0, 0};
fMag = {0, 100000, 0};
gfMag1 = velVec[[1]] * D[fMag[[1]], x[t]] + velVec[[2]] * D[fMag[[1]], y[t]]
   + velVec[[3]] * D[fMag[[1]], z[t]];  
gfMag2 = velVec[[1]] * D[fMag[[2]], x[t]] + velVec[[2]] * D[fMag[[2]], y[t]]
   + velVec[[3]] * D[fMag[[2]], z[t]];  
gfMag3 = velVec[[1]] * D[fMag[[3]], x[t]] + velVec[[2]] * D[fMag[[3]], y[t]]
   + velVec[[3]] * D[fMag[[3]], z[t]];  
gfMag = {gfMag1, gfMag2, gfMag3};
magMomentVec = {0, 0, eMagMom};
(*fMag is a constant field of strength 10,000 in the y-direction, so that all points
in space have this value. This value can be changed in direction and magnitude,
but will still have values throughout all space. Note that the 10,000 is for now to
be considered as 1T.*)
(*gfMag = Dot[velVec, nabla] * fMag*)
(*gfMag is designed to be the Dot[v,nabla]*B component from the total time derivative.
Since we are considering time-independent fields that are not necessarily spatially
independent, we must consider this within each of the terms. For this spatial case of
fMag having a constant value in all points in space, it does not matter, but it could
matter the more that we vary the fields.*)
(*Evolution of Spin*)
evSpin1 = \frac{1}{lorFactors*Light} * Cross[magMomentVec, fMag];
evSpin2 = \frac{lorFactor}{a\text{Light}} * Cross[betaVec, Cross[betaVec, Cross[fMag, magMomentVec]]];
evSpin3 = \frac{2*e\text{Charge}}{e\text{Mass}*Light} * Dot[betaVec, sVec] * Cross[betaVec, fMag];
evSpin = evSpin1 + evSpin2 + evSpin3;
\[
evMass1 = \frac{-1}{sLight} \cdot \text{Dot[magMomentVec, gfMag]};
\]
\[
evMass2 = \frac{eCharge}{eMass \cdot sLight^3} \cdot \text{Dot[fMag, Cross[magMomentVec, Cross[Cross[betaVec, fMag], betaVec]]];}
\]
\[
evMass = \text{evMass1 + evMass2};
\]
\[
(*\text{Force Terms*})
\]
\[
lForceTerm = \frac{eCharge}{sLight} \cdot \text{Cross[velVec, fMag]};
\]
\[
pert1 = \frac{2 \cdot eCharge^2 \cdot \text{lorFactor}^3}{3 \cdot eMass \cdot sLight^3} \cdot \text{Cross[betaVec, gfMag]};
\]
\[
pert2 = \frac{2 \cdot eCharge^4 \cdot \text{lorFactor}^3}{3 \cdot eMass^3 \cdot sLight^3} \cdot \text{Cross[Cross[betaVec, fMag], fMag]};
\]
\[
pert3a = \text{magMomentVec}[1] \cdot \text{D}[fMag[1], x[t]]
\]
\[
+ \text{magMomentVec}[1] \cdot \text{D}[fMag[1], y[t]]
\]
\[
+ \text{magMomentVec}[1] \cdot \text{D}[fMag[1], z[t]];
\]
\[
