1. Many-valued Logics—A Semantic Response to Vagueness

1.1. What are Many-valued Logics? In a nutshell, many-valued logics are logics with more than two truth values. Cashing out the details gets a little complicated, but that’s the basic idea.

1.2. What’s the connection to vagueness? A basic part of the challenge created by the phenomenon of vagueness is just answering the question of what vagueness is. Thus, a basic part of what a theory of vagueness must do is answer this question. Theories can be distinguished, among other things, according to the general nature of their answer. As the territory is typically laid out, there are semantic theories, which say that vagueness is a symptom of semantic (but not epistemic) facts, and there are epistemic theories, which say that vagueness is a symptom of epistemic (but not semantic) facts.

The most significant motivation for developing many-valued logics has been vagueness. Even so, the logics themselves aren’t theories of vagueness. In order to get such a theory, you have to say that one of the many-valued logics is the logic that exhibits the logico-semantic properties of natural language. Further, you have to say that attributing this logico-semantic character to natural language makes possible an adequate solution to sorites paradoxes. If you say this, you’ve given a theory of vagueness with a many-valued logic at its core. What is vagueness on such a theory? It is the feature of having borderline cases, as evidenced a lack of decidable truth-conditions; it is had by certain predicates (or sentences) that have a certain special semantics when these predicates (sentences) are treated as though they had classical semantics. In addition to applying or failing to apply to objects, these predicates can also be semantically related to them in one or more other ways. For instance, they might partially apply (to some degree); or they might stand in some other sort of relation entirely, not applying but not failing to apply either. Corresponding to any of these special semantic relations will be a special additional truth-value. Thus, many-valued theories are semantic theories of vagueness.

1.3. How will Sorites paradoxes be handled? The problem presented by sorites paradoxes is that they seem to constitute logically valid arguments
with true premises and false conclusions. (That’s just the definition of a paradox.) If you want to reject the arguments’ conclusions as false, from a classical perspective there are two ways to respond. You can either deny a premise, or you can reject the relevant argument form as invalid (not preserving truth). If you go for a many-valued theory, your options change depending on how you construe validity. Validity on a many-valued theory is preservation of what are called ‘designated’ truth-values; these need not be limited to (perfect) truth. Regarding validity in this way enables you to say that either that the argument form of the sorites is valid, but the premises do not take designated values, or that the argument form is invalid in that it fails to preserve designated values. Many-valued theories constitute repudiations of the idea that the logico-semantic structure of languages with vague terms, like English, is properly represented with classical logic and its attendant conception of validity. There is more to the story depending on which many value theory is in question, of course, but the general strategy is to reject sorites arguments as invalid. This rejection is of course based on the hypothesis that the proper logic for languages with vague terms is many-valued, and validity (logical consequence) for such languages is thus different than it would be if such languages obeyed classical logic.

2. THREE-VALUED LOGICS

The objection to two-valued logic was the supposed impossibility of classifying all vague propositions as true or false: but the phenomenon of second-order vagueness makes it equally hard to classify all vague propositions as true, false or neither. As grain is piled on grain, we cannot identify a precise point at which ‘That is a heap’ switches from false to true. We are equally unable to identify two precise points, one for a switch from false to neutral, the other for a switch from neutral to true. If two values are not enough, three are not enough. [Vagueness, p. 111]

2.1. Second-Order Vagueness. Since Tarski, philosophers have become accustomed to sharply separating the ‘object language’ from the ‘metalinguage’ when studying the semantic features of a language. The object language is the language whose semantic characteristics are being theorized about. The metalanguage is the language that is used to express the theory about the object language; ordinarily the metalanguage is taken to include the object language as a proper part. This distinction makes most sense in speaking of artificial, formal languages constructed for purposes of studying logical concepts. In natural language semantics, however, it seems both
improper (often) to say either (a) that any one natural language is the lan-
guage under study, or (b) that the object language (whatever it may be) can
be sharply separated from the metalanguage (we seem to theorize about
English in English). These potential irks notwithstanding, ‘second-order
vagueness’, as philosophers such as W speak of it, is supposed to occur
when the metalinguistic theoretical statements that purport to give the se-
matic features of statements in the object language are themselves subject
to vagueness.

