Problem 1. The $Z$ boson is a massive real vector particle that interacts with left-handed fermions and right-handed fermions, but with different coupling strengths, through what is called the weak neutral current interaction. The corresponding interaction Lagrangian for any particular Standard Model Dirac fermion field $f$ is given by eq. (11.1.37) or equivalently eq. (11.1.40) in the notes.

(a) Write down the corresponding Feynman rule for $Z$ boson interactions with $f$.

(b) Treating $f$ as massless, compute the partial decay widths for $Z \rightarrow f\overline{f}$, in terms of $g_V^f$, $g_A^f$, and $m_Z$. (The calculation is similar to that of $W$ decay . . .)

(c) Plug in the numbers and compare with the measured branching ratios for the $Z$ boson. (See page 250 of the notes.)

Problem 2. In a general Yang-Mills theory (including fermion and scalars), the gauge-coupling beta functions at one-loop order can always be written as

$$\frac{d}{d\mu}g_i = \beta(g_i) = \frac{1}{16\pi^2}B_i g_i^3,$$

for each Lie algebra component of the gauge symmetry.

(a) Define $\alpha_i = g_i^2/4\pi$, and show that the quantities $\alpha_i^{-1}$ run linearly with $\ln(\mu/\mu_0)$, in the one-loop running approximation.

(b) In the full Standard Model, there are three gauge couplings $g_i$ with $i = 1, 2, 3$ for the three components of the unbroken gauge symmetry, $SU(3)_c \times SU(2)_L \times U(1)_Y$, with beta function coefficients:

$$B_1 = 41/10, \quad B_2 = -19/6, \quad B_3 = -7.$$  \hspace{1cm} (Standard Model) \hspace{1cm} (0.1)

Here, $g_2 = g$ and $g_1 = \sqrt{5/3}g'$, where $g, g'$ are the $SU(2)_L$ and $U(1)_Y$ couplings in the normalization of section 11.1 of the notes. The choice of normalization for $g_1$ is called the GUT (Grand Unified Theory) normalization. As boundary conditions, take $\alpha_3(M_Z) = 0.1185$, $g_2(M_Z) = 0.652$, and $g_1(M_Z) = 0.461$. Make a graph of $\alpha_i^{-1}(\mu)$ as a function of $\log_{10}(\mu/1\text{GeV})$, for $M_Z \leq \mu \leq 10^{19}$ GeV, using the one-loop running approximation. Make a note of the numerical values of $\alpha_i(\mu)$ at $\mu = 1000$ GeV and $\mu = 5000$ GeV.
In the Minimal Supersymmetric Standard Model (MSSM), the same three gauge couplings appear, but they have different one-loop running coefficients:

\[ B_1 = \frac{33}{5}, \quad B_2 = 1, \quad B_3 = -3 \]  

(MSSM)
due to the fact that the MSSM contains more fields appearing in the loop diagrams. Let us assume that the new particles in the MSSM all have the same masses \( \mu_{\text{SUSY}} \). (This is probably not realistic, but captures the main point of the following.) Then, assuming supersymmetry is correct, the renormalization group running should use the MSSM coefficients for \( \mu > \mu_{\text{SUSY}} \). Starting with the values you found for \( \alpha_i(\mu) \) at \( \mu = \mu_{\text{SUSY}} = 1000 \text{ GeV} \) and \( \mu = \mu_{\text{SUSY}} = 5000 \text{ GeV} \) in part (b), make graphs of \( \alpha_i^{-1}(\mu) \) for \( \mu_{\text{SUSY}} \leq \mu \leq 10^{19} \text{ GeV} \) in the MSSM, as a function of \( \log_{10}(\mu/\text{GeV}) \). You should observe that the three running gauge couplings become approximately equal at a single renormalization group scale \( \mu_{\text{GUT}} \). (To really do this right, one ought to include at least 2-loop RG running, as well as small “threshold” corrections when matching the MSSM onto the Standard Model. But those effects make a rather small difference.) What do you estimate for \( \mu_{\text{GUT}} \)?

This famous unification of gauge couplings is held by many to be a strong, but indirect, piece of evidence in favor of supersymmetry, since it suggests that at very high energies the gauge interaction themselves unify into a “prettier” theory, namely a Grand Unified Theory (GUT), a larger Yang-Mills theory with a single Lie algebra \( SU(5) \) or \( SO(10) \) or \( E_6 \). The unification of gauge couplings can also occur in some versions of superstring theory. Your humble professor’s personal opinion is that this is interesting, but far from compelling, evidence in favor of supersymmetry.