pert3b = \text{magMomentVec}[2] \cdot \text{D}[fMag[2], x[t]]
\]
\[
+ \text{magMomentVec}[2] \cdot \text{D}[fMag[2], y[t]]
\]
\[
+ \text{magMomentVec}[2] \cdot \text{D}[fMag[2], z[t]];
\]
\[
pert3c = \text{magMomentVec}[3] \cdot \text{D}[fMag[3], x[t]]
\]
\[
+ \text{magMomentVec}[3] \cdot \text{D}[fMag[3], y[t]]
\]
\[
+ \text{magMomentVec}[3] \cdot \text{D}[fMag[3], z[t]];
\]
\[
pert3 = \frac{1}{sLight} \cdot \{\text{pert3a, pert3b, pert3c}\};
\]
\[
pert4 = \left(\frac{\text{lorFactor}}{sLight}\right)^2 \cdot \text{Cross[betaVec, Cross[gfMag, Cross[magMomentVec]]]};
\]
\[
pert5 = \left(\frac{\text{lorFactor}}{sLight}\right)^2 \cdot \text{Dot[magMomentVec, gfMag] * betaVec};
\]
\[
pert6 = \left(\frac{\text{lorFactor}^2}{sLight}\right)^2 \cdot \text{Dot[betaVec, Cross[betaVec, Cross[gfMag, Cross[magMomentVec]]] * betaVec};
\]
\[
pert7 = \frac{\text{eCharge}}{eMass \cdot sLight^2} \cdot \text{Cross[Cross[betaVec, fMag], Cross[fMag, Cross[betaVec, Cross[Cross[betaVec, fMag], fMag]]]]};
\]
\[
pert8 = \frac{\text{lorFactor} \cdot \text{eCharge}}{eMass \cdot sLight^2} \cdot \text{Cross[sVec, Cross[betaVec, gfMag]]};
\]
\[
pert9 = \left(\frac{\text{eCharge}}{eMass \cdot sLight}\right)^2 \cdot \text{Cross[sVec, Cross[Cross[betaVec, fMag], fMag]]};
\]
\[
pert10 = \left(\frac{\text{eCharge}}{eMass \cdot sLight}\right) \cdot \text{Cross[evSpin, Cross[betaVec, fMag]]};
\]
\[
\text{pert11} = \frac{\text{eCharge}}{\text{eMass}\times \text{sLight}} \times \text{Dot}\left[\text{Cross}[\text{betaVec}, \text{fMag}], \text{Cross}[\text{evSpin}, \text{betaVec}]\right] \times \text{betaVec};
\]
\[
\text{pert12} = \frac{\text{lorFactor} \times \text{eCharge}}{\text{eMass}\times \text{sLight}} \times \text{Dot}\left[\text{Cross}[\text{betaVec}, \text{gfMag}], \text{Cross}[\text{sVec}, \text{betaVec}]\right] \times \text{betaVec};
\]
\[
\text{pert13} = \frac{\text{lorFactor} \times \text{eCharge}^2}{(\text{eMass}\times \text{sLight})^2} \times \text{Dot}\left[\text{Cross}[\text{betaVec}, \text{fMag}], \text{fMag}], \text{Cross}[\text{sVec}, \text{betaVec}]\right] \times \text{betaVec};
\]
\[
\text{pertFull} = \text{pert1} + \text{pert2} + \text{pert3} + \text{pert4} + \text{pert5} + \text{pert6} + \text{pert7} + \text{pert8} + \text{pert9} + \text{pert10} + \text{pert11} + \text{pert12} + \text{pert13};
\]

