Filter Circuits

Filter Circuits
• Impedances act like resistances in circuits and obey Kirchoff’s laws.
• A voltage divider with resistors has the relationship

\[ v_{out} = \frac{R_2}{R_1 + R_2} v_{in} \]

High Pass Filter
• If \( R_1 \) is replaced by a capacitor the divider can be calculated with impedances:

\[ v_{out} = \frac{Z_2}{Z_1 + Z_2} v_{in} = \frac{R}{j\omega C + R} v_{in} \]
• This can be rewritten in terms of the gain $A$:

$$A = \frac{v_{out}}{v_{in}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega RC + \omega^2 R^2 C^2}{1 + \omega^2 R^2 C^2}$$

• Using the following formula:

$$|A| = |B + jC| = \sqrt{B^2 + C^2}$$

The magnitude of the gain is

$$|A| = \frac{\omega^2 R^2 C^2 + (\omega^2 R^2 C^2)^2}{(1 + \omega^2 R^2 C^2)^2} = \sqrt{\frac{\omega^2 R^2 C^2}{1 + \omega^2 R^2 C^2}}$$

• For $\omega$ much less than $1/RC$, $A$ is very small. $|A| \equiv \omega RC$
• For $\omega$ much greater than $1/RC$, $A$ is nearly 1.
• This is called a high-pass filter with a break frequency $\omega_b = 1/RC$.
• High pass filter graph (Bode plot)

![Bode plot diagram]
Phase Changes

- The phase of the current compared to the voltage can be determined from the complex impedances: the angle in the complex plane is the phase shift

\[ \tan \phi = \frac{C}{B} \]

- For the high pass filter:

\[ jC = \frac{j\omega RC}{1 + \omega^2 R^2 C^2} \]

\[ B = \frac{\omega^2 R^2 C^2}{1 + \omega^2 R^2 C^2} \]

\[ \phi = \tan^{-1} \frac{\omega RC}{\omega^2 R^2 C^2} = \tan^{-1} \frac{1}{\omega RC} \]

- The phase depends on the frequency
  At low frequency, \( \phi \rightarrow 90 \)
  At \( \omega = 1/RC \), \( \phi = 45 \)
  At high frequency, \( \phi \rightarrow 0 \)
**Low Pass Filter**

- If the capacitor replaces $R_2$ in a voltage divider:

\[
v_{out} = \frac{Z_2}{Z_1 + Z_2} v_{in} = \frac{1}{1 + j\omega C} v_{in}
\]

- Again this can be rewritten in terms of the gain $A$:

\[
A = \frac{v_{out}}{v_{in}} = \frac{1}{1 + j\omega RC} = \frac{1 - j\omega RC}{1 + \omega^2 R^2 C^2}
\]

- For $\omega$ much less than $1/RC$, $A$ is nearly 1.
- For $\omega$ much greater than $1/RC$, $A$ is very small.
- This is called a low-pass filter with a break frequency $\omega_b = 1/RC$.

- Low pass graph

\[
\begin{align*}
\frac{v_{out}}{v_{in}} &= 0 \text{ dB} \\
-3 \text{ dB} &= \omega_b \\
-6 \text{ dB/octave}
\end{align*}
\]
Resonance

Series RLC Circuits

- A voltage divider with one impedance due to an inductor and capacitor in series

\[ v_{out} = \frac{Z_{LC}}{R + Z_{LC}}v_{in} \]

- The series impedance can be calculated and inserted to find the gain:

\[ Z_{LC} = \frac{1}{j\omega C} + j\omega L = \frac{1 - \omega^2 LC}{j\omega C} \]

\[ \frac{v_{out}}{v_{in}} = \frac{1 - \omega^2 LC}{j\omega RC + 1 - \omega^2 LC} \]

- Graphically
Parallel RLC Circuits

- A voltage divider with one impedance due to an inductor and capacitor in parallel

\[ v_{out} = \frac{Z_{LC}}{R + Z_{LC}} v_{in} \]

- The impedance can be calculated and inserted to find the gain:

\[ Z_{LC} = \frac{j\omega L / j\omega C}{1/j\omega C + j\omega L} = \frac{j\omega L}{1 - \omega^2 LC} \]

\[ \frac{v_{out}}{v_{in}} = \frac{j\omega L}{R(1 - \omega^2 LC) + j\omega L} = \frac{\omega^2 L^2}{R^2 (1 - \omega^2 LC)^2 + \omega^2 L^2} \]

- Graphically
Phase Changes Through a Resonance

- Use $\tan \phi = C/B$
- For the series RLC circuit:

$$Z_{LC} = 1/j\omega C + j\omega L = \frac{1 - \omega^2 LC}{j\omega C}$$

\[\text{Diagram:} \quad \sqrt{B^2 + C^2} \quad \phi \quad B = R \quad jC = j\frac{\omega^2 LC - 1}{\omega C} \quad \phi = \text{atan} \frac{\omega^2 LC - 1}{\omega RC} \]

- The phase depends on the frequency
  - At low frequency, $\phi \rightarrow -90$
  - At $\omega^2 = 1/LC$, $\phi = 0$
  - At high frequency, $\phi \rightarrow +90$
Quality Factor

Ideal RLC Circuit

- A voltage divider with one impedance due to an inductor and capacitor in parallel, but let the resistance be nearly 0

\[ v_{out} = \frac{Z_L C}{R + Z_L C} v_{in} \]

- The gain in power for the circuit is:

\[ \left| \frac{v_{out}}{v_{in}} \right|^2 = \frac{\omega^2 L^2}{R^2 (1 - \omega^2 LC)^2 + \omega^2 L^2} \]

- At resonance the gain is 1.

- The frequency where the power is halved:

\[ \frac{1}{2} = \frac{\omega^2 L^2}{R^2 (1 - \omega^2 LC)^2 + \omega^2 L^2} \]

\[ \frac{1}{2} R^2 (1 - \omega^2 LC)^2 = \frac{1}{2} \omega^2 L^2 \]

\[ 1 - \omega^2 LC = \frac{\omega L}{R} \]

- This has solutions at roughly:

\[ \omega = \frac{1}{\sqrt{LC}} \pm \frac{R}{2L} = \omega_0 \pm \omega_0 \frac{R}{2\omega_0 L} = \omega_0 \pm \frac{\omega_0}{2Q} \]