Problem 1. In class and in the Primer, we discussed the O’Raifeartaigh model, with the superpotential

\[ W = -k \Phi_1 + m \Phi_2 \Phi_3 + \frac{y}{2} \Phi_1 \Phi_3^2, \]

with \( y, k, m \) each real and positive. We considered the case \( m^2 > yk \). Now, consider the opposite case:

\[ m^2 < yk. \]

(a) Prove that at the minimum of the potential, \( F_3 = 0 \) always.

(b) Find the values of \( \phi_3, F_1, \) and \( F_2 \) at the minimum of the potential. (You should find that they are all non-zero.) What can you say about the values of \( \phi_1 \) and \( \phi_2 \) at the minimum? [Hint: there is a “flat direction”.]

(c) Show that the value of the potential at its minimum is

\[ V_{\text{min}} = (n_1 ykm^2 + n_2 m^4)/y^2, \]

where \( n_1 \) and \( n_2 \) are non-zero integers that you will find.

(d) Find the part of the Lagrangian that involves the fermion fields \( \psi_1, \psi_2, \psi_3 \). In this expression, replace \( \phi_3 \) by its VEV that you found in part (b). Write the resulting Lagrangian in the form:

\[ \mathcal{L} = i \bar{\psi}^i \sigma^\mu \partial_\mu \psi_i + \frac{1}{2} [\bar{\psi}_i M^{ij} \psi_j + \text{c.c.}] \]

where \( M \) is a \( 3 \times 3 \) mass matrix.

(e) Square the mass matrix \( M \) found in part (d), and obtain the eigenvalues of \( M^2 \). For each eigenvalue \( m_i^2 \), find the corresponding eigenvector. Write these as normalized eigenvectors

\[ \psi'_i = N_i^j \psi_j \]

(1)

with

\[ \sum_j |N_i^j|^2 = 1. \]

for each \( i \). This implies that \( N \) is a unitary matrix. [Here \( \psi_{1,2,3} \) are the original fermion fields, and \( \psi'_{1,2,3} \) are the squared-mass eigenstate fields. Four of the entries of \( M^2 \) should
vanish; the same is true of \( N \).] Check your work by rewriting the fermion part of the Lagrangian as:

\[
\mathcal{L} = i\psi_i \gamma^\mu \partial_\mu \psi_i' + \frac{1}{2} \sum_i m_i \psi_i' \psi_i' + \text{c.c.},
\]

(f) What is the goldstino?

(g) Find the part of the Lagrangian involving only the scalars. In it, make the replacement:

\[
\phi \rightarrow \langle \phi \rangle + \phi',
\]

where \( \langle \phi \rangle \) is the VEV that you found in part (b), and then throw away the (scalar)^3 and (scalar)^4 parts of the potential. Write the resulting potential in the form:

\[
V = V_{\text{min}} + (\phi_1^* \phi_2^* - \phi_1 \phi_2) \mu^2 \left( \frac{\phi_1}{\phi_2} \right) + a|\phi_3'|^2 + b(\phi_3^2 + \phi_2^* \phi_3),
\]

where \( \mu^2 \) is a 2 \( \times \) 2 matrix that you will find, and \( a \) and \( b \) are squared mass quantities. (The part of the potential that is linear in the scalar fields should vanish.)

(h) Find the squared mass eigenvalues and eigenvectors for \( \phi_1 \) and \( \phi_2 \), using the matrix \( \mu^2 \) found in part (g). [Hint: this should be remarkably similar to part (e). The correct eigenvectors are linear combinations of \( \phi_1 \) and \( \phi_2 \); there is no need to split them up into real and imaginary parts.]

(i) Write \( \phi_3' = (R_3 + iI_3)/\sqrt{2} \) where \( R_3 \) and \( I_3 \) are real scalar fields. Find squared-mass matrix for these fields, using the results of part (g). What are the squared-mass eigenvalues?

(j) Make a list of all of the real scalar and Weyl fermion squared masses in the theory, as found above, and check for evidence that supersymmetry was indeed broken. Finally, check that the sum rule, equation (6.13) in the Primer, is satisfied.