Physics 751 **Homework 9** **Due Friday, April 19, 2019**

Reading assignment: sections 8.4, 8.5, 8.7, 8.8 of Carroll.

Problem 1. Consider de Sitter space in coordinates where the metric takes the form

$$ds^2 = -dt^2 + e^{2Ht}[dx^2 + dy^2 + dz^2],$$

where H is a constant. Find all of the Christoffel symbols, and obtain the geodesic equations for a non-comoving observer moving in the x direction (x not constant). Assume the observer starts with $dx/d\tau = v_0$ at time $t = 0$. Use the geodesic equations to find a relationship between the coordinate time t and the proper time τ for such an observer. You may use as an additional boundary condition that t and τ coincide in the limit of very large positive t. Show that the observer's geodesic was at $t = -\infty$ at a *finite* proper time ago (and calculate that τ), demonstrating that these coordinates fail to cover the entire manifold.

Problem 2. Before it was understood that the universe is really expanding, Einstein tried to devise a Friedmann-Robertson-Walker universe including matter that was static, i.e. $\dot{a} = 0$ and $\ddot{a} = 0$ at all t. Show that this is possible in a universe filled with just dust and a non-zero cosmological constant, and find the necessary relation between the dust density ρ_{dust} and the cosmological constant Λ. Is the required cosmological constant positive or negative? Also, find the static value of $a = a_{\text{static}}$. Is the spatial curvature in this universe positive, negative, or zero? Consider the stability of the solution, by seeing what happens if the special static solution is perturbed slightly. Do this by taking $a = a_{\text{static}} + \delta a$ and $\rho_{\text{dust}} = \rho_{\text{static}} + \delta\rho$, and solve for $\delta\ddot{a}/\delta a$, eliminating $\delta\rho$ and neglecting everything that is 2nd order in the perturbations, and writing everything in terms of Λ wherever possible. What is the sign of $\delta\ddot{a}/\delta a$, and what does this mean for stability? (This episode shows that even Einstein made mistakes. However, this solution is the reason he invented the cosmological constant in the first place, and that part turned out to not be a mistake.)

Problem 3. Consider the standard cosmological model defined by cosmological constant, dust, radiation and curvature parameters Ω_Λ, Ω_d, Ω_r and Ω_c. Find formulas for these parameters as functions of the scale factor a and their values today $\Omega_{\Lambda,0}$, $\Omega_{d,0}$, $\Omega_{r,0}$ and $\Omega_{c,0}$, taking $a = 1$ today. Now suppose that today the parameters have values given by $\Omega_{d,0} = 0.31$, and $\Omega_{r,0} = 5 \times 10^{-5}$, and (as predicted by inflation, and consistent with observations) $\Omega_{c,0} = 0$. Plot the three quantities Ω_Λ, Ω_d, Ω_r as functions of a, from $a = 10^{-30}$ to $a = 10^{30}$. Use a log scale for the a axis. The reason for choosing this range is that the Planck time is at roughly $a = 10^{-30}$. Also label on your graph the scale factors for nucleosynthesis and recombination, as $a_{\rm BBN}$ and $a_{\rm rec}$ respectively. At what redshift z did the radiation density cease to dominate? At what redshift z did the cosmological constant start to dominate?

[Hint: you do *not* need to solve any differential equations or do any integrals to do this problem; it is entirely algebraic. You are encouraged to use a spreadsheet or any other computer program of your choice to generate the points for your plot.]