(*Evolution of Mass*)
\[
\text{evMass1} = -\frac{1}{\text{sLight}} \times \text{Dot}\left[\text{magMomentVec}, \text{gfMag}\right];
\]
\[
\text{evMass2} = \frac{\text{eCharge}}{\text{eMass}\times \text{sLight}} \times \text{Dot}\left[\text{fMag}, \text{Cross}\left[\text{magMomentVec}, \text{Cross}\left[\text{velVec}, \text{fMag}\right], \text{velVec}\right]\right];
\]
\[
\text{evMass1} = -\frac{1}{\text{sLight}} \times \text{Dot}\left[\text{magMomentVec}, \text{gfMag}\right];
\]
\[
\text{evMass2} = \frac{\text{eCharge}}{\text{eMass}\times \text{sLight}} \times \text{Dot}\left[\text{fMag}, \text{Cross}\left[\text{magMomentVec}, \text{Cross}\left[\text{velVec}, \text{fMag}\right], \text{velVec}\right]\right];
\]
\[
\text{evMass} = \text{evMass1} + \text{evMass2};
\]

(*Calculating Lorentz Force*)
\[
\text{lorPos} = \text{NDSolve} \left[ \left\{ \text{Thread}\left[\text{lorFactor}^2 \times \text{evMass} \times \text{velVec} + \text{lorFactor}^2 \times \text{eMass} \times \text{accVec} - \text{lForceTerm} == 0 \right], x[0] == 400, y[0] == 0, z[0] == 600, x'[0] == 0, y'[0] == 0, z'[0] == 200 \right\}, \left\{ x, y, z, \text{Thread}[\text{sVec}], \text{evMass} \right\}, \{ t, \text{tMin}, \text{tMax} \} \right]
\]
\[
\text{lorForce} = \text{NDSolve} \left[ \left\{ \text{Thread}\left[\text{lorFactor}^2 \times \text{evMass} \times \text{velVec} + \text{lorFactor}^2 \times \text{eMass} \times \text{accVec} - \text{lForceTerm} == 0 \right], x[0] == 400, y[0] == 0, z[0] == 600, x'[0] == 0, y'[0] == 0, z'[0] == 200 \right\}, \left\{ x'', y'', z'', \text{Thread}[\text{D}[\text{sVec}, \ell]], \text{evMass} \right\}, \{ t, \text{tMin}, \text{tMax} \} \right]
\]
\[
\text{lForcePX} = \left\{ \text{posVec}[1]\right\} /\text{lorPos}[1, 1];
\]
\[
\text{lForcePY} = \left\{ \text{posVec}[2]\right\} /\text{lorPos}[1, 2];
\]
Plot[lForcePX, {t, tMin, tMax}, PlotLabel \[Rule] Style["Position, X-Component, Lorentz Force, Non-Relativistic vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[m, 18]},
Plot[lForceFX, {t, tMin, tMax}, PlotLabel \[Rule] Style["Acceleration in X-Direction, Lorentz Force, Non-Relativistic vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[m, 18]},
Plot[lForceFY, {t, tMin, tMax}, PlotLabel \[Rule] Style["Position, Y-Component, Lorentz Force, Non-Relativistic vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[m, 18]},
Plot[lForceFY, {t, tMin, tMax}, PlotLabel \[Rule] Style["Acceleration in Y-Direction, Lorentz Force, Non-Relativistic vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[m, 18]},
Plot[lForcePZ, {t, tMin, tMax}, PlotLabel \[Rule] Style["Position, Z-Component, Lorentz Force, Non-Relativistic vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[m, 18]},
Plot[lForceFZ, {t, tMin, tMax}, PlotLabel \[Rule] Style["Acceleration in Z-Direction, Lorentz
Force, Non-Relativistic, vs. Time", 18], AxesLabel \rightarrow \{\text{Style}[s, 18], \text{Style} \left[ \frac{m}{2}, 18 \right] \},
PlotRange \rightarrow \text{Full}, \text{TicksStyle} \rightarrow \text{Directive} \left[ \text{"Label"}, 18 \right]]

\text{GraphicsGrid}[\{
\{\text{Plot}[l\text{ForceS}[1, 1]], \{t, t\text{Min}, t\text{Max}\}, \text{PlotLabel} \rightarrow \text{Style} \left[ \text{"Spin/ Angular Momentum, X-Component, Non-Relativistic vs. Time"}, 24 \right], \text{AxesLabel} \rightarrow \{\text{Style}[s, 24], \text{Style} \left[ \text{"Spin"}, 24 \right] \}, \text{PlotRange} \rightarrow \text{Full}, \text{TicksStyle} \rightarrow \text{Directive} \left[ \text{"Label"}, 24 \right] \},
\{\text{Plot}[l\text{ForceS}[1, 2]], \{t, t\text{Min}, t\text{Max}\}, \text{PlotLabel} \rightarrow \text{Style} \left[ \text{"Spin/ Angular Momentum, Y-Component, Non-Relativistic vs. Time"}, 24 \right], \text{AxesLabel} \rightarrow \{\text{Style}[s], 24], \text{Style}[\text{"Spin"}, 24] \}, \text{PlotRange} \rightarrow \text{Full}, \text{TicksStyle} \rightarrow \text{Directive} \left[ \text{"Label"}, 24 \right] \},
\{\text{Plot}[l\text{ForceS}[1, 3]], \{t, t\text{Min}, t\text{Max}\}, \text{PlotLabel} \rightarrow \text{Style} \left[ \text{"Spin/ Angular Momentum, Z-Component, Non-Relativistic vs. Time"}, 24 \right], \text{AxesLabel} \rightarrow \{\text{Style}[s], 24], \text{Style}[\text{"Spin"}, 24] \}, \text{PlotRange} \rightarrow \text{Full}, \text{TicksStyle} \rightarrow \text{Directive} \left[ \text{"Label"}, 24 \right] \}}

