Problem 1. On page 14 of the Phys 586 notes, the Lorentz transformation properties of Dirac spinors was discussed. Translate this into a discussion of the Lorentz transformation properties of Weyl two-component left-handed spinors and right-handed spinors, in terms of \( \bar{\sigma} \) and \( \sigma \) matrices.

Problem 2. Suppose we have a Lagrangian consisting of Weyl two-component fermions \( \psi_i \) (where \( i = 1, \ldots n \)) and complex scalar fields \( \phi_I \) (where \( I = 1, \ldots N \)). The Lagrangian is:

\[
\mathcal{L} = i \psi i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i + \bar{\sigma}^\mu \phi_i^\dagger \partial_\mu \phi_I - \frac{1}{2} (M^{ij} \psi_i \psi_j + \text{c.c.}) - [m^2]^I_j \phi^\dagger I \phi_J - \frac{1}{6} (y^{ijk} \phi_I \psi_j \psi_k + \text{c.c.}).
\]

Find the necessary and sufficient conditions on each of the masses \( M^{ij} \) and \( [m^2]^I_j \) and the coupling \( y^{ijk} \) so that the Lagrangian is invariant under the global symmetry transformation:

\[
\phi_I \rightarrow \phi_I + i e^a T^a I \phi_J \\
\psi_i \rightarrow \psi_i + i e^a t^{ij} \psi_j.
\]

(Here, \( T^a \) and \( t^a \) are the symmetry generators for the representations of the scalars and fermions, respectively.)

Problem 3. Consider the see-saw model for a single neutrino species, given in terms of two-component left-handed Weyl spinors \( \nu \) and \( N \) by the Lagrangian:

\[
\mathcal{L} = i \nu^\dagger \bar{\sigma}^\mu \partial_\mu \nu + i N^\dagger \bar{\sigma}^\mu \partial_\mu N - \left( m \nu N + \frac{M}{2} NN + \text{c.c.} \right)
\]

where "c.c." means complex conjugate, and \( m \) and \( M \) are mass parameters. We want to find the mass eigenstates, so we will do a field redefinition of the form:

\[
\begin{pmatrix} \nu' \\ N' \end{pmatrix} = U \begin{pmatrix} \nu \\ N \end{pmatrix}
\]

(1)

where \( U \) is a unitary \( 2 \times 2 \) matrix.

(a) Assume that \( m \) and \( M \) are positive and real, but do not assume anything about their
relative sizes (yet). Find an appropriate $U$ to put the Lagrangian in the form:

$$\mathcal{L} = i\nu'^\mu \sigma^\mu \partial_\mu \nu' + iN'^\mu \sigma^\mu \partial_\mu N' - \frac{1}{2} (m_{\nu'} \nu' \nu' + M_{N'} N' N' + \text{c.c.)}$$ (2)

where $m_{\nu'}$ and $M_{N'}$ are positive quantities that you will find.

(b) Now suppose that $m \ll M$, and show that the masses are given by:

$$m_{\nu'} = m^2 / M + \ldots$$
$$m_{N'} = M + \ldots,$$

and find the next-order correction in each case.

(c) The interaction of the gauge-eigenstate neutrino $\nu_e$ with the $W$ boson and the electron is given by a term in the Lagrangian:

$$\mathcal{L} = -\frac{g}{2} \left( e^\dagger \sigma^\mu \nu W^-_\mu + \text{c.c.} \right).$$

(This is given in two-component Weyl notation; see equation (12.16) in the Phys 586 notes for the four-component spinor version.) By what proportion is the interaction strength for the mass eigenstate $\nu'$ smaller than that for $\nu$? (Give your answer to lowest non-trivial order in $m/M$.) What is your answer for $m = 100$ GeV and $M = 10^{16}$ GeV? This explains the statement that in the see-saw model, the neutrinos interact almost exactly the same way as in the Standard Model.

(d) The assumption that the original parameter $m$ and $M$ are real is actually completely unnecessary. Explain why, by describing how to modify your unitary matrix $U$ so that $m_{\nu'}$ and $m_{N'}$ will be real and positive, no matter what the complex phases of the original parameters $m$ and $M$ were.