So suppose Jones is a borderline case of being bald and consider the
statement ‘Jones is bald’. That statement is a constituent of our ‘object
language’—insofar as we are going to study its semantics. Now, what a
three-value theorist will say in giving its semantics is that the meaning of
‘is bald’ is such that under certain well-defined conditions, the statement
is true, under certain other conditions false, and under certain other condi-
tions it is indeterminate. What it is for ‘Jones is bald’ to be (seem) vague
on this theory is just for its actual semantics to be such that there are con-
ditions under which it is indeterminate and for us to mistakenly treat it as
though there are no such conditions. But, W asks, what do we say about the
statement “Jones is bald’ is true’? It seems that that statement (which is a
statement in the metalanguage with respect to ‘Jones is bald’) is going to be
true iff the well-defined truth condition of ‘Jones is bald’ obtains, and false
otherwise. But here we have reintroduced bivalence in giving the semantics
of our metalinguistic statement.¹ Let’s step back.

According to the three-valued theory of vagueness, a bivalent treatment
of certain statements is what makes them subject to vagueness, and the rem-
edy is to treat them as trivalent. The symptom of vagueness is having bor-
derline cases—cases where all the underlying facts can be known while the
status of the statement remains undecideable. Going trivalent is supposed to
alleviate this symptom by grouping these cases together into a set and treat-
ing this set as a new condition corresponding to a third type of status that
the statement can have: indeterminacy. Given this adjustment in our con-
ception of the statement’s semantics, the statement’s vagueness is supposed
to be removed, and hence, the decideability of the status of the statement
(as T, F, or I) is supposed to be restored.

Here’s the problem. We clearly seem to be faced with a borderline case
with “Jones is bald’ is true’, hence it is vague. But then it follows that
the borderline between the truth condition and the indeterminacy condition
for ‘Jones is bald’ is itself subject to vagueness. This is so because the
borderline between truth and indeterminacy for ‘Jones is bald’ is the same

¹ Note: We are here using a third level language—a metalanguage of level 2—to give
these semantics.
as the borderline between truth and falsity for “Jones is bald’ is true’—if this latter statement is treated as bivalent.

Treating “Jones is bald’ is true’ as trivalent doesn’t seem to remove the problem, either. To do this is to say that there is a set of those cases that lie “between” the truth condition and the indeterminacy condition for ‘Jones is bald’; these “between” cases are cases where it is indeterminate whether ‘Jones is bald’ is true or indeterminate. But then what are we to say about the status of ‘Jones is bald’ when one of these “between” cases obtains? It cannot be that it is true or false. It seems that the only options are to say is that it is indeterminate, or that it is undefined. But to say the former is just to group these “between” cases in with the indeterminacy condition for ‘Jones is bald’, and this is just to deny that the border between truth and indeterminacy is vague. But it seems that these borders are vague. To say the latter is just to admit that the trivalent treatment doesn’t make the status of the statement a function of the facts—which is contrary to part of the original goal. In sum, if we admit that the margins separating truth and indeterminacy and falsity are subject to borderline cases, then we grant that the trivalent treatment of ‘Jones is bald’ fails to remove its vagueness—fails, that is, to make the status of the statement with respect to the facts decidable.

Here’s a final question: Does it make sense to talk about cases where we remain undecided about whether we are undecided on our judgment or not? Aren’t these just cases where we are undecided?

3. Fuzzy Logics

Any treatment of the case within a semantics of finitely many values must divide the continuous process of darkening into a finite sequence of discrete segments, corresponding to the different values though which the sentence ‘The patch is dark’ is supposed to pass. Any particular choice of segments seems arbitrary. A continuum of degrees of truth is attractive because it promises to avoid such arbitrary choices. The appearance of continuity can be taken at face value. [p. 114]

3.1. Basics of Fuzzy Logics. On fuzzy logics, there are an uncountable infinity of truth-values. Applied to natural language statements, this means roughly that the truth-condition for a statement amounts to a mapping into the closed real number interval [0, 1], each member of which represents a distinct truth-value. How this mapping will be construed precisely will be a task for the philosopher who wishes to treat natural language as having fuzzy semantics.
In addition to utilizing a continuum of truth-values, fuzzy logic seeks to retain truth functionality for the standard logical connectives. So the truth-value of a logically complex expression is supposed to be a function of the truth-values of its constituents.