\text{pertPos} = \text{NDSolve} \left[ \{\text{Thread}[l\text{orFactor}^2 \ast e\text{Mass} \ast \text{velVec}
\hspace{1cm} + l\text{orFactor}^2 \ast e\text{Mass} \ast \text{accVec} - l\text{ForceTerm} - \text{pertFull} == 0] \},
\hspace{1cm} x[0] == 400, y[0] == 0, z[0] == 600, x'[0] == 0, y'[0] == 0,
\hspace{1cm} z'[0] == 200 \}, \{x, y, z, \text{Thread}[s\text{Vec}, \text{evMass}], \{t, t\text{Min}, t\text{Max}\} \)
\text{pertForce} = \text{NDSolve} \left[ \{\text{Thread}[l\text{orFactor}^2 \ast e\text{Mass} \ast \text{velVec}
\hspace{1cm} + l\text{orFactor}^2 \ast e\text{Mass} \ast \text{accVec} - l\text{ForceTerm} - \text{pertFull} == 0] \},
\hspace{1cm} x[0] == 400, y[0] == 0, z[0] == 600, x'[0] == 0, y'[0] == 0,
\hspace{1cm} z'[0] == 200 \}, \{x", y", z" \}, \text{Thread}[D[s\text{Vec}, t], \text{evMass}] \},
\{t, t\text{Min}, t\text{Max}\} \)
\text{pertForcePX} = \{\text{posVec}[1]\}/.\text{pertPos}[1, 1] \};
\text{pertForcePY} = \{\text{posVec}[2]\}/.\text{pertPos}[1, 2] \};
\text{pertForcePZ} = \{\text{posVec}[3]\}/.\text{pertPos}[1, 3] \};
\text{pertForceS} = \{\text{Thread}[s\text{Vec}]\}/.\text{pertPos}[1, 4] \};
\text{pertForceMass} = \{e\text{Mass}\}/.\text{pertPos}[1, 5] \};
pertForceFX = {accVec[[1]]}/.pertForce[[1, 1]];  
pertForceFX1 = pertForceFX + 400;  
pertForceFY = {accVec[[2]]}/.pertForce[[1, 2]];  
pertForceFZ = {accVec[[3]]}/.pertForce[[1, 3]];  
pertForceFZ1 = pertForceFZ + 400;
Plot[pertForcePX, \{t, tMin, tMax\}, PlotLabel \rightarrow Style["Position, X-Component, Lorentz and Perturbation vs. Time", 18], AxesLabel \rightarrow \{Style[s, 18], Style[m, 18]\}, PlotRange \rightarrow Full, TicksStyle \rightarrow Directive["Label", 18]]
Plot[pertForceFX, \{t, tMin, tMax\}, PlotLabel \rightarrow Style["Acceleration in X-Direction, Lorentz and Perturbation vs. Time", 18], AxesLabel \rightarrow \{Style[s, 18], Style["m \, m/s^2", 18]\}, PlotRange \rightarrow Full, TicksStyle \rightarrow Directive["Label", 18]]
Plot[pertForcePY, \{t, tMin, tMax\}, PlotLabel \rightarrow Style["Position, Y-Component, Lorentz and Perturbation vs. Time", 18], AxesLabel \rightarrow \{Style[s, 18], Style[m, 18]\}, PlotRange \rightarrow Full, TicksStyle \rightarrow Directive["Label", 18]]
Plot[pertForceFY, \{t, tMin, tMax\}, PlotLabel \rightarrow Style["Acceleration in Y-Direction, Lorentz and Perturbation vs. Time", 18], AxesLabel \rightarrow \{Style[s, 18], Style["m \, m/s^2", 18]\}, PlotRange \rightarrow Full, TicksStyle \rightarrow Directive["Label", 18]]
Plot[pertForcePZ, \{t, tMin, tMax\}, PlotLabel \rightarrow Style["Position, Z-Component, Lorentz and Perturbation vs. Time", 18], AxesLabel \rightarrow \{Style[s, 18], Style[m, 18]\}, PlotRange \rightarrow Full, TicksStyle \rightarrow Directive["Label", 18]]
Plot[pertForceFZ1, \{t, tMin, tMax\}, PlotLabel \rightarrow Style["Acceleration in Z-Direction, Lorentz and Perturbation vs. Time", 18], AxesLabel \rightarrow \{Style[s, 18], Style["m \, m/s^2", 18]\}, PlotRange \rightarrow \{200, 600\}, TicksStyle \rightarrow Directive["Label", 18]]
GraphicsGrid[{{
Plot[pertForceS[[1, 1]], \{t, tMin, tMax\}, PlotLabel \rightarrow Style["Spin/Angular Momentum, X-Component vs. Time", 18], AxesLabel \rightarrow \{Style[s, 18], Style["Spin", 18]\},
PlotRange → Full, TicksStyle → Directive["Label", 18]],
{Plot[pertForceS[[1, 2]], \{t, tMin, tMax\}, PlotLabel → Style["Spin/Angular Momentum,
Y-Component vs. Time", 18], AxesLabel → \{Style[s, 18], Style["Spin", 18]\},
PlotRange → Full, TicksStyle → Directive["Label", 18]],
{Plot[pertForceS[[1, 3]], \{t, tMin, tMax\}, PlotLabel → Style["Spin/Angular Momentum,
Z-Component vs. Time", 18], AxesLabel → \{Style[s, 18], Style["Spin", 18]\},
PlotRange → Full, TicksStyle → Directive["Label", 18]]}

