Reading assignment: sections 3.1 through 3.4 of Carroll.

Problem 1. Consider the special case of a *diagonal* metric $g_{\mu\nu}$. (This special case comes up surprisingly often, so it is a very useful thing to consider.) Show that the Christoffel symbols are given by:

\[ \Gamma^\rho_{\mu\nu} = 0, \]
\[ \Gamma^\nu_{\mu\mu} = -\frac{1}{2} g_{\nu\nu} \partial_\nu g_{\mu\mu}, \]
\[ \Gamma^\nu_{\mu\nu} = \Gamma^\nu_{\nu\mu} = \frac{1}{2} g_{\nu\nu} \partial_\mu g_{\nu\nu} = \partial_\mu (\ln \sqrt{|g_{\nu\nu}|}), \]
\[ \Gamma^\mu_{\mu\mu} = \frac{1}{2} g_{\mu\mu} \partial_\mu g_{\mu\mu} = \partial_\mu (\ln \sqrt{|g_{\mu\mu}|}). \]

In these expressions, $\mu \neq \nu \neq \rho$, and repeated indices are not summed over.

Problem 2. Consider a 2-sphere with coordinates $(\theta, \phi)$ and metric

\[ ds^2 = d\theta^2 + \sin^2 \theta \, d\phi^2. \]

(a) Compute all of the Christoffel symbols.

(b) Write down the geodesic equations, which involve $d^2 \theta / d\lambda^2$ and $d^2 \phi / d\lambda^2$.

(c) Show that lines of constant longitude ($\phi =$ constant) are geodesics. Also show that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).

(d) Take a vector with components $V^\mu = (1, 0)$ in the $(\partial_\theta, \partial_\phi)$ basis, and parallel transport it once around a circle of constant latitude with $\theta = \theta_0$. What are the components of the resulting vector, as a function of $\theta_0$? What is the magnitude of the resulting vector?

(e) Extra credit, and only if you are very brave, and have lots of excess time: for two arbitrary given points $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$, find the geodesic between them. (Trying to solve the geodesic equation directly may not be the best way to go!)