Here are the rules for ‘and’ and ‘or’:

\[(\&) \ [p \& q] = \min \{[p], [q]\}\]
\[(\lor) \ [p \lor q] = \max \{[p], [q]\}\]

Universal quantification is treated in terms of the rule for ‘and’; existential quantification is treated in terms of the rule for ‘or’.

Truth-functional treatments of negation, the material conditional, and the biconditional are, as W says, “less straightforward.”

Negation seems straightforward enough to me to require little explanation:

\[(\neg) \ [\neg p] = 1 - [p]\]

The biconditional is a little trickier. Here is what W says:

Consider the biconditional. If p and q have exactly the same degree of truth, then p \equiv q should be perfectly true. If p is perfectly true and q is perfectly false, or vice versa, then p \equiv q should be perfectly false. When the truer component decreases in degree of truth and the less true component increases, the biconditional should increase in truth, but at what rate? The simplest assumption is that the degree of truth of the biconditional is perfect truth minus the difference between the degrees of truth of its components. [p. 116-17]

Hence, the function for ‘if and only if’ is the following:

\[(\equiv) \ [p \equiv q] = 1 - (\max \ {[p], [q]} - \min \ {[p], [q]} )\]

The conditional is then defined using the biconditional and conjunction in the following way:

\[(\supset) \ [p \supset q] = [p \equiv (p \& q)]
= 1 - ([p] - \min \ {[p], [q]} )\]

These truth functions were devised by the Polish logician Łukasiewicz. Ł also developed the notion of validity in fuzzy logic; in his logic, validity is preservation of perfect truth under all interpretations. There are few things to note.
While, as W points out, most of the valid formulas of classical logic will remain valid on Ł’s logic, excluded middle fails. If \( p \) is neither perfectly true nor perfectly false, \( (p \lor \neg p) \) is not perfectly true. It is at best never less than half true.

In addition, some classical tautologies can be less than half true. The example W uses is \( \neg(p \equiv \neg p) \). This classical tautology will be perfectly false when \( p \) is half true—the embedded contradiction will be perfectly true.

A few additional non-classical connectives are legitimized by fuzzy logic; notably a conditional which is perfectly true if its consequent is at least as true as its antecedent, and perfectly false otherwise.

3.2. Handling the Sorites. On a fuzzy logic, a sorites sequence of a sufficiently hefty size can have the feature that the atomic premise is perfectly true, and all of the conditional premises are true to the same very high degree, and the conclusion is perfectly false. There two notable ways to handle this.

On the one hand, validity can be regarded as preservation of perfect truth. In this case the modus-ponens-based form of the sorites sequence will be valid, but this will not yield the result that its conclusion must, paradoxically, be true—because not all the premises are perfectly true. On the other hand, validity can be regarded as preservation of the least degree of truth in the premises. In this case modus-ponens will be invalid, and hence, so will the sorites sequence.

Side-note: Validity could be regarded as itself admitting of degrees, but modus ponens would have a very low degree of validity on such a view—we can discuss this if there is interest.

3.3. Second-Order Vagueness Yet Again. W’s main objection to a fuzzy-logic-based theory of vagueness is that it is, in a way analogous to the trivalent theories, subject to higher order vagueness. Let’s see if we can understand this. On the one hand, in the case of trivalent theories of vagueness, higher order vagueness was shown to attack the boundaries between the new sets of truth-conditions. But the idea that motivates fuzzy theories of vagueness is that there are no definite boundaries between truth conditions of vague statements—the truth-values form a perfect continuum. So one’s first thought might be that there are no definite boundaries in the truth conditions of vague statements for second-order vagueness to target. However, in the case of trivalent theories, the import of higher order vagueness was shown to be that it purports to undermine the hypothesis that additional truth values make vague statements decidable in every case. Here we can see the problem clearly.

If we recall that decideability is supposed to be the hallmark of non-vagueness that many-valued treatments are aiming for in reconstruing the
truth conditions of vague statements, we can see that the problem of higher order vagueness takes an extreme form in the case of fuzzy theories. There are two aspects to the problem: arbitrariness, and undecideability. These can be drawn out separately by considering cases where the vague predicate’s degree of application supervenes on a discrete underlying factor (as with ‘bald’, perhaps, or ‘a swarm’), and cases where the underlying factor is itself continuous (as with ‘a child’, or ‘tall’).