pertOnlyPX = \frac{pertForcePX}{lForcePX} - 1;
pertOnlyPY = -lForcePY + pertForcePY;
pertOnlyPY1 = \frac{pertOnlyPY}{200};
pertOnlyPZ = \frac{pertForcePZ}{lForcePZ} - 1;
pertOnlyFX = \frac{pertForceFX1}{lForceFX1} - 1;
pertOnlyFY = \frac{(-lForceFY+pertForceFY)}{200};
pertOnlyFZ = \frac{pertForceFZ1}{lForceFZ1} - 1;
pertOnlyS = -lForceS + pertForceS;
pertOnlyMass = -lForceMass + pertForceMass;
Plot[pertOnlyPX, \{t, tMin, tMax\}, PlotLabel → Style["Position, X-Component,
Perturbation Magnitude vs. Time", 18], AxesLabel → \{Style[\text{s}, 18]\}, PlotRange
→ Full, TicksStyle → Directive["Label", 18]]

Plot[pertOnlyFX, \{t, tMin, tMax\}, PlotLabel → Style["Acceleration in X-Direction,
Perturbation Magnitude vs. Time", 18], AxesLabel → \{Style[\text{s}, 18]\}, PlotRange
→ Full, TicksStyle → Directive["Label", 18]]

Plot[pertOnlyPY1, \{t, tMin, tMax\}, PlotLabel → Style["Position, Y-Component,
Perturbation Magnitude vs. Time", 18], AxesLabel → \{Style[\text{s}, 18]\}, PlotRange
→ Full, TicksStyle → Directive["Label", 18]]
Plot[pertOnlyFZ, {t, tMin, tMax}, PlotLabel \[Rule] Style["Acceleration in Z-Direction, Perturbation Magnitude vs. Time", 18], AxesLabel \[Rule] {Style[s, 18]}, PlotRange \[Rule] \{-0.07, 0.2\}, TicksStyle \[Rule] Directive["Label", 18]]
GraphicsGrid[
{Plot[pertOnlyS[[1, 1]], {t, tMin, tMax}, PlotLabel \[Rule] Style["Spin/Angular Momentum, X-Component", 18], AxesLabel \[Rule] {Style[s, 18]}, PlotRange \[Rule] Full, TicksStyle \[Rule] Directive["Label", 18]]},
{Plot[pertOnlyS[[1, 2]], {t, tMin, tMax}, PlotLabel \[Rule] Style["Spin/Angular Momentum, Y-Component", 18], AxesLabel \[Rule] {Style[s, 18]}, PlotRange \[Rule] Full, TicksStyle \[Rule] Directive["Label", 18]]},
GraphicsGrid[
{Plot[pertOnlyMass, {t, tMin, tMax}, PlotLabel \[Rule] "Evolution of Mass, Perturbation"}
Only”, AxesLabel –> {“Time”}, PlotRange –> Full}
]

(*Now that the non-relativistic part is done, we can move onto the relativistic case.

Rather than trying to type this all out in separate notebooks, we will just add
"Rel" to the end of each quantity.*)

lorPosRel = NDSolve[{Thread[lorFactor^2*evMass*velVec 
  + lorFactor^2*evMass*accVec - lForceTerm == 0],
  x[0] == 0, y[0] == 0, z[0] == 10000, \(
  \frac{x'[0]}{c} == 0, \frac{y'[0]}{c} == 0, \frac{z'[0]}{c} == 0, \)
  \}, \{x, y, z, Thread[sVec], evMass\},
  \{t, tMin, tMax\}]

