Physics 751 **Homework 4** **Due Friday, February 22, 2019**

Reading assignment: sections 3.5 through 3.7 of Carroll.

Problem 1. The metric near the surface of the Earth in coordinates (t, r, θ, ϕ) is given to a good approximation by

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2,$$

where

$$\Phi = -\frac{GM}{r}$$

is the familiar Newtonian potential, with $G =$ Newton's constant and M the mass of the Earth. You should neglect anything quadratic or higher order in Φ, or equivalently quadratic or higher order in GM. [So, for example, $1/(1 + 2\Phi) = (1 - 2\Phi)$ within this approximation.] Also, neglect the rotation of the Earth.

(a) Consider a clock fixed on the surface of the Earth at distance R_1 from the center, and another clock above it in a tall building at a distance R_2 from the center. Calculate the time elapsed on each clock (proper time τ) as a function of the elapsed coordinate time t. Which clock moves faster?

(b) Compute all of the components of the Christoffel symbol, being sure to work to linear order in GM. [Hint: this is a good chance to make use of the result of Problem 1 on the last homework set.]

(c) Consider the geodesic equation in terms of $U^\mu = dx^\mu/d\tau$, see eq. (3.60) in Carroll. Write the components of this geodesic equation out explicitly for the special case of a geodesic corresponding to an equatorial circular orbit at fixed radius R, so that U^t and U^ϕ are both constants and $U^r = U^\theta = 0$. Solve the geodesic equations for the components of U. Solve for $\omega^2 = (d\phi/dt)^2$, which should give you the same result in terms of G, M, and R as one would get from Newtonian gravity.

(d) How much proper time elapses for a satellite at radius R to complete one equatorial circular orbit? For the special case that R is chosen to be equal to the radius of the Earth (neglecting air resistance, bumping into mountains, and such), compute the difference between this proper time and that for a clock held stationary at the equator. What is it, numerically?

Problem 2. Consider the two-dimensional Lorentzian spacetime with coordinates

$$-\infty < u < \infty, \qquad 0 \leq \phi \leq 2\pi$$

and metric

$$ds^2 = A^2(-du^2 + \cosh^2 u \, d\phi^2),$$

where A is a constant. Do not use pre-configured software for this problem.

(a) Compute the Christoffel symbols.

(b) Compute the Riemann tensor, the Ricci tensor, and the Ricci scalar. You should find that this is a manifold with constant curvature (in other words, the Ricci scalar is constant). This manifold-with-metric is called two-dimensional de Sitter spacetime. We will encounter four-dimensional de Sitter spacetime later.

Problem 3. Starting from the definition of the Weyl tensor, eq. (3.147) in Carroll, prove that it obeys the trace-free condition:

$$C^{\mu}{}_{\sigma\mu\nu} = 0.$$