3.4. **Problems with Truth-functionality?** W also objects to fuzzy theories on the basis of their degree-functional treatment of the connectives. I don’t think these objections are all that forceful. It seems that W simply begs the question. I’ll explain why I think this in class.

4. **Additional Thoughts**

4.1. **What could an additional truth-value be?** Here is a style of objection to many-valued theories that W doesn’t consider. We seem to have an intuitive grasp of Truth (capital T) and Falsity (capital F). But one might stress that we lack any intuitive notion of what it might be to be “half-true,” or to be “indeterminate” (where this doesn’t signify an incompleteness of meaning). How much is there to this?

You might think that the intuition we have concerning truth might be fleshed out as follows: a statement is true when things are as it states, and false otherwise. Thus, it seems that in a loose sense, perhaps, our clear grasp of truth is based on our conception of meaningful statements as representing the world as being a certain definite way. Truth is just the status of those statements that are such that the world is the way they represent it. Falsity is the status of all the other statements. Consider a fuzzy theory. It seems on first glance that on such a theory, contrary to our intuition, a vague statement does not represent the world as instantiating a certain definite state of affairs, and this isn’t because it is deficient in meaning. Rather, it is because there is a continuum of states of affairs which would be compatible to varying degrees with the statement. The statement itself does not represent the world as instantiating any one of these states of affairs in particular (that would make it bivalent), nor does it represent it as instantiating them all at once (this is obviously incoherent). We might say that it represents the world as instantiating some unspecified one of them. But if this is how our statements mean, then where do we get the original intuition of a definite constraint, which still seems so clear? More pointedly, what are we doing when we assert a vague statement?

Perhaps the degree-based MV-theorist can say the following. Though the statement does not represent the world as instantiating a definite state of affairs, the speaker certainly does represent the world as instantiating
a definite state of affairs when he asserts the statement. Part of asserting is pragmatically conveying complete assent, or certainty; and if we do not qualify the statement as “partly right,” or something similar, there is a pragmatic assumption in play that we regard it as perfectly true. So this response would be to say that to assert a vague statement is to assert that it is perfectly true. But to say this is to say that what we assert is a second-order statement that is not vague unless there is second-order vagueness. Will we say the same thing about other speech acts and propositional attitudes? Can none of these coherently take first-order vague content either?

One might try to reject this line of reasoning in the following way. One might say that a singular statement with a vague predicate represents the relevant object as being a member of the set that corresponds to the predicate. It’s just that being a member is a matter of degree. But this response contains the ground for its own rejection. Presumably ‘being a member’ here means being a perfect member. But given that there are degrees of membership, vague statements cannot represent their subjects as being perfect members of the relevant predicate sets, as that would make all such statements about objects that are partial members of these sets perfectly false iff there is no second-order vagueness. This reasoning generalizes. Vague statements cannot represent their subjects as being members-to-degree-

It seems correct to say that we should not assert things that are only partly true (and hence “borderline”); therefore it is not wholly implausible prima facie to say that when we make assertions of vague statements, what we say is a second-order content to effect that the first order statement which we uttered is perfectly true. But it seems quite problematic to say that when we doubt whether Bobby is a child, we are always thereby doubting whether Bobby is a clear case of a child. We may grant that he clearly isn’t a clear case of a child, but still withhold judgment on whether he is a child. What is the content of what we are doubting here? Are we doubting whether Bobby is a child to degree .568? Degree .4392? Are we doubting whether he is a child to any degree? If this latter thing is what’s going on, then we are doubting whether he is so much as a child to .000000000001 degree.

4.2. Do fuzzy theories entail ontological vagueness or nihilism? Consider kind-concepts like ‘person’. Kind-concepts are bound up with identity conditions for the things that instantiate them, hence to say that they are
vague is tantamount to saying that certain objects have vague identity conditions. Making sense of this would require endorsing degrees of existence.

4.3. **Fuzzy Epistemicism?** Could second-order vagueness on the fuzzy theory be handled in a way similar to the way that W handles first-order vagueness?