lorForceRel = NDSolve[{Thread[lorFactor^2*evMass*velVec 
  + lorFactor^2*evMass*accVec - lForceTerm == 0],
  x[0] == 0, y[0] == 0, z[0] == 10000, \(
  \frac{x'[0]}{c} == 0, \frac{y'[0]}{c} == 0, \frac{z'[0]}{c} == 0, \)
  \}, \{x, y, z, Thread[sVec], evMass\},
  \{t, tMin, tMax\}]

pertPosRel = NDSolve[{Thread[lorFactor^2*evMass*velVec 
  + lorFactor^2*evMass*accVec - lForceTerm - pertFull == 0],
  x[0] == 0, y[0] == 0, z[0] == 10000, \(
  \frac{x'[0]}{c} == 0, \frac{y'[0]}{c} == 0, \frac{z'[0]}{c} == 0, \)
  \}, \{x, y, z, Thread[sVec], evMass\},
  \{t, tMin, tMax\}]

pertForceRel = NDSolve[{Thread[lorFactor^2*evMass*velVec 
  + lorFactor^2*evMass*accVec - lForceTerm - pertFull == 0],
  x[0] == 0, y[0] == 0, z[0] == 10000, \(
  \frac{x'[0]}{c} == 0, \frac{y'[0]}{c} == 0, \frac{z'[0]}{c} == 0, \)
  \}, \{x, y, z, Thread[sVec], evMass\},
  \{t, tMin, tMax\}]

Thread[sVec] = {x, y, z} 
Thread[sLight] = {sLight, 0} 
Thread[lForceTerm] = 0 
Thread[pertFull] = 0 
Thread[velVec] = {x, y, z} 
Thread[accVec] = {x', y', z'} 
Thread[D[sVec, t]] = \( \frac{d}{dt} \) sVec
lForcePXRel = {posVec[[1]]}/.lorPosRel[[1, 1]];
lForcePXRel1 = lForcePXRel + 10^{15};
lForcePYRel = {posVec[[2]]}/.lorPosRel[[1, 2]];
lForcePYRel1 = lForcePYRel + 10^{15};
lForcePZRel = {posVec[[3]]}/.lorPosRel[[1, 3]];
lForcePZRel1 = lForcePZRel + 10^{15};
lForceSRel = {Thread[sVec]}/.lorPosRel[[1, 4]];
lForceMassRel = {eMass}/.lorForceRel[[1, 5]];
lForceFXRel = {accVec[[1]]}/.lorForceRel[[1, 1]];
lForceFXRel1 = lForceFXRel + 10^{15};
lForceFYRel = {accVec[[2]]}/.lorForceRel[[1, 2]];
lForceFYRel1 = lForceFYRel + 10^{15};
lForceFZRel = {accVec[[3]]}/.lorForceRel[[1, 3]];
lForceFZRel1 = lForceFZRel + 10^{15};
pertForcePXRel = {posVec[[1]]}/.pertPosRel[[1, 1]];
pertForcePXRel1 = pertForcePXRel + 10^{15};
pertForcePYRel = {posVec[[2]]}/.pertPosRel[[1, 2]];
pertForcePYRel1 = pertForcePYRel + 10^{15};
pertForcePZRel = {posVec[[3]]}/.pertPosRel[[1, 3]];
pertForcePZRel1 = pertForcePZRel + 10^{15};
pertForceSRel = {Thread[sVec]}/.pertPosRel[[1, 4]];
pertForceMassRel = {eMass}/.lorForceRel[[1, 5]];
pertForceFXRel = {accVec[[1]]}/.pertForceRel[[1, 1]];
pertForceFXRel1 = pertForceFXRel + 10^{15};
pertForceFYRel = {accVec[[2]]}/.pertForceRel[[1, 2]];
pertForceFYRel1 = pertForceFYRel + 10^{15};
pertForceFZRel = {accVec[[3]]}/.pertForceRel[[1, 3]]; 
pertForceFZRel1 = pertForceFZRel + 10^15;

Plot[lForcePXRel, {t, tMin, tMax}, PlotLabel \[Rule] Style["Position, X-Component, Lorentz Force, \[Beta]=0.99 vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[m, 18]}, PlotRange \[Rule] Full, TicksStyle \[Rule] Directive["Label", 18]]
Plot[lForceFXRel, {t, tMin, tMax}, PlotLabel \[Rule] Style["Acceleration in X-Direction, Lorentz Force, \[Beta]=0.99 vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[F, 18]}, PlotRange \[Rule] Full, TicksStyle \[Rule] Directive["Label", 18]]
Plot[lForcePYRel, {t, tMin, tMax}, PlotLabel \[Rule] Style["Position, Y-Component, Lorentz Force, \[Beta]=0.99 vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[m, 18]}, PlotRange \[Rule] Full, TicksStyle \[Rule] Directive["Label", 18]]
Plot[lForcePZRel, {t, tMin, tMax}, PlotLabel \[Rule] Style["Position, Z-Component, Lorentz Force, \[Beta]=0.99 vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style[m, 18]}, PlotRange \[Rule] Full, TicksStyle \[Rule] Directive["Label", 18]]
Plot[lForceSRel[[1]], {t, tMin, tMax}, PlotLabel \[Rule] Style["Spin/Angular Momentum, X-Component, Lorentz Force, \[Beta]=0.99 vs. Time", 18], AxesLabel \[Rule] {Style[s, 18], Style["Spin", 18]}, PlotRange \[Rule] Full, TicksStyle \[Rule] Directive["Label", 18]]
Plot[lForceSRel[[2]], {t, tMin, tMax}, PlotLabel \[Rule] Style["Spin/Angular Momentum,
Y-Component, Lorentz Force, $\beta=0.99$ vs. Time”, 18], AxesLabel $\rightarrow \{\text{Style}[s, 18],$
Style[“Spin”, 18]\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$ Directive[“Label”, 18]]
Plot[pertForceSRel[[1, 3]], \{t, tMin, tMax\}, PlotLabel $\rightarrow$ Style[“Spin/Angular Momentum,
Z-Component, Lorentz Force, $\beta=0.99$ vs. Time”, 18], AxesLabel $\rightarrow \{\text{Style}[s, 18],$
Style[“Spin”, 18]\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$ Directive[“Label”, 18]]
Plot[pertForcePXRel, \{t, tMin, tMax\}, PlotLabel $\rightarrow$ Style[“Position, X-Component,
Lorentz and Perturbation, $\beta=0.99$ vs. Time”, 18], AxesLabel $\rightarrow \{\text{Style}[s, 18],$
Style[m, 18]\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$ Directive[“Label”, 18]]
Plot[pertForceFXRel, \{t, tMin, tMax\}, PlotLabel $\rightarrow$ Style[“Acceleration in X-Direction,
Lorentz and Perturbation, $\beta=0.99$ vs. Time”, 18], AxesLabel $\rightarrow \{\text{Style}[s, 18],$
Style[F, 18]\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$ Directive[“Label”, 18]]
Plot[pertForcePYRel, \{t, tMin, tMax\}, PlotLabel $\rightarrow$ Style[“Position, Y-Component,
Lorentz and Perturbation, $\beta=0.99$ vs. Time”, 18], AxesLabel $\rightarrow \{\text{Style}[s, 18],$
Style[m, 18]\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$ Directive[“Label”, 18]]
Plot[pertForceFYRel, \{t, tMin, tMax\}, PlotLabel $\rightarrow$ Style[“Acceleration in Y-Direction,
Lorentz and Perturbation, $\beta=0.99$ vs. Time”, 18], AxesLabel $\rightarrow \{\text{Style}[s, 18],$
Style[F, 18]\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$ Directive[“Label”, 18]]
Plot[pertForcePZRel, \{t, tMin, tMax\}, PlotLabel $\rightarrow$ Style[“Position, Z-Component,
Lorentz and Perturbation, $\beta=0.99$ vs. Time”, 18], AxesLabel $\rightarrow \{\text{Style}[s, 18],$
Style[m, 18]\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$ Directive[“Label”, 18]]
Plot[pertForceFZRel, \{t, tMin, tMax\}, PlotLabel $\rightarrow$ Style[“Acceleration in Z-Direction,
Lorentz and Perturbation, $\beta=0.99$ vs. Time”, 18], AxesLabel $\rightarrow \{\text{Style}[s, 18],$
Style[F, 18]\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$ Directive[“Label”, 18]]
Plot[pertForceSRel[[1, 1]], \{t, tMin, tMax\}, PlotLabel $\rightarrow$ Style[“Spin/Angular
Momentum, X-Component, Lorentz and Perturbation, $\beta=0.99$ vs. Time”, 18],
AxesLabel $\rightarrow \{\text{Style}[s, 18], \text{Style[“Spin”, 18]}\}, PlotRange $\rightarrow$ Full, TicksStyle $\rightarrow$
Directives

Plot[pertForceSRel[[1, 2]], {t, tMin, tMax}, PlotLabel -> Style["Spin/Angular Momentum, Y-Component, Lorentz and Perturbation, \(\beta=0.99\) vs. Time", 18],
AxesLabel -> {Style[s, 18], Style["Spin", 18]}, PlotRange -> Full, TicksStyle -> Directive["Label", 18]]

Plot[pertForceSRel[[1, 3]], {t, tMin, tMax}, PlotLabel -> Style["Spin/Angular Momentum, Z-Component, Lorentz and Perturbation, \(\beta=0.99\) vs. Time", 18],
AxesLabel -> {Style[s, 18], Style["Spin", 18]}, PlotRange -> Full, TicksStyle -> Directive["Label", 18]]

pertOnlyPXRel = \[fraction\] pertForceFXRel[lForceFXRel] - 1;
pertOnlyPYRel = \[fraction\] pertForceFYRel[lForceFYRel] - 1;
pertOnlyPZRel = \[fraction\] pertForceFZRel[lForceFZRel] - 1;
pertOnlyFXRel = \[fraction\] pertForceFXRel[lForceFXRel] - 1;
pertOnlyFYRel = \[fraction\] pertForceFYRel[lForceFYRel] - 1;
pertOnlyFZRel = \[fraction\] pertForceFZRel[lForceFZRel] - 1;
pertOnlySRel = -lForceSRel + pertForceSRel;
pertOnlyMassRel = lForceMassRel - pertForceMassRel;

Plot[pertOnlyPXRel, {t, tMin, tMax}, PlotLabel -> Style["Position, X-Component, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel -> {Style[s, 18]}, PlotRange -> Full, TicksStyle -> Directive["Label", 18]]

Plot[pertOnlyFXRel, {t, tMin, tMax}, PlotLabel -> Style["Acceleration in X-Direction, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel -> {Style[s, 18]}, PlotRange -> Full, TicksStyle -> Directive["Label", 18]]

Plot[pertOnlyPYRel, {t, tMin, tMax}, PlotLabel -> Style["Position, Y-Component, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel -> {Style[s, 18]}, PlotRange
Plot[pertOnlyFYRel, {t, tMin, tMax}, PlotLabel → Style["Acceleration in Y-Direction, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel → {Style[s, 18]}, PlotRange → Full, TicksStyle → Directive["Label", 18]]

Plot[pertOnlyFZRel, {t, tMin, tMax}, PlotLabel → Style["Acceleration in Z-Direction, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel → {Style[s, 18]}, PlotRange → Full, TicksStyle → Directive["Label", 18]]

Plot[pertOnlyFZRel, {t, tMin, tMax}, PlotLabel → Style["Position, Z-Component, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel → {Style[s, 18]}, PlotRange → Full, TicksStyle → Directive["Label", 18]]

Plot[pertOnlySRel[[1, 1]], {t, tMin, tMax}, PlotLabel → Style["Spin/Angular Momentum, X-Component, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel → {Style[s, 18]}, PlotRange → Full, TicksStyle → Directive["Label", 18]]

Plot[pertOnlySRel[[1, 2]], {t, tMin, tMax}, PlotLabel → Style["Spin/Angular Momentum, Y-Component, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel → {Style[s, 18]}, PlotRange → Full, TicksStyle → Directive["Label", 18]]

Plot[pertOnlySRel[[1, 3]], {t, tMin, tMax}, PlotLabel → Style["Spin/Angular Momentum, Z-Component, Perturbation Only, \(\beta=0.99\) vs. Time", 18], AxesLabel → {Style[s, 18]}, PlotRange → Full, TicksStyle → Directive["Label", 18]]

Plot[lForceMassRel, {t, tMin, tMax}, PlotLabel → Style["Evolution of Mass, Lorentz", 18], AxesLabel → {Style[s, 18], Style["Mass", 18]}, PlotRange → Full, TicksStyle → Directive["Label", 18]]

Plot[pertForceMassRel, {t, tMin, tMax}, PlotLabel → Style["Evolution of Mass, Lorentz and Perturbation", 18], AxesLabel → {Style[s, 18], Style["Mass", 18]}, PlotRange → Full, TicksStyle → Directive["Label", 18]]

Plot[pertOnlyMassRel, {t, tMin, tMax}, PlotLabel → Style["Evolution of Mass,
Perturbation Only", 18], AxesLabel -> {Style[s, 18]}, PlotRange -> Full,
TicksStyle -> Directive["Label", 18]]